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Existence and multiplicity of periodic solutions for a class of second-order ordinary differential equations

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Abstract

In this paper, we study the existence of positive periodic solutions for a class of non-autonomous second-order ordinary differential equations

$$x'' + \alpha x' + a(t)x^n - b(t)x^{n+1} + c(t)x^{n+2} = 0,$$

where $\alpha \in \mathbb{R}$ is a constant, n is a finite positive integer, and $a(t), b(t), c(t)$ are continuous periodic functions. By using Mawhin's continuation theorem, we prove the existence and multiplicity of positive periodic solutions for these equations.

Keywords Second-order ordinary differential equations · Positive periodic solutions · Mawhin's continuation theorem

Mathematics Subject Classification 34B18 · 34K13 · 34C25

1 Introduction and main results

In the past few years, scholars have become more and more interested in the study of differential equations in some mathematical models that arise in Biology and Physics, such as the equations

$$x'' + a(t)x - b(t)x^2 + c(t)x^3 = 0, \tag{1.1}$$

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where $a(t), b(t), c(t)$ are positive continuous periodic functions. Eq. (1.1) comes from a biomathematics model and was suggested by Cronin in [1] and Austin in [2]. Equation (1.1) description of some of the properties of an aneurysm of the circle of Willis, where x is the velocity of blood flow in the aneurysm, $a(t), b(t), c(t)$ are coefficient functions related to aneurysm. For more equations related to the model, see [3–5].

Equation (1.1) have been studied by several authors, see [6–8]. The main tools used by these authors for obtaining their results are variational method and coincidence degree theories. At the same time, the existence of periodic solutions of nonlinear differential equations has been studied, see for instance the papers [9–17].

In this paper, our purpose is to establish the existence and multiplicity of positive periodic solutions of the non-autonomous second-order nonlinear ordinary differential equations

$$x'' + \alpha x' + a(t)x^n - b(t)x^{n+1} + c(t)x^{n+2} = 0, \tag{1.2}$$

where n is a positive integer, $\alpha \in \mathbb{R}$ is a constant, and $a(t), b(t), c(t)$ are continuous T-periodic functions on $\mathbb{R}$, subject to the constraints $0 < a \le a(t) \le A,\ 0 < b \le b(t) \le B,\ 0 < c \le c(t) \le C$, or $-A \le a(t) \le -a < 0,\ -B \le b(t) \le -b < 0,\ -C \le c(t) \le -c < 0$.

We also consider the following particular case of Eq. (1.2)

$$x'' + \alpha x' + a(t)x^n - b(t)x^{n+1} = 0, \tag{1.3}$$

namely, coefficient function $c(t) \equiv 0$ of Eq. (1.2).

We will use coincidence degree theories to prove the existence of at least two positive periodic solutions for Eq. (1.2) and at least one positive periodic solution for Eq. (1.3), under some specific assumptions on a, A, b, B, c, C, n, T to be given later, and we will calculate the exact interval of the existence of the solutions and one of the least upper bound of the period T. It is worth noting that when $\alpha = 0, n = 1$ and $0 < a \le a(t) \le A,\ 0 < b \le b(t) \le B,\ 0 < c \le c(t) \le C$, Eq. (1.2) reduce to Eq. (1.1).

Our main results are as following theorems.

Theorem 1.1 *Let $a(t), b(t), c(t)$ be continuous T-periodic functions with*

$$0 < a \le a(t) \le A, \quad 0 < b \le b(t) \le B, \quad 0 < c \le c(t) \le C, \tag{1.4}$$

or

$$-A \le a(t) \le -a < 0, \quad -B \le b(t) \le -b < 0, \quad -C \le c(t) \le -c < 0,$$

where a, A, b, B, c, C be positive constants such that

$$b^2 - 4AC > 0, \tag{1.5}$$

$$\frac{B - \sqrt{b^2 - 4AC}}{2c} < \frac{b + \sqrt{b^2 - 4AC}}{2C}. \tag{1.6}$$

If the period T satisfies

$$0 < T \le \frac{1}{\beta^2(AN_1^{n-1} + BN_1^n + CN_1^{n+1} + 1)},$$

where β is the constant immersion of $H^1(0, T)$ in $C([0, T])$, $N_1 = \frac{B+\sqrt{B^2-4ac}}{2c} + 1/2$, and n is a finite positive integer. Then Eq. (1.2) has at least two positive T-periodic solutions.

In Theorem 1.1, we assume that the coefficient functions $a(t)$, $b(t)$ and $c(t)$ have no zero and have same sign, if one of them is identical to zero, Theorem 1.1 will not hold. In the following Theorem, we give the case when $c(t) \equiv 0$.

Theorem 1.2 *When $c(t) \equiv 0$, let $a(t)$, $b(t)$ be continuous T-periodic functions with*

$$0 < a \le a(t) \le A, \quad 0 < b \le b(t) \le B \tag{1.7}$$

or

$$-A \le a(t) \le -a < 0, \quad -B \le b(t) \le -b < 0,$$

where a, A, b, B are positive constants. If the period T satisfies

$$0 < T \le \frac{1}{\beta^2(AF^{n-1} + BF^n + 1)},$$

where β is the constant immersion of $H^1(0, T)$ in $C([0, T])$, $F = \frac{A}{b} + \epsilon > 0$, where $\epsilon > 0$ small enough such that $\frac{a}{B} - \epsilon > 0$, and n is a finite positive integer. Then Eq. (1.3) *has at least one positive T-periodic solution.*

Remark 1.3 In this case, we can only get the existence of one positive periodic solution.

Remark 1.4 Theorem 1.2 also holds in the case of the coefficient function $a(t) \equiv 0$ of Eq. (1.2).

Remark 1.5 There is no result when $b(t) \equiv 0$ of Eq. (1.2).

2 Preliminaries

In this section, we given some notations and preliminary results which paly important roles in the prove of our main result. For more details see [18].

Definition 2.1 Let X, Y be real Banach spaces, $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping. The mapping L is said to be a Fredholm mapping of index zero if

(a) $\operatorname{Im} L$ is closed in Y;
(b) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$.

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$\begin{aligned} \operatorname{Im} P &= \operatorname{Ker} L, \\ \operatorname{Ker} Q &= \operatorname{Im} L = \operatorname{Im}(I - Q). \end{aligned}$$

It follows that the restriction L_P of L to $\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of L_P by K_P.

Definition 2.2 If Ω is a bounded open subset of X, N is called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.3 *(Mawhin's Continuation Theorem). Let L be a Fredholm mapping of index zero, $\Omega \subset X$ is an open bounded set and let N is $L-$compact on $\overline{\Omega}$. If all the following conditions hold:*

(1) $Lx \neq \lambda Nx$ *for all* $x \in \partial\Omega \cap \operatorname{Dom} L$, *and all* $\lambda \in (0, 1)$*;*
(2) $QNx \neq 0$, *for all* $x \in \partial\Omega \cap \operatorname{Ker} L$*;*
(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$.

Consider the following Banach spaces

$$X = Y = \{x | x \in C(\mathbb{R}, \mathbb{R}), x(t + T) = x(t), \forall t \in \mathbb{R}\}$$

with the norm $\|x\|_X = |x|_\infty$ *and* $\|x\|_Y = |x|_\infty$, *where* $|x|_\infty = \max\limits_{t \in [0,T]} |x(t)|$.

Define a linear operator $L : \operatorname{Dom} L \subset X \to Y$ *by setting*

$$Lx = x'' + \alpha x', \quad x \in \operatorname{Dom} L,$$

where

$$\operatorname{Dom} L = \{x | x \in X, x'' \in C(\mathbb{R}, \mathbb{R}), x'(t + T) = x'(t)\}.$$

It is immediate to prove that $\operatorname{Ker} L = \mathbb{R}$ *and*

$$\operatorname{Im} L = \left\{ x \mid x \in Y, \quad \int_0^T x(s)ds = 0 \right\}.$$

It is not difficult to see that $\operatorname{Im} L$ *is a closed set in* Y *and*

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1.$$

Thus the operator L is a Fredholm operator with index zero.

Define a nonlinear operator $N : X \to Y$ *by setting*

$$Nx = -a(t)x^n + b(t)x^{n+1} - c(t)x^{n+2}.$$

Now we define the projector $P : X \to \operatorname{Ker} L$ *and the projector* $Q : Y \to Y$ *by setting*

$$Px(t) = \frac{1}{T}\int_0^T x(s)ds$$

and

$$Qx(t) = \frac{1}{T}\int_0^T x(s)ds.$$

Hence, $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$.

Lemma 2.4 *Let* L *and* N *be as before and assume that* $a(t), b(t), c(t)$ *satisfy the assumptions of Theorem 1.1. Then* N *is* $L-$*compact on* $\overline{\Omega}$ *with any open bounded subset* $\Omega \subset X$.

Proof Clearly, operator $QN : X \to Y$ by setting

$$QNx = \frac{1}{T}\int_0^T [-a(s)x^n(s) + b(s)x^{n+1}(s) - c(s)x^{n+2}(s)]ds.$$

Obviously, $QN(\overline{\Omega})$ is bounded. It is readily seen that when $a(t) \equiv 0$ or $c(t) \equiv 0$, $QN(\overline{\Omega})$ also is bounded.

Let $G(t, s)$ be the Green's function of

$$x''(t) + \alpha x'(t) = 0, \quad t \in [0, T],$$
$$\int_0^T x(t)dt = 0, \quad x^{(i)}(0) = x^{(i)}(T), \quad i = 0, 1.$$

When $\alpha = 0$, we obtain that

$$G(s,t) = \begin{cases} -\dfrac{(T-t)s}{T}, & 0 \le s \le t \le T, \\ -\dfrac{(T-s)t}{T}, & 0 \le t \le s \le T. \end{cases}$$

When $\alpha \neq 0$, we obtain that

$$G(s,t) = \begin{cases} \dfrac{1}{\alpha}(1 - e^{\alpha(s-t)}) - \dfrac{e^{\alpha(s-t)}}{\alpha(e^{T\alpha}-1)}, & 0 \le s \le t \le T, \\ -\dfrac{e^{\alpha(s-t)}}{\alpha(e^{T\alpha}-1)}, & 0 \le t \le s \le T. \end{cases}$$

Then $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ can be given by

$$(K_P y)(t) = \int_0^T G(t,s)y(s)ds.$$

It is immediate to prove that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Furthermore, N is $L-$compact on $\overline{\Omega}$ with any open bounded subset $\Omega \subset X$.

3 Proof of the main result

Proof of Theorem 1.1. In the preceding assumption, we assume that the coefficient functions $a(t)$, $b(t)$ and $c(t)$ have the same sign, which include both positive and both negative cases.

Case 1: $0 < a \leq a(t) \leq A$, $0 < b \leq b(t) \leq B$, $0 < c \leq c(t) \leq C$.

In this case, Eq. (1.2) is equivalent to equation

$$x'' + \alpha x' + a(t)x^n - b(t)x^{n+1} + c(t)x^{n+2} = 0, \tag{3.1}$$

where $0 < a \leq a(t) \leq A$, $0 < b \leq b(t) \leq B$, $0 < c \leq c(t) \leq C$.

Let

$$\Omega_1 := \{x \in X \mid M < x(t) < N_1\}, \tag{3.2}$$

which is an open set in X, where

$$N_1 := N + 1/2, \quad N := \frac{B + \sqrt{B^2 - 4ac}}{2c} > 0,$$
$$M := \frac{B - \sqrt{b^2 - 4AC}}{4c} + \frac{b + \sqrt{b^2 - 4AC}}{4C} > 0.$$

By (1.4), M and N are well defined.

By (1.5) and (1.6), we obtain

$$M < \frac{b + \sqrt{b^2 - 4AC}}{2C} \leq \frac{b(t) + \sqrt{b(t)^2 - 4a(t)c(t)}}{2c(t)} \leq \frac{B + \sqrt{B^2 - 4ac}}{2c} < N_1 \tag{3.3}$$

and

$$0 < \frac{2ac}{C(B + \sqrt{B^2 - 4ac})} \leq \frac{b(t) - \sqrt{b(t)^2 - 4a(t)c(t)}}{2c(t)}$$
$$\leq \frac{B - \sqrt{b^2 - 4AC}}{2c} < M \tag{3.4}$$

uniformly in t.

Let $0 < \lambda < 1$ and x be such that

$$x'' + \alpha x' + \lambda a(t)x^n - \lambda b(t)x^{n+1} + \lambda c(t)x^{n+2} = 0.$$

Multiplying by x and the integrating from 0 to T, we have that

$$\int_0^T [(x')^2 - \lambda a(t)x^{n+1} + \lambda b(t)x^{n+2} - \lambda c(t)x^{n+3}]dt = 0.$$

By (3.2), if $x \in \partial\Omega_1$, we have $M \leq |x|_\infty \leq N_1$. Then

$$\begin{aligned}
0 &= \int_0^T [(x')^2 - \lambda a(t)x^{n+1} + \lambda b(t)x^{n+2} - \lambda c(t)x^{n+3}]dt \\
&> \int_0^T (x')^2 dt - \int_0^T [a(t)x^{n+1} + b(t)x^{n+2} + c(t)x^{n+3}]dt \\
&= \int_0^T [(x')^2 + x^2]dt - \int_0^T [a(t)x^{n+1} + b(t)x^{n+2} + c(t)x^{n+3}]dt - \int_0^T x^2 dt \\
&\geq \|x\|^2_{H^1(0,T)} - \int_0^T x^2(A|x|_\infty^{n-1} + B|x|_\infty^n + C|x|_\infty^{n+1} + 1)dt \\
&\geq \|x\|^2_{H^1(0,T)} - \int_0^T x^2(AN_1^{n-1} + BN_1^n + CN_1^{n+1} + 1)dt \\
&\geq \frac{|x|_\infty^2}{\beta^2} - T|x|_\infty^2(AN_1^{n-1} + BN_1^n + CN_1^{n+1} + 1) \\
&= \left[\frac{1}{\beta^2} - T(AN_1^{n-1} + BN_1^n + CN_1^{n+1} + 1)\right]|x|_\infty^2 \\
&\geq 0,
\end{aligned}$$

where β is the immersion constant of $H^1(0, T)$ in $C([0, T])$, but this is contradiction. So

$$x'' + \alpha x' + \lambda a(t)x^n - \lambda b(t)x^{n+1} + \lambda c(t)x^{n+2} \neq 0 \ \text{ for } x \in \partial\Omega_1 \text{ and } \lambda \in (0, 1).$$

Therefore condition (1) of Lemma 2.3 holds for Ω_1.

By (3.3) and (3.4), we have that

$$\begin{aligned}
&-a(t) + b(t)N_1 - c(t)N_1^2 \\
&= -c(t)\left(N_1 - \frac{b(t) + \sqrt{b(t)^2 - 4a(t)c(t)}}{2c(t)}\right)\left(N_1 - \frac{b(t) - \sqrt{b(t)^2 - 4a(t)c(t)}}{2c(t)}\right) \\
&\leq -\frac{1}{2}c(t)\left(N_1 - \frac{b(t) - \sqrt{b(t)^2 - 4a(t)c(t)}}{2c(t)}\right) \\
&< 0.
\end{aligned}$$

$$-a(t) + b(t)M - c(t)M^2$$

$$= -c(t)\left(M - \frac{b(t)+\sqrt{b(t)^2-4a(t)c(t)}}{2c(t)}\right)\left(M - \frac{b(t)-\sqrt{b(t)^2-4a(t)c(t)}}{2c(t)}\right)$$
$$> 0.$$

That is

$$-a(t) + b(t)N_1 - c(t)N_1^2 < 0 \tag{3.5}$$

and

$$-a(t) + b(t)M - c(t)M^2 > 0 \tag{3.6}$$

uniformly in t.

Take $x \in \partial\Omega_1 \cap \operatorname{Ker} L$, we have $x = M$ or $x = N_1$. By (3.5) and (3.6), we know that for $\forall x \in \partial\Omega_1 \cap \operatorname{Ker} L$, we have that

$$QNx = \frac{1}{T}\int_0^T x^n(-a(t) + b(t)x - c(t)x^2)dt \neq 0.$$

Therefore condition (2) of Lemma 2.3 holds for Ω_1.

Now we consider $\frac{M+N_1}{2}$, the arithmetic mean of M and N_1. We define a continuous function

$$\begin{aligned} H(x,\mu) = &-(1-\mu)\left(x - \frac{M+N_1}{2}\right) \\ &+\mu\frac{1}{T}\int_0^T x^n(-a(t)+b(t)x-c(t)x^2)dt, \quad \mu \in [0,1]. \end{aligned}$$

Obviously, we obtain

$$H(x,\mu) \neq 0, \quad \forall x \in \partial\Omega_1 \cap \operatorname{Ker} L.$$

By using the homotopy invariance theorem, we find that

$$\begin{aligned} \deg(QN, \Omega_1 \cap \operatorname{Ker} L, 0) &= \deg\,(H(x,1), \Omega_1 \cap \operatorname{Ker} L, 0) \\ &= \deg\,(H(x,0), \Omega_1 \cap \operatorname{Ker} L, 0) \\ &= -1 \neq 0. \end{aligned}$$

Therefore condition (3) of Lemma 2.3 holds for Ω_1.

In view of all the discussion above, we conclude from Lemma 2.3 that Eq. (3.1) has a solution in $\overline{\Omega}_1$.

Now, we will prove the existence of the second solution for Eq. (3.1). By (3.4), there exists an $\epsilon > 0$ small enough that

$$
\begin{aligned}
0 < H &:= \frac{2ac}{C(B+\sqrt{B^2-4ac})} - \epsilon < \frac{b(t)-\sqrt{b(t)^2-4a(t)c(t)}}{2c(t)} \\
&\leq \frac{B-\sqrt{b^2-4AC}}{2c} < M
\end{aligned}
$$

uniformly in t.

Let

$$
\Omega_2 := \{x \in X \mid H < x(t) < M\}, \tag{3.7}
$$

which is an open set in X.

Let $0 < \lambda < 1$ and x be such that

$$
x'' + \alpha x' + \lambda a(t)x^n - \lambda b(t)x^{n+1} + \lambda c(t)x^{n+2} = 0.
$$

Multiplying by x and the integrating from 0 to T, we have

$$
\int_0^T [(x')^2 - \lambda a(t)x^{n+1} + \lambda b(t)x^{n+2} - \lambda c(t)x^{n+3}]dt = 0.
$$

By (3.7), if $x \in \partial\Omega_2$, we have $H \leq |x|_\infty \leq M$. Then

$$
\begin{aligned}
0 &= \int_0^T (x')^2 dt + \int_0^T [-\lambda a(t)x^{n+1} + \lambda b(t)x^{n+2} - \lambda c(t)x^{n+3}]dt \\
&> \int_0^T [(x')^2 + x^2]dt - \int_0^T [a(t)x^{n+1} + b(t)x^{n+2} + c(t)x^{n+3}]dt - \int_0^T x^2 dt \\
&\geq \|x\|^2_{H^1(0,T)} - \int_0^T x^2(A|x|_\infty^{n-1} + B|x|_\infty^n + C|x|_\infty^{n+1} + 1)dt \\
&\geq \|x\|^2_{H^1(0,T)} - \int_0^T x^2(AM^{n-1} + BM^n + CM^{n+1} + 1)dt \\
&> \|x\|^2_{H^1(0,T)} - \int_0^T x^2(AN_1^{n-1} + BN_1^n + CN_1^{n+1} + 1)dt \\
&\geq \left[\frac{1}{\beta^2} - T(AN_1^{n-1} + BN_1^n + CN_1^{n+1} + 1)\right]|x|_\infty^2 \\
&\geq 0,
\end{aligned}
$$

where β is the immersion constant of $H^1(0,T)$ in $C([0,T])$, but this is contradiction. Therefore condition (1) of Lemma 2.3 holds for Ω_2.

It may easily be shown that

$$-a(t)+b(t)H-c(t)H^2<0 \tag{3.8}$$

uniformly in t. By (3.6), we have

$$-a(t)+b(t)M-c(t)M^2>0 \tag{3.9}$$

uniformly in t.

Take $x\in\partial\Omega_2\cap\operatorname{Ker}L$, we have $x=H$ or $x=M$. By (3.8) and (3.9), we know that for $\forall x\in\partial\Omega_2\cap\operatorname{Ker}L$, we have that

$$QNx=\frac{1}{T}\int_0^T x^n(-a(t)+b(t)x-c(t)x^2)dt\neq 0.$$

Therefore condition (2) of Lemma 2.3 holds for Ω_2.

Now we consider $\frac{H+M}{2}$, the arithmetic mean of M and H. We define a continuous function

$$\begin{aligned}H(x,\mu)=&(1-\mu)\left(x-\frac{H+M}{2}\right)\\&+\mu\frac{1}{T}\int_0^T x^n(-a(t)+b(t)x-c(t)x^2)dt,\quad \mu\in[0,1].\end{aligned}$$

Obviously, we obtain

$$H(x,\mu)\neq 0,\quad \forall x\in\partial\Omega_2\cap\operatorname{Ker}L.$$

By using the homotopy invariance theorem, we find that

$$\begin{aligned}\deg(QN,\Omega_2\cap\operatorname{Ker}L,0)&=\deg(H(x,1),\Omega_2\cap\operatorname{Ker}L,0)\\&=\deg(H(x,0),\Omega_2\cap\operatorname{Ker}L,0)\\&=1\neq 0.\end{aligned}$$

Therefore condition (3) of Lemma 2.3 holds for Ω_2.

In view of all the discussion above, we conclude from Lemma 2.3 that Eq. (3.1) has a solution in $\overline{\Omega}_2$.

Since $\overline{\Omega}_1\cap\overline{\Omega}_2=\{x=M\}$, and by (3.9), we know that M does not satisfy Eq. (3.1), namely, Eq. (3.1) has at least two T-periodic solutions.

Case 2: $0<-A\le a(t)\le -a\le 0, -B\le b(t)\le -b\le 0, -C\le c(t)\le -c\le 0$. Let $a'(t)=-a(t), b'(t)=-b(t), c'(t)=-c(t)$, then

$$0<a\le a'(t)\le A, 0<b\le b'(t)\le B, 0<c\le c'(t)\le C. \tag{3.10}$$

In this case, Eq. (1.2) is equivalent to equation

$$x'' + \alpha x' - a'(t)x^n + b'(t)x^{n+1} - c'(t)x^{n+2} = 0. \tag{3.11}$$

Let $0 < \lambda < 1$ and x be such that

$$x'' + \alpha x' - \lambda a'(t)x^n + \lambda b'(t)x^{n+1} - \lambda c'(t)x^{n+2} = 0.$$

Multiplying by x and the integrating from 0 to T, we have that

$$\int_0^T [(x')^2 + \lambda a'(t)x^{n+1} - \lambda b'(t)x^{n+2} + \lambda c'(t)x^{n+3}]dt = 0.$$

By (3.2), if $x \in \partial\Omega_1$, we have $M \leq |x|_\infty \leq N_1$. Then

$$\begin{aligned}
0 &= \int_0^T [(x')^2 + \lambda a'(t)x^{n+1} - \lambda b'(t)x^{n+2} + \lambda c'(t)x^{n+3}]dt \\
&> \int_0^T [(x')^2 + x^2]dt - \int_0^T [a'(t)x^{n+1} + b'(t)x^{n+2} + c'(t)x^{n+3}]dt - \int_0^T x^2 dt \\
&\geq \|x\|^2_{H^1(0,T)} - \int_0^T x^2(A|x|^{n-1}_\infty + B|x|^n_\infty + C|x|^{n+1}_\infty + 1)dt \\
&\geq \|x\|^2_{H^1(0,T)} - \int_0^T x^2(AN_1^{n-1} + BN_1^n + CN_1^{n+1} + 1)dt \\
&\geq 0,
\end{aligned}$$

where β is the immersion constant of $H^1(0, T)$ in $C([0, T])$, but this is contradiction. Therefore condition (1) of Lemma 2.3 holds for Ω_1.

It is readily seen that

$$a'(t) - b'(t)N_1 + c'(t)N_1^2 > 0$$

and

$$a'(t) - b'(t)M + c'(t)M^2 < 0 \tag{3.12}$$

uniformly in t.

The remaining proof is similar to the proof of case 1, and so we omit it.

In view of all the discussion above, we conclude from Lemma 2.3 that Eq. (3.11) has a solution in $\overline{\Omega}_1$.

Now, we will prove the existence of a second solution for Eq. (3.11).

Let $0 < \lambda < 1$ and x be such that

$$x'' + \alpha x' - \lambda a'(t)x^n + \lambda b'(t)x^{n+1} - \lambda c'(t)x^{n+2} = 0.$$

Multiplying by x and the integrating from 0 to T, we have

$$\int_0^T [(x')^2 + \lambda a'(t)x^{n+1} - \lambda b'(t)x^{n+2} + \lambda c'(t)x^{n+3}]dt = 0.$$

By (3.7), if $x \in \partial\Omega_2$, we have $H \le |x|_\infty \le M$. Then

$$\begin{aligned}
0 &= \int_0^T [(x')^2 + \lambda a'(t)x^{n+1} - \lambda b'(t)x^{n+2} + \lambda c'(t)x^{n+3}]dt \\
&> \int_0^T [(x')^2 + x^2]dt - \int_0^T [a'(t)x^{n+1} + b'(t)x^{n+2} + c'(t)x^{n+3}]dt - \int_0^T x^2 dt \\
&\ge \|x\|^2_{H^1(0,T)} - \int_0^T x^2(AM^{n-1} + BM^n + CM^{n+1} + 1)dt \\
&> \|x\|^2_{H^1(0,T)} - \int_0^T x^2(AN_1^{n-1} + BN_1^n + CN_1^{n+1} + 1)dt \\
&\ge \frac{|x|^2_\infty}{\beta^2} - T|x|^2_\infty(AN_1^{n-1} + BN_1^n + CN_1^{n+1} + 1) \\
&\ge 0,
\end{aligned}$$

where β is the immersion constant of $H^1(0, T)$ in $C([0, T])$, but this is contradiction. Therefore condition (1) of Lemma 2.3 holds for Ω_2.

It may easily be shown that

$$a'(t) - b'(t)H + c'(t)H^2 > 0$$

and

$$a'(t) - b'(t)M + c'(t)M^2 < 0$$

uniformly in t.

The remaining proof is similar to the proof of case 1, and so we omit it.

In view of all the discussion above, we conclude from Lemma 2.3 that Eq. (3.11) has a solution in $\overline{\Omega}_2$.

Since $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \{x = M\}$, and by (3.12), we know that M does not satisfy Eq. (3.11). Then Eq. (3.11) has at least two $T-$periodic solutions.

In view of all the discussion above, Eq. (1.2) has at least two T-periodic solutions. Theorem 1.1 is proved.

Proof of Theorem 1.2. The coefficient functions $a(t)$ and $b(t)$ have the same sign, which include both positive and negative cases.

Case 1: When $c(t) \equiv 0$, $0 < a \le a(t) \le A$, $0 < b \le b(t) \le B$.

In this case, Eq. (1.3) is equivalent to equation

$$x'' + \alpha x' + a(t)x^n - b(t)x^{n+1} = 0, \tag{3.13}$$

where $0 < a \le a(t) \le A$, $0 < b \le b(t) \le B$.

Let

$$\Omega_3 := \{x \in X \mid E < x(t) < F\}, \tag{3.14}$$

which is an open set in X, where

$$E = \frac{a}{B} - \epsilon, \tag{3.15}$$

$$F = \frac{A}{b} + \epsilon, \tag{3.16}$$

where $\epsilon > 0$ small enough such that $\frac{a}{B} - \epsilon > 0$. By (1.7), E and F are well defined.

By (3.15), (3.16) and (1.7), we have that

$$a(t) - b(t)E > 0$$

and

$$a(t) - b(t)F < 0$$

uniformly in t.

The remaining proof is similar to the proof of Theorem 1.1, and so we omit it.

In view of all the discussion above, we conclude from Lemma 2.3 that Eq. (3.13) has a solution in $\overline{\Omega}_3$.

Case 2: When $c(t) \equiv 0$, $-A \le a(t) \le -a < 0 \le B$, $-B \le b(t) \le -b < 0$.

Let $a'(t) = -a(t)$, $b'(t) = -b(t)$, then

$$0 < a \le a'(t) \le A, \quad 0 < b \le b'(t) \le B.$$

In this case, Eq. (1.3) is equivalent to equation

$$x'' + \alpha x' - a'(t)x^n + b'(t)x^{n+1} = 0, \tag{3.17}$$

where $0 < a \le a'(t) \le A$, $0 < b \le b'(t) \le B$.

Similarly, we can prove Eq. (3.17) has at least one positive T-periodic solutions in $\overline{\Omega}_3$.

In view of all the discussion above, we conclude from Lemma 2.3 that Eq. (1.3) has a solution in $\overline{\Omega}_3$. Theorem 1.2 is proved.

4 Example

Example 4.1 Consider Eq. (1.2) with $a(t) = \cos(\frac{2\pi t}{T}) + 3$, $b(t) = \sin(\frac{2\pi t}{T}) + 11$ and $c(t) = |\cos(\frac{2\pi t}{T})| + 3$. Define $a = 2$, $A = 4$, $b = 10$, $B = 12$, $c = 3$, $C = 4$, $n = 1$ and $\epsilon = \frac{1}{12+2\sqrt{30}}$. We have that

$$\begin{aligned}&b^2-4AC=100-64=36>0,\\&\frac{B-\sqrt{b^2-4AC}}{2c}=\frac{12-6}{6}<\frac{10+6}{8}=\frac{b+\sqrt{b^2-4AC}}{2C},\end{aligned}$$

and

$$0<T\le\frac{3}{\beta^2(220+32\sqrt{30})}.$$

Theorem 1.1 guarantees that the equations

$$\begin{aligned}&x''+\alpha x'+\left(\cos\left(\frac{2\pi t}{T}\right)+3\right)x-\left(\sin\left(\frac{2\pi t}{T}\right)+11\right)x^2\\&\quad+\left(\left|\cos\left(\frac{2\pi t}{T}\right)\right|+3\right)x^3=0\end{aligned}$$

has at least two positive T-periodic solutions in $\overline{\Omega}_1\cup\overline{\Omega}_2$, where $\Omega_1=\{x(t)\in X\mid \frac{3}{2}<x(t)<\frac{15+2\sqrt{30}}{6}\}$ and $\Omega_2=\{x(t)\in X\mid \frac{2}{12+2\sqrt{30}}<x(t)<\frac{3}{2}\}$.

Example 4.2 Consider Eq. (1.3) with $a(t)=\cos(\frac{2\pi t}{T})+6$ and $b(t)=\sin(\frac{2\pi t}{T})+9$. Define $a=5$, $A=7$, $b=8$, $B=10$, $n=1$ and $\epsilon=\frac{1}{8}$. We have $E=\frac{3}{8}$, $F=1$, and

$$0<T\le\frac{1}{18\beta^2}.$$

Theorem 1.2 guarantees that the equations

$$x''+\alpha x'+\left(\cos\left(\frac{2\pi t}{T}\right)+6\right)x-\left(\sin\left(\frac{2\pi t}{T}\right)+9\right)x^2=0$$

has at least one positive T-periodic solution in $\overline{\Omega}_3$, where $\Omega_3=\{x(t)\in X\mid \frac{3}{8}<x(t)<1\}$.

Example 4.3 Consider Eq. (1.3) with $a(t)=-\left(\cos(\frac{2\pi t}{T})+7\right)$, and $b(t)=-\left(\sin(\frac{2\pi t}{T})+10\right)$.

Define $a=-6$, $A=-8$, $b=-10$, $B=-12$, $n=2$ and $\epsilon=\frac{1}{5}$. We have $E=\frac{3}{10}$, $F=1$, and

$$0<T\le\frac{1}{21\beta^2}.$$

Theorem 1.2 guarantees that the equations

$$x''+\alpha x'-\left(\cos\left(\frac{2\pi t}{T}\right)+7\right)x^2+\left(\sin\left(\frac{2\pi t}{T}\right)+10\right)x^3=0$$

has at least one positive T-periodic solution in $\overline{\Omega}_4$, where $\Omega_4 = \{x(t) \in X \mid \frac{3}{10} < x(t) < 1\}$.

It is worth noting that the case of Eq. (1.2) when $a(t) = 0$.

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