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On the reducibility systems of two linear first-order ordinary differential equations

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Abstract
Some global solvability criteria for the scalar Riccati equations are used to establish new reducibility criteria for systems of two linear first-order ordinary differential equations. Some examples are presented.

Keywords The Riccati equation · Global solvability · Linear systems of ordinary differential equations · Reducibility

Mathematics Subject Classification 34C10 · 34C11 · 34C99

1 Introduction

Let $a(t), b(t), c(t)$ and $d(t)$ be real valued continuous and bounded functions on $[t_0, +\infty)$. Consider the linear system of ordinary differential equations

$$\begin{cases} \phi' = a(t)\phi + b(t)\psi, \\ \psi' = c(t)\phi + d(t)\psi, \quad t \geq t_0. \end{cases} \tag{1.1}$$

Introduce new unknowns by equalities

$$\begin{cases} \phi_1 = z_{11}(t)\phi + z_{12}(t)\psi, \\ \psi_1 = z_{21}(t)\phi + z_{22}(t)\psi, \end{cases} \tag{1.2}$$

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where ϕ_1, ψ_1 are new unknowns, $z_{jk}(t)$, $j, k = 1, 2$ are the coefficients of a transformation. For ϕ_1 and ϕ_2 from (1.1) and (1.2) we obtain a new linear system of equations

$$\begin{cases} \phi_1' = a_1(t)\phi_1 + b_1(t)\psi_1, \\ \psi_1' = c_1(t)\phi_1 + d_1(t)\psi_1, \quad t \geq t_0. \end{cases} \tag{1.3}$$

Definition 1.1 The system (1.1) is called reducible if there exists a bounded matrix function $Z(t) \equiv (z_{jk}(t))_{j,k=1}^2$ of transformation (1.2) for the system (1.1) such that $Z'(t), Z^{-1}(t)$ exist and are bounded with $det Z^{-1}(t)$ and the coefficients $a_1(t), b_1(t), c_1(t)$ and $d_1(t)$ of the system (1.3) are constants.

Study the reducibility behavior of systems of linear ordinary differential equation, in particular of the system (1.1), is an important problem of qualitative theory of differential equations and many works are devoted to it (see [1–6], and cited works therein) The reducible systems (see [7]) play an important role in the study of stability of solutions of nonlinear systems, for which the first approximation contains the time. They play also an important role in the '.... study of stability of quasi-periodic motion and preservation of invariant tori in Hamiltonian mechanics (where the reducibility of linear equations with quasi-periodic coefficients play an important role)" (see [1]). In this paper some new reducibility criteria for the system (1.1) are obtained.

2 Auxiliary propositions

Let $f(t), g(t), h(t)$ be real valued continuous functions on $[t_0, +\infty)$. Consider the Riccati equation

$$y' + f(t)y^2 + g(t)y + h(t) = 0, \quad t \geq t_0. \tag{2.1}$$

In this section we represent some global existence criteria for Eq. (2.1) proved in [8] and [9]. They will be used in the Sect. 3 to obtain new reducibility criteria for the system (1.1).

Theorem 2.1 *Let $f_i(t)$ and $h_i(t)$ be continuously differentiable functions on $[t_0, +\infty)$ such that $(-1)^i f_i(t) > 0, (-1)^i h_i(t) > 0, \quad t \geq t_0, i = 1, 2$. If $f_1(t) \leq f(t) \leq f_2(t), h_1(t) \leq h(t) \leq h_2(t), g(t) \geq \frac{1}{2}\left(\frac{f_i'(t)}{f_i(t)} - \frac{h_i'(t)}{h_i(t)}\right) + 2(-1)^i\sqrt{f_i(t)h_i(t)}, i = 1, 2, \quad t \geq t_0$, then for every $y_{(0)} \in \left[-\sqrt{\frac{h_2(t_0)}{f_2(t_0)}}, \sqrt{\frac{h_1(t_0)}{f_1(t_0)}}\right]$ Eq. (2.1) has a solution $y(t)$ on $[t_0, +\infty)$ with $y(t_0) = y_{(0)}$ and*

$$-\sqrt{\frac{h_2(t)}{f_2(t)}} \leq y(t) \leq \sqrt{\frac{h_1(t)}{f_1(t)}}, \quad t \geq t_0. \tag{2.2}$$

See the proof in [8].

Theorem 2.2 *Let $f_i(t)$ and $h_i(t)$ be continuously differentiable functions on $[t_0, +\infty)$ such that $(-1)^i f_i(t) > 0, (-1)^i h_i(t) > 0, \quad t \ge t_0, i = 1, 2$. If $f_1(t) \le f(t) \le f_2(t), h_1(t) \le h(t) \le h_2(t), g(t) \le \frac{1}{2}\left(\frac{f_i'(t)}{f_i(t)} - \frac{h_i'(t)}{h_i(t)}\right) - 2(-1)^i\sqrt{f_i(t)h_i(t)}, i = 1, 2, \quad t \ge t_0$, then Eq. (2.1) has a solution $y(t)$ on $[t_0, +\infty)$ and*

$$-\sqrt{\frac{h_1(t)}{f_1(t)}} \le y(t) \le \sqrt{\frac{h_2(t)}{f_2(t)}}, \quad t \ge t_0.$$

Proof In Eq. (2.1) substitute $y = y(t) = z(-t), t \ge t_0$. We come to the equation

$$z' + \widetilde{f}(t)z^2 + \widetilde{g}(t)z + \widetilde{h}(t) = 0, t \le -t_0, \tag{2.3}$$

where $\widetilde{f}(t) \equiv -f(-t), \widetilde{g}(t) \equiv -g(-t), \widetilde{h}(t) \equiv -h(-t), t \le -t_0$. Set: $\widetilde{f}_i(t) \equiv -f_i(-t), \widetilde{h}_i(t) \equiv -h_i(-t), t \le -t_0, i = 1, 2$. From the condition of the theorem it follows

$$\begin{aligned}
&(-1)^i \widetilde{f}_i(t) < 0, \quad (-1)^i \widetilde{h}_i(t) < 0, i = 1, 2,\\
&\widetilde{f}_2(t) \le f(t) \le \widetilde{f}_1(t), \quad \widetilde{h}_2(t) \le h(t) \le \widetilde{h}_1(t),\\
&\widetilde{g}(t) \ge \frac{1}{2}\left[\frac{\widetilde{f}_i'(t)}{\widetilde{f}_i(t)} - \frac{\widetilde{h}_i'(t)}{\widetilde{h}_i(t)}\right] + 2(-1)^i\sqrt{\widetilde{f}_i(t)\widetilde{h}_i(t)}, i = 1, 2, t \le -t_0.
\end{aligned}$$

Then by Theorem 2.1 for evwry $T > t_0$ and for every $z_{(0)} \in \left[\sqrt{\frac{\widetilde{h}_1(-T)}{\widetilde{f}_1(-T)}}, \sqrt{\frac{\widetilde{h}_2(-T)}{\widetilde{f}_2(-T)}}\right]$ Eq. (3.3) has a solution $z_0(t)$ on $[-T, -t_0]$ with $z_0(-T) = z_{(0)}$ and

$$\sqrt{\frac{\widetilde{h}_1(-t)}{\widetilde{f}_1(-t)}} \le z_0(t) \le \sqrt{\frac{\widetilde{h}_2(-t)}{\widetilde{f}_2(-t)}}, \quad t \in [-T, -t_0].$$

Denote by I_T the set of that values $z(-t_0)$ of of the solutions $z(t)$ of Eq. (2.3) for which

$$z(-T) \in \left[\sqrt{\frac{\widetilde{h}_1(-T)}{\widetilde{f}_1(-T)}}, \sqrt{\frac{\widetilde{h}_2(-T)}{\widetilde{f}_2(-T)}}\right].$$

Obviously I_T is a finite and close interval and if $T_1 > T_2 > t_0$ then

$$I_{T_1} \subset I_{T_2}. \tag{2.4}$$

Let $t_0 < T_1 < T_2 < .... < T_n < ...$ be a infinitely large sequence. By (2.4) we have

$$I \equiv \bigcap_{n=1}^{+\infty} T_{T_n} \ne \emptyset.$$

Take $y_{(0)} \in I$. Then (by already proven) for every $n = 1, 2, \ldots$ Eq. (2.2) has a solution $z_n(t)$ on $[-T_n, -t_0]$ with $z_n(-t_0) = y_{(0)}$ and

$$\sqrt{\frac{\widetilde{h}_1(-t)}{\widetilde{f}_1(-t)}} \le z_n(t) \le \sqrt{\frac{\widetilde{h}_2(-t)}{\widetilde{f}_2(-t)}}, t \in [-T_n, -t_0], n = 1, 2, \ldots.$$

Therefore $y_n(t) \equiv z_n(-t)$ is a solution of Eq. (2.1) on $[t_0, T_n]$ with $y_n(t_0) = y_{(0)}$ and

$$\sqrt{\frac{\widetilde{h}_1(t)}{\widetilde{f}_1(t)}} \le y_n(t) \le \sqrt{\frac{\widetilde{h}_2(t)}{\widetilde{f}_2(t)}}, \quad t \in [t_0, T_n], n = 1, 2, \ldots.$$

By virtue of the uniqueness theorem from here it follows that Eq. (2.1) has a solution $y_0(t)$ on $[t_0, +\infty)$ and (2.2) is valid. The theorem is proved. □

Theorem 2.3 *Let $f_1(t)$ and $h_1(t)$ be continuously differentiable functions on $[t_0, +\infty)$ such that $f_1(t) > 0, h_1(t) > 0, t \ge t_0$. If $0 \le f(t) \le f_1(t), h(t) \le h_1(t), g(t) \le \frac{1}{2}\Big[\frac{f_1'(t)}{f_1(t)} - \frac{h_1'(t)}{h_1(t)}\Big] - 2\sqrt{f_1(t)h_1(t)}, t \ge t_0$ then for every $y_{(0)} \ge \sqrt{\frac{h_1(t_0)}{f_1(t_0)}}$ Eq. (2.1) has a solution $y(t)$ on $[t_0, +\infty)$ with $y(t_0) = y_{(0)}$ and*

$$y(t) \ge \sqrt{\frac{h_1(t)}{f_1(t)}}, \quad t \ge t_0.$$

See the proof in [9].

Theorem 2.4 *Let $f_1(t)$ and $h_1(t)$ be the same as in Theorem 2.2. If $0 \le f(t) \le f_1(t), h(t) \le h_1(t), g(t) \ge \frac{1}{2}\Big[\frac{f_1'(t)}{f_1(t)} - \frac{h_1'(t)}{h_1(t)}\Big] + 2\sqrt{f_1(t)h_1(t)}, f(t)f_1(t) + 2f_1(t)h_1(t) + h_1(t)h(t) \ge 0, \quad t \ge t_0$, then for every $y_{(0)} \in \Big[-\sqrt{\frac{h_1(t_0)}{f_1(t_0)}}, \sqrt{\frac{h_1(t_0)}{f_1(t_0)}}\Big]$ Eq. (2.1) has a solution $y(t)$ on $[t_0, +\infty)$ with $y(t_0) = y_{(0)}$ and*

$$-\sqrt{\frac{h_1(t)}{f_1(t)}} \le y(t) \le \sqrt{\frac{h_1(t)}{f_1(t)}}, \quad t \ge t_0.$$

See the proof in [9].

Theorem 2.5 *Let $f_1(t), g_1(t)$ and $h_1(t)$ be real valued continuous functions such that $f_1(t) > 0, \frac{g_1(t)}{f_1(t)}$ is continuously differentiable on $[t_0, +\infty)$. If $0 \le f(t) \le f_1(t), \lambda(g(t) - g_1(t))g_1(t) \ge 0, h(t) \le \lambda\Big[\Big(\frac{g_1(t)}{f_1(t)}\Big) + (1-\lambda)\frac{g_1^2(t)}{f_1(t)}\Big], \lambda = const, \quad t \ge t_0$, then for every $y_{(0)} \ge -\lambda\frac{g_1(t_0)}{f_1(t_0)}$ Eq. (2.1) has a solution $y(t)$ on $[t_0, +\infty)$ with $y(t_0) = t_{(0)}$ and*

$$y(t) \ge -\lambda\frac{g_1(t)}{f_1(t)}, \quad t \ge t_0.$$

See the proof in [9].

Theorem 2.6 *Let* $f_1(t)$ *and* $h_1(t)$ *be continuous functions on* $[t_0, +\infty)$. *If* $0 \leq f(t) \leq f_1(t), \left[\lambda + \int_{t_0} t h_1(\tau)d\tau\right]\left(f_1(t)\left[\lambda + \int_{t_0} t h_1(\tau)d\tau\right] - g(t)\right) \leq 0, h(t) \leq h_1(t), \lambda = const, \quad t \geq t_0$, *then for every* $y_{(0)} \geq -\lambda$ *Eq.* (2.1) *has a solution* $y(t)$ *with* $y(t_0) = y_{(0)}$ *and*

$$y(t) \geq -\lambda - \int_{t_0}^{t} h_1(\tau)d\tau, \quad t \geq t_0.$$

If in addition for some $\mu \leq \lambda$ *the inequality* $\left[\mu + \int_{t_0} t h_1(\tau)d\tau\right]\left(f_1(t)\left[\mu + \int_{t_0} t h_1(\tau)d\tau\right] - g(t)\right) \geq 0, t \geq t_0$ *is satisfied and* $y_{(0)} \leq -\mu$, *then*

$$y(t) \leq -\mu - \int_{t_0}^{t} h_1(\tau)d\tau, \quad t \geq t_0.$$

See the proof in [9].

Theorem 2.7 *Let for some* $\lambda \in \mathbb{R}$ *and continuous on* $[t_0, +\infty)$ *functions* $g_1(t)$ *and* $h_1(t)$ *the following conditions be satisfied.*
$f(t) \geq 0, \int_{t_0} t \exp\left\{\int_{t_0} \tau\left[f(s)\big(\eta_0(s) + \eta_1(s)\big) + g(s)\right]ds\right\}\left[f(\tau)\eta_1^2(\tau) + (g(\tau) - g_1(\tau))\eta_1(\tau) + h(\tau) - h_1(\tau)\right]d\tau \leq 0, \quad t \geq t_0$, *where*

$$\eta_0(t) \equiv \lambda \exp\left\{-\int_{t_0}^{t} g(\tau)d\tau\right\} - \int_{t_0}^{t} \exp\left\{-\int_{\tau}^{t} g(s)ds\right\}h(\tau)d\tau,$$

$$\eta_1(t) \equiv \lambda \exp\left\{-\int_{t_0}^{t} g_1(\tau)d\tau\right\} - \int_{t_0}^{t} \exp\left\{-\int_{\tau}^{t} g_1(s)ds\right\}h_1(\tau)d\tau.$$

Then for every $y_{(0)} \geq \lambda$ *Eq.* (2.1) *has a solution* $y_0(t)$ *on* $[t_0, +\infty)$ *with* $y_0(t_0) = y_{(0)}$ *and* $y_0(t) \geq \eta_1(t), t \geq t_0$.

See the proof in [9].

3 Reducibility criteria for the system (1.1)

Consider the Riccati equation

$$y' + b(t)y^2 + E(t)y - c(t) = 0, \quad t \ge t_0, \tag{3.1}$$

where $E(t) \equiv a(t) - d(t), \quad t \ge t_0$. In [2] it was established that the system (1.1) is reducible provided:
α) Eq. (3.1) has a bounded solution on $[t_0, +\infty)$
or
β) Eq. (3.1) has a solution $y(t)$ on $[t_0, +\infty)$ such that $\lim_{t\to+\infty} y(t) = \infty$.
We will use this fact with the theorems from Sect. 2 to prove reducibility criteria for the system (1.1).

Theorem 3.1 *Let $b_i(t)$ and $c_i(t)$ be continuously differentiable functions on $[t_0, +\infty)$ such that $(-1)^i b_i(t) > 0, (-1)^i c_i(t) > 0, \quad t \ge t_0, i = 1, 2$. If $b_1(t) \le b(t) \le b_2(t), c_1(t) \le -c(t) \le c_2(t), E(t) \ge \frac{1}{2}\Big(\frac{b_i'(t)}{b_i(t)} - \frac{c_i'(t)}{c_i(t)}\Big) + 2(-1)^i\sqrt{b_i(t)c_i(t)}, i = 1, 2, \quad t \ge t_0$,*
A_1) $\frac{c_i(t)}{b_i(t)}$ is bounded on $[t_0, +\infty), i = 1, 2$,
then the system (1.1) is reducible.

Proof By virtue of Theorem 2.1 from the conditions of the theorem it follows that Eq. (3.1) has a solution $y(t)$ on $[t_0, +\infty)$ such that

$$-\sqrt{\frac{c_2(t)}{b_2(t)}} \le y(t) \le \sqrt{\frac{c_1(t)}{b_1(t)}}, \quad t \ge t_0.$$

From here and from the condition A_1) it follows that $y(t)$ is bounded. Then by α) the system (1.1) is reducible. The theorem is proved. □

Example 3.1 Let $\nu(t)$ be a continuous and bounded function on $[t_0, +\infty)$. Consider the system

$$\begin{cases} \phi' = (2 + \nu(t))\phi + \sin t^4 \psi, \\ \psi' = \cos e^t \phi + (\nu(t) - \sin^2 t)\psi, \quad t \ge t_0. \end{cases}$$

It is not difficult to verify that for $b_i(t) = c_i(t) = (-1)^i, i = 1, 2, \quad t \ge t_0$ the conditions of Theorem 3.1 for this system are satisfied.

By analogy using Theorem 2.2 in place of Theorem 2.1 can be proved

Theorem 3.2 *Let $b_i(t)$ and $c_i(t)$ be continuously differentiable functions on $[t_0, +\infty)$ such that $(-1)^i b_i(t) > 0, (-1)^i c_i(t) > 0, \quad t \ge t_0, i = 1, 2$. If $b_1(t) \le b(t) \le b_2(t), c_1(t) \le -c(t) \le c_2(t), E(t) \le \frac{1}{2}\Big(\frac{b_i'(t)}{b_i(t)} - \frac{c_i'(t)}{c_i(t)}\Big) - 2(-1)^i\sqrt{b_i(t)c_i(t)}, i =$*

$1, 2,\quad t \geq t_0$, *the function* $\frac{c_i(t)}{b_i(t)}$ *is bounded on* $[t_0, +\infty)$, $i = 1, 2$, *then the system (1.1) is reducible.*

Example 3.2 Let $\nu(t)$ be the same as in Example 3.1. Consider the system

$$\begin{cases} \phi' = (\nu(t) - 2)\phi + \cos\frac{1}{\sqrt{1+t^2}}\psi, \\ \psi' = \sin e^t \phi + (\nu(t) + \cos^2 t^7)\psi, \quad t \geq t_0. \end{cases}$$

One can readily check that for $b_i(t) = c_i(t) = (-1)^i$, $i = 1, 2, \quad t \geq t_0$ the conditions of Theorem 3.2 for this system are satisfied.

Theorem 3.3 *Let* $b_1(t)$ *and* $c_1(t)$ *be continuously differentiable functions on* $[t_0, +\infty)$ *such that* $b_1(t) > 0, c_1(t) > 0, t \geq t_0$. *If* $0 \leq b(t) \leq b_1(t), -c(t) \leq c_1(t), E(t) \leq \frac{1}{2}\left(\frac{b_1'(t)}{b_1(t)} - \frac{c_1'(t)}{c_1(t)}\right) - 2\sqrt{b_1(t)c_1(t)}, t \geq t_0$.
A_2) $\lim_{t\to+\infty}\frac{c_1(t)}{b_1(t)} = +\infty$,
then the system (1.1) is reducible.

Proof In virtue of Theorem 2.3 from the conditions of the thorem it follows that Eq. (3.1) has a solution $y(t)$ on $[t_0, +\infty)$ such that

$$y(t) \geq \sqrt{\frac{c_1(t)}{b_1(t)}}, \quad t \geq t_0.$$

From here and from A_2) it follows that

$$\lim_{t\to+\infty} y(t) = +\infty,$$

By β) from here it follows the reducibility of the system (1.1). The theorem is proved. □

Example 3.3 Let $\nu(t)$ be the same as in Example 3.1. Consider the system

$$\begin{cases} \phi' = (\nu(t) - \frac{1}{2t})\phi + \frac{|cos^3 t|}{t}\psi, \\ \psi' = -\sin^4\frac{1}{t}\phi + (\nu(t) + \frac{2}{\sqrt{t}})\psi, \quad t \geq 1. \end{cases}$$

One can readily check that for $b_1(t) = \frac{1}{t}, c_1(t) = 1, t \geq 1$ the conditions of Theorem 3.3 for this system are satisfied.

Theorem 3.4 *Let* $b_1(t)$ *and* $c_1(t)$ *be the same as in Theorem 3.3. If* $0 \leq b(t) \leq b_1(t), -c(t) \leq c_1(t), E(t) \geq \frac{1}{2}\left(\frac{b_1'(t)}{b_1(t)} - \frac{c_1'(t)}{c_1(t)}\right) + 2\sqrt{b_1(t)c_1(t)}, b(t)b_1(t) + 2b_1(t)c_1(t) - c_1(t)c(t) \geq 0, \quad t \geq t_0$,
A_3) $\frac{c_1(t)}{b_1(t)}$ *is bounded on* $[t_0, +\infty)$,
then the system (1.1) is reducible.

Proof By Theorem 2.4 from the conditions of the theorem it follows that Eq. (3.1) has a solution $y(t)$ on $[t_0, +\infty)$ such that

$$-\sqrt{\frac{c_1(t)}{b_1(t)}} \le y(t) \le \sqrt{\frac{c_1(t)}{b_1(t)}}, \quad t \ge t_0.$$

From here and from A_3) it follows that $y(t)$ is bounded on $[t_0, +\infty)$. Then by α) the system (1.1) is reducible. The theorem is proved. □

Example 3.4 Consider the system

$$\begin{cases} \phi' = [3 + \nu(t)]\phi + e^{\sin 2t}\psi, \\ \psi' = e^{-\sin 2t}\phi + [\nu(t) - 3\cos 2t]\psi, \quad t \ge t_0, \end{cases} \tag{3.2}$$

where $\nu(t)$ is the same as in Example 3.1. It is not difficult to verify that for $b_1(t) = e^{\sin 2t}, c_1(t) = e^{-\sin 2t}, \quad t \ge t_0$ the conditions of Theorem 3.4 for this system are fulfilled. Hence this system is reducible.

Remark 3.1 The reducibility of the system (3.2) in the case when $\nu(t) \equiv 0$ is evident.

Theorem 3.5 *Let $b_1(t)$ and $E_1(t)$ be continuous functions on $[t_0, +\infty)$ such that $b_1(t) > 0, \quad t \ge t_0, \frac{E_1(t)}{b_1(t)}$ is continuously differentiable on $[t_0, +\infty)$. If $0 \le b(t) \le b_1(t), \lambda(E(t) - E_1(t))E_1(t) \ge 0, -c(t) \le \lambda\left[\left(\frac{E_1(t)}{b_1(t)}\right)' + (1-\lambda)\frac{E_1^2(t)}{b_1(t)}\right], \lambda = const, \quad t \ge t_0$,*
A_4) $\lim_{t\to+\infty} -\lambda\frac{E_1(t)}{b_1(t)} = +\infty$,
then the system (1.1) is reducible.

Proof In virtue of Theorem 2.5 from the conditions of the theorem it follows that Eq. (3.1) has a solution $y(t)$ on $[t_0, +\infty)$ such that $y(t) \ge -\lambda\frac{E_1(t)}{b_1(t)}, \quad t \ge t_0$. From here and from the condition A_4) it follows that

$$\lim_{t\to+\infty} y(t) = +\infty.$$

Consequently by β) the system (1.1) is reducible. The theorem is proved. □

Example 3.5 Let $\nu(t)$ be the same as in Example 3.1. Consider the system

$$\begin{cases} \phi' = [\nu(t) + 5arctan(t^5 \sin t)]\phi + \frac{100\sin^2 e^t}{t}\psi, \\ \psi' = 5\cos t^5\phi + [\nu(t) + \sin t^{10}]\psi, \quad t \ge 5. \end{cases}$$

It is not difficult to verify that for $b_1(t) = \frac{100}{t}, E_1(t) = 10, \lambda = -1$ the conditions of Theorem 3.5 for this system are satisfied.

Theorem 3.6 *Let* $b_1(t)$ *and* $c_1(t)$ *be continuous functions on* $[t_0, +\infty)$ *such that* $\int_{t_0} tc_1(\tau)d\tau, \quad t \geq t_0,$ *is bounded on* $[t_0, +\infty)$. *If* $0 \leq b(t) \leq b_1(t), \Big[\lambda + \int_{t_0} tc_1(\tau)d\tau\Big]\Big(b_1(t)\Big[\lambda + \int_{t_0} tc_1(\tau)d\tau\Big] - E(t)\Big) \leq 0, -c(t) \leq c_1(t), \Big[\mu + \int_{t_0} tc(\tau)d\tau\Big]\Big(b(t)\Big[\mu + \int_{t_0} tc(\tau)d\tau\Big] - E(t)\Big) \geq 0, \lambda = const, \mu = const, \mu \leq \lambda, \quad t \geq t_0,$ *then the system (1.1) is reducible.*

Proof By Theorem 2.6 from the conditions of the theorem it follows that Eq. (3.1) has a solution $y(t)$ such that

$$-\lambda - \int_{t_0}^{t} c_1(\tau)d\tau \leq y(t) \leq \mu - \int_{t_0}^{t} c_1(\tau)d\tau, \quad t \geq t_0.$$

Then since $\int_{t_0} tc_1(\tau)d\tau$ is bounded $y(t)$ is also bounded. Therefore according to $\alpha)$ the system (1.1) is reducible. The theorem is proved. □

Example 3.6 Let $\nu(t)$ be the same as in Example 3.1. Consider the system

$$\begin{cases} \phi' = [sint + \nu(t)]\phi + |sint^2|\psi, \\ \psi' = -2\cos t\phi + [\nu(t) - \sin t]\psi, \quad t \geq t_0. \end{cases}$$

One can readily check that for $c_1(t) = 2\cos t, b_1(t) \equiv 1, \lambda = \sin t_0$ the conditions of Theorem 3.6 for this system are satisfied.

Theorem 3.7 *Let for some* $\lambda \in \mathbb{R}$ *and continuous functions* $E_1(t)$ *and* $c_1(t)$ *the following conditions be satisfied*
$b(t) \geq 0,$

$$\int_{t_0}^{t} \exp\Big\{\int_{t_0}^{\tau}\Big[b(s)\big(\chi_0(s) + \chi_1(s)\big) + E(s)\Big]ds\Big\}\Big[b(\tau)\chi_1^2(\tau) + \\ +(E(\tau) - E_1(\tau))\chi_1(\tau) + c_1(\tau) - c(\tau)\Big]d\tau \leq 0, \quad t \geq t_0, \tag{3.3}$$

$\lim_{t\to+\infty} \chi_1(t) = +\infty$, *where*

$$\chi_0(t) \equiv \lambda \exp\Big\{-\int_{t_0}^{t} E(\tau)d\tau\Big\} + \int_{t_0}^{t} \exp\Big\{-\int_{\tau}^{t} E(s)ds\Big\}c(\tau)d\tau,$$

$$\chi_1(t) \equiv \lambda \exp\Big\{-\int_{t_0}^{t} E_1(\tau)d\tau\Big\} + \int_{t_0}^{t} \exp\Big\{-\int_{\tau}^{t} E_1(s)ds\Big\}c_1(\tau)d\tau.$$

Then the system (1.1) is reducible.

Proof By virtue of Theorem 2.7. from the conditions of the theorem it follows that Eq. (2.1) has a solution $y_0(t)$ on $[t_0, +\infty)$ such that $\lim_{t\to+\infty} y_0(t) = +\infty$. By β) from here it follows the reducibility of the system (1.1). The theorem is proved. □

Remark 3.2 The condition (3.3) of Theorem 3.7 is satisfied if in particular $\lambda = 0$ and

$$b(t)I^2_{E_1,c_1}(t) + (E(t) - E_1(t))I_{E_1,c_1}(t) + c_1(t) - c(t) \le 0, t \ge t_0,$$

where $I_{E_1,c_1}(t) \equiv \int_{t_0} t \exp\left\{-\int_\tau tE_1(s)ds\right\}c_1(\tau)d\tau, \quad t \ge t_0.$

Example 3.7 Let $\nu(t)$ be the same as in Example 3.1. Consider the system

$$\begin{cases} \phi' = [\nu(t) - \sin^2 t^4]\phi + \frac{\sin^2 e^t}{t}\psi, \\ \psi' = [1 + cos^4\sqrt{t}]\phi + [\nu(t) - 1]\psi, \quad t \ge 1. \end{cases}$$

Using Remark 3.2 one can readily check that for $c_1(t) \equiv 1$, $E_1(t) \equiv 0$, $\lambda = 0$ the conditions of Theorem 3.7 for this system are satisfied.

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