Monatshefte für Mathematik (2020) 193:783–806
https://doi.org/10.1007/s00605-020-01440-5

Landau-type theorems and bi-Lipschitz theorems for bounded biharmonic mappings

Shi-Fei Chen[1] · **Ming-Sheng Liu**[1]

Received: 26 May 2020 / Accepted: 18 June 2020 / Published online: 1 July 2020

Abstract
In this paper, we first establish five versions of Landau-type theorems for five classes of bounded biharmonic mappings $F(z) = |z|^2G(z) + H(z)$ on the unit disk $\mathbb{D}$ with $G(0) = H(0) = J_F(0) - 1 = 0$, which improve the related results of earlier authors. In particular, two versions of those Landau-type theorems are sharp. Then we derive five bi-Lipschitz theorems for these classes of bounded and normalized biharmonic mappings.

Keywords Biharmonic mappings · Harmonic mappings · Landau-type theorems · Bi-Lipschitz theorems · Univalent

Mathematics Subject Classification Primary 30C99; Secondary 30C62

1 Introduction

Let $\mathbb{D}$=$\{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk with center at the origin and radius 1. For $r > 0$, let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. A function $f(z) = u(z) + iv(z)$, $z = x + iy$ is a harmonic mapping on the unit disk $\mathbb{D}$ if and only if F is twice continuously differentiable and satisfies the Laplacian equation

$$\Delta f = 4f_{z\bar{z}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Communicated by Adrian Constantin.

✉ Ming-Sheng Liu
liumsh65@163.com

Shi-Fei Chen
csf1377539656@163.com

[1] School of Mathematical Sciences, South China Normal University, Guangzhou 510631, Guangdong, People's Republic of China

for $z \in \mathbb{D}$, where the formal derivatives of f are defined by

$$f_z = \frac{1}{2}\left(f_x - i f_y\right), \quad f_{\bar{z}} = \frac{1}{2}\left(f_x + i f_y\right).$$

A function $F(z) = U(z) + iV(z)$ is a biharmonic mapping on $\mathbb{D}$ if and only if F is four times continuously differentiable and satisfies the biharmonic equation $\Delta(\Delta F) = 0$ for $z \in \mathbb{D}$. In other words, $F(z)$ is biharmonic on $\mathbb{D}$ if and only if ΔF is harmonic on $\mathbb{D}$.

It is known [1] that a mapping F is biharmonic on $\mathbb{D}$ if and only if F can be represented as follow:

$$F(z) = |z|^2 G(z) + H(z),\ z \in \mathbb{D}, \tag{1.1}$$

where $G(z)$ and $H(z)$ are complex-valued harmonic mappings on $\mathbb{D}$.

In [15], it's known that a harmonic mapping $f(z)$ is locally univalent on $\mathbb{D}$ if and only if its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0$ for any $z \in \mathbb{D}$. Since $\mathbb{D}$ is simply connected, $f(z)$ can be written as $f = h + \overline{g}$ with $f(0) = h(0)$, h and g are analytic on $\mathbb{D}$. Thus, we have

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

For such function f, we define

$$\Lambda_f(z) = \max_{0 \le \theta \le 2\pi} |e^{i\theta} f_z(z) + e^{-i\theta} f_{\bar{z}}(z)| = |f_z(z)| + |f_{\bar{z}}(z)|,$$

and

$$\lambda_f(z) = \min_{0 \le \theta \le 2\pi} |e^{i\theta} f_z(z) + e^{-i\theta} f_{\bar{z}}(z)| = ||f_z(z)| - |f_{\bar{z}}(z)||.$$

Recall that a mapping $\omega : \mathbb{D} \to \Omega$ is said to be L_1-Lipschitz ($L_1 > 0$) (l_1-co-Lipschitz ($l_1 > 0$)) if

$$|\omega(z_1) - \omega(z_2)| \le L_1 |z_1 - z_2|, \quad z_1, z_2 \in \mathbb{D}, \tag{1.2}$$

$$(|\omega(z_1) - \omega(z_2)| \ge l_1 |z_1 - z_2|, \quad z_1, z_2 \in \mathbb{D}). \tag{1.3}$$

A mapping ω is bi-Lipschitz if it is Lipschitz and co-Lipschitz (see [14]). In [13], the Lipschitz character of q.c. harmonic self-mappings of the unit disk was established with respect to the hyperbolic metric and this was generalized to an arbitrary domain in [25].

Harmonic mappings techniques have been used to study and solve fluid flow problems (see [4,11]). For example, in 2012, Aleman and Constantin [4] developed ingenious technique to solve the incompressible two dimensional Euler equations in terms of univalent harmonic mappings. More precisely, the problem of finding all solutions which in Lagrangian variables describing the particle paths of the flow present a labelling by harmonic mappings is reduced to solve an explicit nonlinear differential system in $\mathbb{C}^n$ (please refer to [11]).

The classical Landau's theorem states that if f is an analytic function on the unit disk $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in \mathbb{D}$, then f is univalent in the disk $\mathbb{D}_{r_0}$ with $r_0 = \frac{1}{M+\sqrt{M^2-1}}$ and $f(\mathbb{D}_{r_0})$ contains a disk $|w| < R_0$ with $R_0 = Mr_0^2$. This result is sharp, with the extremal function $f_0(z) = Mz\frac{1-Mz}{M-z}$. The Bloch theorem asserts the existence of a positive constant number b such that if f is an analytic function on the unit disk $\mathbb{D}$ with $f'(0) = 1$, then $f(\mathbb{D})$ contains a schlicht disk of radius b, that is, a disk of radius b which is the univalent image of some region on $\mathbb{D}$. The supremum of all such constants b is called the Bloch constant (see [6,12]).

For harmonic mappings on $\mathbb{D}$, under suitable restriction, Chen, Gauthier and Hengartner [6] obtained two versions of Landau's theorems. In 2008, Abdulhadi and Muhanna proved the following Landau-type theorem of certain bounded biharmonic mappings in [2].

Theorem A (Abdulhadi and Muhanna [2]) *Let $f(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk $\mathbb{D}$, as in (1.1), with $f(0) = h(0) = J_f(0) - 1 = 0$ and $|g(z)| \leq M$, $|h(z)| \leq M$ for $z \in \mathbb{D}$. Then there is a constant $0 < r_1 < 1$ so that f is univalent in the disk $\mathbb{D}_{r_1}$. In specific r_1 satisfies the following equation*

$$\frac{\pi}{4M} - 2r_1 M - \frac{2Mr_1^2}{(1-r_1)^2} - 2M \cdot \frac{2r_1 - r_1^2}{(1-r_1)^2} = 0, \tag{1.4}$$

and $f(\mathbb{D}_{r_1})$ contains a schlicht disk $\mathbb{D}_{R_1}$ with

$$R_1 = \frac{\pi}{4M} r_1 - 2M \frac{r_1^3 + r_1^2}{1 - r_1}. \tag{1.5}$$

From that on, many authors considered the Landau-type theorems for certain bounded biharmonic mappings (see [5,7–9,16,18–23,26]). Liu et al. improved Theorem A by establishing the following theorem.

Theorem B (Liu [16]) *Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk $\mathbb{D}$, as in (1.1), with $F(0) = h(0) = J_F(0) - 1 = 0$ and $|g(z)| \leq M_1$, $|h(z)| \leq M_2$ for $z \in \mathbb{D}$. Then, F is univalent in the disk $\mathbb{D}_{r_2}$, and $F(\mathbb{D}_{r_2})$ contains a schlicht disk $\mathbb{D}_{R_2}$, where r_2 is the minimum positive root of the following equation*

$$\lambda_0(M_2) - 2rM_1 - \frac{2M_1 r^2}{(1-r)^2} - \sqrt{2M_2^2 - 2} \cdot \frac{2r - r^2}{(1-r)^2} = 0, \tag{1.6}$$

and

$$R_2 = \lambda_0(M_2) r_2 - 2M_1 \cdot \frac{r_2^3}{1 - r_2} - \sqrt{2M_2^2 - 2} \cdot \frac{r_2^2}{1 - r_2}, \tag{1.7}$$

where $\lambda_0(M)$ *is defined by*

$$\lambda_0(M)=\begin{cases}\frac{\sqrt{2}}{\sqrt{M^2-1}+\sqrt{M^2+1}} & \text{if } 1\le M\le M_0=\frac{\pi}{2\sqrt[4]{2\pi^2-16}},\\ \frac{\pi}{4M} & \text{if } M>M_0=\frac{\pi}{2\sqrt[4]{2\pi^2-16}}\approx 1.1296\,.\end{cases} \tag{1.8}$$

Chen et al. established the following theorem, which improved Theorems A and B for the case $M_1=M_2=M$.

Theorem C (Chen et al. [8]) *Let* $F(z)=|z|^2g(z)+h(z)$ *be a biharmonic mapping of the unit disk* $\mathbb{D}$*, as in* (1.1)*, with* $F(0)=h(0)=J_F(0)-1=0$ *and* $|g(z)|\le M_1$, $|h(z)|\le M_2$ *for* $z\in\mathbb{D}$*. Then,* F *is univalent in the disk* $\mathbb{D}_{r_3}$*, and* $F(\mathbb{D}_{r_3})$ *contains a schlicht disk* $\mathbb{D}_{R_3}$*, where* r_3 *is the minimum positive root of the following equation*

$$\frac{\pi}{4M_2}-2rM_1-\frac{4M_1r^2}{\pi(1-r)^2}-\sqrt{2M_2^2-2}\cdot\frac{2r-r^2}{(1-r)^2}=0, \tag{1.9}$$

and

$$R_3=\frac{\pi}{4M_2}r_3-\frac{r_3^2(4M_1r_3+\pi\sqrt{2M_2^2-2})}{\pi(1-r_3)}\,, \tag{1.10}$$

Zhu et al. improved Theorems A, B and C by establishing the following theorem:

Theorem D (Zhu and Liu [26]) *Suppose that* $F(z)=|z|^2g(z)+h(z)$ *is a biharmonic mapping in the unit disk* $\mathbb{D}$ *such that* $|g(z)|\le M_1$ *and* $|h(z)|\le M_2$ *for* $z\in\mathbb{D}$ *with* $|J_F(0)|=1$.

(i) If $M_2>1$ *or* $M_2=1$ *and* $M_1>0$*, then* F *is univalent in the disk* $\mathbb{D}_{r_4}$*, and* $F(\mathbb{D}_{r_4})$ *contains a schlicht disk* $\mathbb{D}_{R_4}(F(0))$*, where* $r_4=r_4(M_1,M_2)$ *is the minimum positive root of the following equation:*

$$\lambda_0(M_2)-2M_1r-\frac{4M_1r^2}{\pi(1-r^2)}-\lambda_0(M_2)\sqrt{M_2^4-1}\cdot\frac{r\sqrt{4-3r^2+r^4}}{(1-r^2)^{3/2}}=0, \tag{1.11}$$

and

$$R_4=\lambda_0(M_2)r_4-M_1r_4^2-\lambda_0(M_2)\sqrt{M_2^4-1}\cdot\frac{r_4^2}{(1-r_4^2)^{1/2}}, \tag{1.12}$$

where $\lambda_0(M)$ *is given by* (1.8).

(ii) If $M_2=1$ *and* $M_1=0$*, then* F *is univalent in* $\mathbb{D}$ *and* $F(\mathbb{D})=\mathbb{D}$.

For the biharmonic mappings with $\lambda_F(0)=1$*, many versions of Landau-type theorems, even sharp results have been found. In 2019, Liu and Luo proved the following sharp results.*

Theorem E (Liu and Luo [20]) *Suppose that* $\Lambda_1 \geq 0$ *and* $\Lambda_2 > 1$. *Let* $F(z) = |z|^2 G(z) + H(z)$ *be a biharmonic mapping of the unit disk* $\mathbb{D}$, *where* $G(z)$ *and* $H(z)$ *are harmonic in* $\mathbb{D}$, *satisfying* $G(0) = H(0) = 0, \lambda_F(0) = 1, \Lambda_G(z) \leq \Lambda_1$ *and* $\Lambda_H(z) < \Lambda_2$ *for all* $z \in \mathbb{D}$. *Then* $F(z)$ *is univalent on the disk* $\mathbb{D}_{r_5}$ *and* $F\left(\mathbb{D}_{r_5}\right)$ *contains a Schlicht disk* $\mathbb{D}_{R_5}$, *where* r_5 *is the unique root in* $(0, 1)$ *of the equation*

$$\Lambda_2 \frac{1 - \Lambda_2 r}{\Lambda_2 - r} - 3\Lambda_1 r^2 = 0, \tag{1.13}$$

and

$$R_5 = \Lambda_2^2 r_5 + \left(\Lambda_2^3 - \Lambda_2\right) \ln\left(1 - \frac{r_5}{\Lambda_2}\right) - \Lambda_1 r_5^3. \tag{1.14}$$

This result is sharp, with an extremal function given by

$$\begin{aligned} F_0(z) &= \Lambda_2 \int_{[0,z]} \frac{1 - \Lambda_2 z}{\Lambda_2 - z}\, dz - \Lambda_1 |z|^2 z \\ &= \Lambda_2^2 z - \Lambda_1 |z|^2 z + \left(\Lambda_2^3 - \Lambda_2\right) \ln\left(1 - \frac{z}{\Lambda_2}\right), \quad z \in \mathbb{D}. \end{aligned} \tag{1.15}$$

Theorem F (Liu and Luo [20]) *Suppose that* $\Lambda \geq 0$. *Let* $F(z) = |z|^2 G(z) + H(z)$ *be a biharmonic mapping of* $\mathbb{D}$, *where* $G(z), H(z)$ *are harmonic in* $\mathbb{D}$, *satisfying* $G(0) = H(0) = 0$, $\lambda_F(0) = 1$, $\Lambda_G(z) \leq \Lambda$, *and* $\Lambda_H(z) \leq 1$ *or* $|H(z)| < 1$ *for all* $z \in \mathbb{D}$. *Then* F *is univalent on the disk* $\mathbb{D}_{\rho_1}$, *and* $F\left(\mathbb{D}_{\rho_1}\right)$ *contains a schlicht disk* $\mathbb{D}_{\sigma_1}$, *where*

$$\rho_1 = \begin{cases} 1 & \text{if } 0 \leq \Lambda \leq \frac{1}{3}, \\ \frac{1}{\sqrt{3\Lambda}} & \text{if } \Lambda > \frac{1}{3}, \end{cases} \tag{1.16}$$

and

$$\sigma_1 = \rho_1 - \Lambda \rho_1^3 = \begin{cases} 1 - \Lambda, & \text{if } 0 \leq \Lambda \leq \frac{1}{3}, \\ \frac{2}{3\sqrt{3\Lambda}}, & \text{if } \Lambda > \frac{1}{3}. \end{cases} \tag{1.17}$$

This result is sharp.

It is natural raise the following.

Problem 1 *If* $\lambda_F(0) = 1$ *is replaced by* $J_F(0) = 1$ *in Theorems* E *and* F, *can we obtain sharp versions of Landau-type theorems for such bounded and normalized biharmonic mappings?*

Problem 2 *Can we improve Theorem* D*?*

In this paper, we first establish several new lemmas (see Lemmas 2.1, 2.2, 2.4, 2.5 and 2.9). Then, using these estimates, we prove several new versions of Landau-type theorems of bounded biharmonic mappings $F(z)$ with $J_F(0) = 1$. In particular, the results of Theorems 3.1 and 3.2 are sharp, which gives part of affirmative answer to the first question, Theorem 3.5 improves Theorems A, B, C and D, which gives an affirmative answer to the second question. Moreover, we can verify that these biharmonic mappings $F(z)$ are bi-Lipschitz on the univalent disks without changing the hypothesis of the theorems in Sect. 3.

2 Preliminaries

In this section, we establish some lemmas needed in the proof of the main results.

Lemma 2.1 *Suppose* $\Lambda > 1$. *Let* $H(z)$ *be a harmonic mapping of the unit disk* $\mathbb{D}$ *with* $J_H(0) = 1$ *and* $\Lambda_H(z) < \Lambda$ *for all* $z \in \mathbb{D}$. *Then for all* $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < 1, z_1 \neq z_2)$, *we have*

$$|H(z_2) - H(z_1)| = \left| \int_{\overline{z_1 z_2}} H_z(z)dz + H_{\bar{z}}(z)d\bar{z} \right| \geq \Lambda \frac{\lambda_H(0) - \Lambda r}{\Lambda - \lambda_H(0) r} |z_1 - z_2| \,. \quad (2.1)$$

Proof Following the idea from [17] (see also [20, Proof of Lemma 2.2]), let $\theta_0 = \arg(z_2 - z_1)$. Since $H(z)$ is a harmonic mapping in the unit disk $\mathbb{D}$, $H(z)$ can be written as $H(z) = H_1(z) + \overline{H_2(z)}$ for $z \in \mathbb{D}$, where H_1 and H_2 are analytic in $\mathbb{D}$. Since $J_H(0) = |H_1'(0)|^2 - |H_2'(0)|^2 = 1$, we have $|H_1'(0)| > |H_2'(0)|$, and

$$\begin{aligned} &\Delta_{0 \leq \theta \leq 2\pi} \arg\left\{ H_1'(0) e^{i(\theta_0 + \theta)} + H_2'(0) e^{i(\theta_0 - \theta)} \right\} \\ &= \Delta_{0 \leq \theta \leq 2\pi} \arg\left\{ H_1'(0) e^{i(\theta_0 + \theta)} \right\} = 2\pi, \end{aligned}$$

where $\Delta_{0 \leq \theta \leq 2\pi}$ denotes the increment of the succeeding function as θ increasing from 0 to 2π. Thus there exists a $\theta_1 \in [0, 2\pi]$ such that

$$H_1'(0) e^{i(\theta_0 + \theta_1)} + H_2'(0) e^{i(\theta_0 - \theta_1)} > 0.$$

Since $\Lambda_H(0) < \Lambda$, we have

$$\lambda_H(0) = \frac{J_H(0)}{\Lambda_H(0)} > \frac{1}{\Lambda} > 0.$$

For $z \in \mathbb{D}$, let

$$\omega(z) = \frac{H_1'(z) e^{i(\theta_0 + \theta_1)} + H_2'(z) e^{i(\theta_0 - \theta_1)}}{\Lambda}.$$

Then $\omega(z)$ is analytic with $|\omega(z)| \le \Lambda_H(z)/\Lambda < 1$ for $z \in \mathbb{D}$ and

$$\alpha := \omega(0) = \frac{H_1'(0)e^{i(\theta_0+\theta_1)} + H_2'(0)e^{i(\theta_0-\theta_1)}}{\Lambda} \ge \frac{\lambda_H(0)}{\Lambda}.$$

Using Schwarz–Pick Lemma, we have

$$\operatorname{Re}\omega(z) \ge \frac{\alpha - r}{1-\alpha r} \ge \frac{\frac{\lambda_H(0)}{\Lambda} - r}{1 - \frac{\lambda_H(0)r}{\Lambda}} = \frac{\lambda_H(0) - \Lambda r}{\Lambda - \lambda_H(0)r}, \quad z \in \mathbb{D}_r.$$

Then

$$\begin{aligned}\left|\int_{\overline{z_1z_2}} H_z(z)dz + H_{\bar z}(z)d\bar z\right| &= \left|\int_{\overline{z_1z_2}} \left(H_1'(z)e^{i(\theta_0+\theta_1)} + \overline{H_2'(z)}e^{-i(\theta_0-\theta_1)}\right)|dz|\right| \\ &\ge \int_{\overline{z_1z_2}} \operatorname{Re}\left\{H_1'(z)e^{i(\theta_0+\theta_1)} + \overline{H_2'(z)}e^{-i(\theta_0-\theta_1)}\right\}|dz| \\ &= \int_{\overline{z_1z_2}} \operatorname{Re}\left\{H_1'(z)e^{i(\theta_0+\theta_1)} + H_2'(z)e^{i(\theta_0-\theta_1)}\right\}|dz| \\ &\ge \int_{\overline{z_1z_2}} \Lambda\frac{\lambda_H(0) - \Lambda r}{\Lambda - \lambda_H(0)r}|dz| = \Lambda\frac{\lambda_H(0) - \Lambda r}{\Lambda - \lambda_H(0)r}|z_1 - z_2|.\end{aligned}$$

□

Applying the analogous proof of Lemma 2.3 in [20], we have the following lemma.

Lemma 2.2 *Suppose $\Lambda > 1$. Let $H(z)$ be a harmonic mapping of the unit disk $\mathbb{D}$ with $J_H(0) = 1$ and $\Lambda_H(z) < \Lambda$ for all $z \in \mathbb{D}$. Set $\gamma = H^{-1}(\overline{ow'})$ with $w' \in H(\partial\mathbb{D}_r)$ $(0 < r \le 1)$ and $\overline{ow'}$ denotes the closed line segment joining the origin and w', then*

$$\left|\int_\gamma H_\zeta(\zeta)d\zeta + H_{\bar\zeta}(\zeta)d\bar\zeta\right| \ge \Lambda\int_0^r \frac{\lambda_H(0) - \Lambda t}{\Lambda - \lambda_H(0)t}dt. \tag{2.2}$$

Lemma 2.3 [16] *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $|f(z)| \le 1$. If $J_f(0) = 1$, then $f(z) = \alpha z$, where $|\alpha| = 1$.*

Lemma 2.4 *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $J_f(0) = 1$. Then $|f(z)| \le 1$ for all $z \in \mathbb{D}$ if and only if $\Lambda_f(z) \le 1$ for all $z \in \mathbb{D}$.*

Proof If $|f(z)| \le 1$ for all $z \in \mathbb{D}$, it follows from Lemma 2.3 that $f(z) = \alpha z$, where $|\alpha| = 1$. Hence

$$\Lambda_f(z) = |f_z(z)| + |f_{\bar z}(z)| = |\alpha| = 1 \le 1$$

for all $z \in \mathbb{D}$. Conversely, if $\Lambda_f(z) \le 1$ for all $z \in \mathbb{D}$, then for each $z \in \mathbb{D}$, we have

$$|f(z)| = \left| \int_{[0,z]} f_z(z)dz + f_{\bar{z}}(z)d\bar{z} \right| \le \int_{[0,z]} \left|\Lambda_f(z)\right| |dz| \le |z| \le 1. \qquad \square$$

Because of its independent interest, we establish the following estimates of coefficients of harmonic mapping f with $f(0) = J_f(0) - 1 = 0$ and $\Lambda_f(z) \le \Lambda$ for all $z \in \mathbb{D}$.

Lemma 2.5 *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping on $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on $\mathbb{D}$, and $f(0) = J_f(0) - 1 = 0$, $\Lambda_f(z) \le \Lambda$ for all $z \in \mathbb{D}$, then $\Lambda \ge 1$, $|a_1| + |b_1| \le \Lambda$, and*

$$|a_n| + |b_n| \le \frac{\Lambda^4 - 1}{n\Lambda^3}, \quad n = 2, 3, \ldots. \tag{2.3}$$

and

$$\frac{1}{\Lambda} \le \lambda_f(0) \le 1. \tag{2.4}$$

When $\Lambda = 1$, then $f(z) = a_1 z$ with $|a_1| = 1$.

Proof Since $J_f(0) = (|a_1| + |b_1|)(|a_1| - |b_1|) = 1$ and $\Lambda_f(z) \le \Lambda$ for all $z \in \mathbb{D}$, we have

$$0 < \frac{1}{|a_1| + |b_1|} = |a_1| - |b_1| \le |a_1| + |b_1| = \Lambda_f(0) \le \Lambda.$$

which implies that $\Lambda \ge 1$,

$$\lambda_f(0) = ||a_1| - |b_1|| = \frac{1}{|a_1| + |b_1|} \ge \frac{1}{\Lambda}. \tag{2.5}$$

and

$$\lambda_f(0) = ||a_1| - |b_1|| \le |a_1| + |b_1| = \frac{1}{||a_1| - |b_1||} \Longrightarrow \lambda_f(0) = ||a_1| - |b_1|| \le 1. \tag{2.6}$$

Fixed $n \in \mathbb{N} - \{1\} = \{2, 3, \ldots\}$, we choose a real number α such that $|a_n + e^{i\alpha} b_n| = |a_n| + |b_n|$, and set

$$F(z) = \frac{1}{\Lambda}[h'(z) + e^{i\alpha} g'(z)] = \frac{a_1 + e^{i\alpha} b_1}{\Lambda} + \sum_{n=2}^{\infty} \frac{k(a_k + e^{i\alpha} b_k)}{\Lambda} z^{k-1}.$$

Since $g(z)$ and $h(z)$ are analytic and $\Lambda_f(z) = |h'(z)| + |g'(z)| \leq \Lambda$ on $\mathbb{D}$, we get $F(z)$ is analytic and $|F(z)| \leq \frac{|h'(z)|+|g'(z)|}{\Lambda} \leq 1$ on $\mathbb{D}$. By Lemma 1.3 in [16] and (2.5), we have

$$\left|\frac{k(a_k + e^{i\alpha}b_k)}{\Lambda}\right| \leq 1 - \left|\frac{a_1 + e^{i\alpha}b_1}{\Lambda}\right|^2 \leq 1 - \frac{||a_1| - |b_1||^2}{\Lambda^2} \leq 1 - \frac{1}{\Lambda^4}$$

for $k = 2, 3, \ldots$. In particular, we have

$$n(|a_n| + |b_n|) = n|a_n + e^{i\alpha}b_n| \leq \Lambda(1 - \frac{1}{\Lambda^4}) = \frac{\Lambda^4 - 1}{\Lambda^3},$$

which implies that

$$|a_n| + |b_n| \leq \frac{\Lambda^4 - 1}{n\Lambda^3}, \quad n = 2, 3, \ldots.$$

When $\Lambda = 1$, we have $|a_n| + |b_n| \leq \frac{\Lambda^4 - 1}{n\Lambda^3} = 0$ for $n = 2, 3, \ldots$, which implies $a_n = b_n = 0$ for $n = 2, 3, \ldots$.

Since $0 \leq |a_1| - |b_1| \leq |a_1| + |b_1| \leq 1$ and $J_f(0) = (|a_1| - |b_1|)(|a_1| + |b_1|) = 1$, we have $|a_1| - |b_1| = |a_1| + |b_1| = 1$. Hence $|a_1| = 1$, $b_1 = 0$, and $f(z) = a_1 z$ with $|a_1| = 1$. □

Lemma 2.6 [3,16,26] *Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ be a harmonic mapping on the unit disk $\mathbb{D}$.*

(i) If $|f(z)| < M$, then

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|)^2 \leq 2M^2.$$

(ii) If $J_f(0) = 1$ and $|f(z)| < M$, then

$$\sqrt{\sum_{n=2}^{\infty} (|a_n| + |b_n|)^2} \leq \sqrt{M^4 - 1} \cdot \lambda_f(0),$$

and $\lambda_f(0) \geq \lambda_0(M)$, where $\lambda_0(M)$ is given by (1.8).

Applying the analogous proof of Lemma 2.5 in [20], we have the following lemma.

Lemma 2.7 *Suppose $\Lambda \geq 0$. Let $F(z) = a\Lambda|z|^2 z + bz$ be a biharmonic mapping of the unit disk $\mathbb{D}$ with $|a| = |b| = 1$. Then F is univalent on the disk $\mathbb{D}_{\rho_1}$, and $F\left(\mathbb{D}_{\rho_1}\right)$ contains a Schlicht disk $\mathbb{D}_{\sigma_1}$, where ρ_1 and σ_1 are given by (1.16) and (1.17) respectively. This result is sharp.*

Lemma 2.8 [20] *Let $F(z)=|z|^2G(z)+H(z)$ be a biharmonic mapping of the unit disk $\mathbb{D}$ with $G(0)=H(0)=0$ and $\Lambda_G(z)\le\Lambda$ for all $z\in\mathbb{D}$, where $G(z)=G_1(z)+\overline{G_2(z)}=\sum_{n=1}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b_n}\bar{z}^n$, $H(z)=H_1(z)+\overline{H_2(z)}=\sum_{n=1}^{\infty}c_nz^n+\sum_{n=1}^{\infty}\overline{d_n}\bar{z}^n$ are harmonic mappings on $\mathbb{D}$. Then for all $z_1,z_2\in\mathbb{D}_r(0<r<1)$ with $z_1\ne z_2$, we have*

$$|F(z_1)-F(z_2)|\ge|z_1-z_2|\left[\||c_1|-|d_1|\|-\sum_{n=2}^{\infty}(|c_n|+|d_n|)nr^{n-1}-3\Lambda r^2\right]. \tag{2.7}$$

Lemma 2.9 *Let $F(z)=|z|^2G(z)+H(z)$ be a biharmonic mapping of the unit disk $\mathbb{D}$ with $G(0)=H(0)=0$ and $\Lambda_G(z)\le\Lambda$ for all $z\in\mathbb{D}$, where $G(z)=G_1(z)+\overline{G_2(z)}=\sum_{n=1}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b_n}\bar{z}^n$, $H(z)=H_1(z)+\overline{H_2(z)}=\sum_{n=1}^{\infty}c_nz^n+\sum_{n=1}^{\infty}\overline{d_n}\bar{z}^n$ are harmonic mappings on $\mathbb{D}$. Then for all $z_1,z_2\in\mathbb{D}_r(0<r<1)$ with $z_1\ne z_2$, we have*

$$|F(z_1)-F(z_2)|\le|z_1-z_2|\left(\Lambda_H(z)+3\Lambda r^2\right). \tag{2.8}$$

Proof For any $z_1,z_2\in\mathbb{D}_r$ $(0<r<1, z_1\ne z_2)$, we have

$$|G(z)|=\left|\int_{[0,z]}G_z(z)dz+G_{\bar z}(z)d\bar z\right|\le\int_{[0,z]}|\Lambda_G(z)\||dz|\le\Lambda\,|z|, \tag{2.9}$$

and

$$\begin{aligned}|F(z_1)-F(z_2)|&=\left|\int_{\overline{z_1,z_2}}F_z(z)dz+F_{\bar z}(z)d\bar z\right|\\&=\left|\int_{\overline{z_1,z_2}}(\bar zG(z)+|z|^2G_1'(z)\right.\\&\quad\left.+H_z(z))dz+(zG(z)+|z|^2\overline{G_2'(z)}+H_{\bar z}(z))d\bar z\right|\\&\le\left|\int_{\overline{z_1,z_2}}H_z(z)dz+H_z(z)d\bar z\right|\\&\quad+\left|\int_{\overline{z_1,z_2}}\left(\bar zG(z)+|z|^2G_1'(z)\right)dz+(zG(z)+|z|^2\overline{G_2'(z)})d\bar z\right|\\&\le\int_{\overline{z_1,z_2}}(|H_z(z)|+|H_{\bar z}(z)|)|dz|\\&\quad+\int_{\overline{z_1,z_2}}(2|z||G(z)|+|z|^2|G_1'|+|z|^2|G_2'|)|dz|\\&\le|z_1-z_2|\left(\Lambda_H(z)+3\Lambda r^2\right).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.10 [24] *Let $G(z)$ be a harmonic mapping of the unit disk $\mathbb{D}$ with $G(0) = 0$ and $|G(z)| \leq M$. Then for all $z_1, z_2 \in \mathbb{D}_r (0 < r < 1)$ with $z_1 \neq z_2$, we have*

$$||z_2|^2 G(z_2) - |z_1|^2 G(z_1)| \leq \frac{4M(3r^2 - 2r^4)}{\pi(1 - r^2)} |z_1 - z_2|.$$

Lemma 2.11 [6] *Let G be a harmonic mapping of the unit disk $\mathbb{D}$ with $G(0) = 0$ and $G(\mathbb{D}) \subset \mathbb{D}$. Then*

$$|G(z)| \leq \frac{4}{\pi} \arctan |z| \leq \frac{4}{\pi} |z|, \ \text{ for } z \in \mathbb{D}.$$

Lemma 2.12 [10] *Suppose that $f(z) = f_1(z) + \overline{f_2(z)}$ is a harmonic mapping with $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n z^n$ being analytic in $\mathbb{D}$. If $|f(z)| \leq M$ for all $z \in \mathbb{D}$, then*

$$\Lambda_f(z) \leq \frac{4M}{\pi(1 - |z|^2)}. \tag{2.10}$$

3 Landau-type theorems of biharmonic mappings

We first prove a new version of Landau-type theorem for biharmonic mappings $F(z)$ under the assumptions $G(0) = H(0) = J_F(0) - 1 = 0$, $\Lambda_G(z) \leq \Lambda_1$ and $\Lambda_H(z) < \Lambda_2$ for all $z \in \mathbb{D}$, which is one of the main results in this paper.

Theorem 3.1 *Suppose that $\Lambda_1 \geq 0$ and $\Lambda_2 > 1$. Let $F(z) = |z|^2 G(z) + H(z)$ be a biharmonic mapping of the unit disk $\mathbb{D}$, where $G(z)$ and $H(z)$ are harmonic on $\mathbb{D}$, satisfying $G(0) = H(0) = J_F(0) - 1 = 0$, $\Lambda_G(z) \leq \Lambda_1$ and $\Lambda_H(z) < \Lambda_2$ for all $z \in \mathbb{D}$. Then $\frac{1}{\Lambda_2} < \lambda_F(0) \leq 1$, $F(z)$ is univalent on the disk $\mathbb{D}_{\rho_0}$ and $F\left(\mathbb{D}_{\rho_0}\right)$ contains a schlicht disk $\mathbb{D}_{\sigma_0}$, where ρ_0 is the unique root in $(0, 1)$ of the equation*

$$\Lambda_2 \frac{\lambda_H(0) - \Lambda_2 r}{\Lambda_2 - \lambda_H(0) r} - 3\Lambda_1 r^2 = 0, \tag{3.1}$$

and

$$\sigma_0 = \frac{\Lambda_2^2}{\lambda_H(0)} \rho_0 + \left(\frac{\Lambda_2^3}{\lambda_H^2(0)} - \Lambda_2 \right) \ln \left(1 - \frac{\lambda_H(0) \rho_0}{\Lambda_2} \right) - \Lambda_1 \rho_0^3. \tag{3.2}$$

This result is sharp for the biharmonic mapping given by (1.15).

Proof We first prove that F is univalent in the disk $\mathbb{D}_{\rho_0}$. Indeed, for all $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < \rho_0)$ with $z_1 \neq z_2$, note that $J_H(0) = J_F(0) = 1$ and $\Lambda_H(z) < \Lambda_2$ for all $z \in \mathbb{D}$, we obtain from (2.4), (2.9) and Lemma 2.1 that

$$0 < \frac{1}{\Lambda_2} < \lambda_F(0) = \lambda_H(0) \leq 1, \tag{3.3}$$

and

$$\begin{aligned} |F(z_2) - F(z_1)| &= \left| \int_{\overline{z_1 z_2}} F_z(z)dz + F_{\bar{z}}(z)d\bar{z} \right| \\ &= \left| \int_{\overline{z_1 z_2}} \left(\bar{z}G(z) + |z|^2 G_z(z) + H_z(z) \right) dz \right. \\ &\quad \left. + \left(zG(z) + |z|^2 G_{\bar{z}}(z) + H_{\bar{z}}(z) \right) d\bar{z} \right| \\ &\geq \left| \int_{\overline{z_1 z_2}} H_z(z)dz + H_{\bar{z}}(z)d\bar{z} \right| - \int_{\overline{z_1 z_2}} 3\Lambda_1 r^2 |dz| \\ &\geq |z_1 - z_2| \left(\Lambda_2 \frac{\lambda_H(0) - \Lambda_2 r}{\Lambda_2 - \lambda_H(0) r} - 3\Lambda_1 r^2 \right). \end{aligned} \tag{3.4}$$

It is easy to verify that the function

$$g_0(r) := \Lambda_2 \frac{\lambda_H(0) - \Lambda_2 r}{\Lambda_2 - \lambda_H(0) r} - 3\Lambda_1 r^2$$

is continuous and strictly decreasing on $[0, 1]$, $g_0(0) = \lambda_H(0) > \frac{1}{\Lambda_2} > 0$, and

$$g_0(1) = -(\Lambda_2 + 3\Lambda_1) < 0.$$

Therefore, by the mean value theorem, there is a unique real $\rho_0 \in (0, 1)$ such that $g_0(\rho_0) = 0$. Then, for any $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < \rho_0)$ with $z_1 \neq z_2$, we obtain that

$$|F(z_2) - F(z_1)| \geq |z_1 - z_2| \left(\Lambda_2 \frac{\lambda_H(0) - \Lambda_2 r}{\Lambda_2 - \lambda_H(0) r} - 3\Lambda_1 r^2 \right) > |z_1 - z_2|\, g_0(\rho_0) = 0.$$

This implies $F(z_1) \neq F(z_2)$, which proves the univalence of F in the disk $\mathbb{D}_{\rho_0}$.

Next, we prove that $\sigma_0 > 0$ and $F\left(\mathbb{D}_{\rho_0}\right) \supseteq \mathbb{D}_{\sigma_0}$.

In fact, considering the real differentiable function

$$h(x) = \frac{\Lambda_2^2}{\lambda_H(0)} x + \left(\frac{\Lambda_2^3}{\lambda_H^2(0)} - \Lambda_2 \right) \ln \left(1 - \frac{\lambda_H(0) x}{\Lambda_2} \right) - \Lambda_1 x^3, x \in [0, 1]. \tag{3.5}$$

Since the continuous function

$$h'(x) = \frac{\Lambda_2^2}{\lambda_H(0)} - 3\Lambda_1 x^2 + \frac{\Lambda_2 \lambda_H(0)^2 - \Lambda_2^3}{\lambda_H(0)(\Lambda_2 - \lambda_H^2(0) x)} \tag{3.6}$$

is strictly decreasing on $[0, 1]$ and $h'(\rho_0) = g_0(\rho_0) = 0$, we see that $h'(x) = 0$ for $x \in [0, 1]$ if and only if $x = \rho_0$. Thus $h(x)$ is strictly increasing on $[0, \rho_0]$ and strictly decreasing on $[\rho_0, 1]$. Since $h(0) = 0$, we have

$$\sigma_0 = \frac{\Lambda_2^2}{\lambda_H(0)}\rho_0 + \left(\frac{\Lambda_2^3}{\lambda_H^2(0)} - \Lambda_2\right)$$
$$\ln\left(1 - \frac{\lambda_H(0)\rho_0}{\Lambda_2}\right) - \Lambda_1\rho_0^3 = h(\rho_0) > h(0) = 0. \tag{3.7}$$

In addition, note that $F(0) = 0$, for any $z \in \partial\mathbb{D}_{\rho_0}$, taking $z_0 = \rho_0 e^{i\theta} \in \partial\mathbb{D}_{\rho_0}$ with $w_0 = F(z_0) \in F\left(\partial\mathbb{D}_{\rho_0}\right)$ and $|w_0| = \min\left\{|w| : w \in F\left(\partial\mathbb{D}_{\rho_0}\right)\right\}$. Let $\gamma = F^{-1}(\overline{ow})$, by Lemma 2.2 and (2.9), we have

$$\begin{aligned}
|F(z) - F(0)| &\geq |w_0| = ||\rho_0 e^{i\theta}|^2 G(\rho_0 e^{i\theta}) + H(\rho_0 e^{i\theta})| \geq |H(\rho_0 e^{i\theta})| - \Lambda_1\rho_0^3 \\
&= \left|\int_\gamma H_\zeta(\zeta)d\zeta + H_{\bar{\zeta}}(\zeta)d\bar{\zeta}\right| - \Lambda_1\rho_0^3 \\
&\geq \Lambda_2\int_0^{\rho_0} \frac{\lambda_H(0) - \Lambda_2 t}{\Lambda_2 - \lambda_H(0)t}dt - \Lambda_1\rho_0^3 \\
&= \frac{\Lambda_2^2}{\lambda_H(0)}\rho_0 + \left(\frac{\Lambda_2^3}{\lambda_H^2(0)} - \Lambda_2\right)\ln\left(1 - \frac{\lambda_H(0)\rho_0}{\Lambda_2}\right) - \Lambda_1\rho_0^3 = \sigma_0.
\end{aligned}$$

which implies that $F\left(\mathbb{D}_{\rho_0}\right) \supseteq \mathbb{D}_{\sigma_0}$.

Now, we prove the sharpness of ρ_0 and σ_0 for the biharmonic mapping $F_0(z)$ given by (1.15). In fact, it is easy to verify that $F_0(z)$ satisfies the hypothesis of Theorem 3.1, and thus, we have that $F_0(z)$ is univalent in the disk $\mathbb{D}_{\rho_0}$, and $F_0(\mathbb{D}_{\rho_0}) \supseteq \mathbb{D}_{\sigma_0}$.

Note that for the biharmonic mapping $F_0(z)$, $\lambda_H(0) = \lambda_F(0) = J_F(0) = 1$, the Eqs. (3.1), (3.2) reduce to (1.13) and (1.14) respectively. Thus we obtain $\rho_0 = r_5$ and $\sigma_0 = R_5$. By Theorem E, we conclude that ρ_0 and σ_0 are sharp. This completes the proof. □

Now we give an example to show that for each a value $\alpha \in (1/\Lambda_2, 1)$, there exits a biharmonic mapping F satisfying the hypothesis of Theorem 3.1 such that $\lambda_F(0) = \alpha$.

Example 3.1 Suppose that $\Lambda_1 \geq 0$, $\Lambda_2 > 1$ and $\alpha \in (1/\Lambda_2, 1)$. Let $|a| = 1$, $|b| = \frac{1}{2}(1/\alpha - \alpha)$, and

$$|c| = \frac{\sqrt{|b|^2 + 1}}{|b|} = \frac{1 + \alpha^2}{1 - \alpha^2}.$$

Consider the biharmonic mapping

$$F(z) = a\Lambda_1|z|^2 z + b(cz + \bar{z}), \quad z \in \mathbb{D}.$$

Then $F(z)$ satisfies the hypothesis of Theorem 3.1, $\lambda_F(0) = \alpha$, $F(z)$ is univalent on $\mathbb{D}_{\rho_0''}$, and $F(\mathbb{D}_{\rho_0''})$ contains a Schlicht disk $\mathbb{D}_{\sigma_0''}$, where

$$\rho_0'' = \begin{cases} 1, & \text{if } \Lambda_1 \leq \frac{\alpha}{3}, \\ \sqrt{\frac{\alpha}{3\Lambda_1}}, & \text{if } \Lambda_1 > \frac{\alpha}{3}, \end{cases} \tag{3.8}$$

and

$$\sigma_0'' = \begin{cases} \alpha - \Lambda_1, & \text{if } \Lambda_1 \le \frac{\alpha}{3}, \\ \frac{2\alpha}{3}\sqrt{\frac{\alpha}{3\Lambda_1}}, & \text{if } \Lambda_1 > \frac{\alpha}{3}, \end{cases} \tag{3.9}$$

and when $\arg c = \pi - \arg \frac{b}{a}$, this result is sharp.

Proof Set $G(z) = a\Lambda_1|z|^2 z$, $H(z) = b(cz + \overline{z})$. Direct computation yields

$$G(0) = H(0) = 0,\ \Lambda_G(z) = |a\Lambda_1| = \Lambda_1 \le \Lambda_1,$$
$$\Lambda_H(z) = |bc| + |c| = |b|(|c| + 1) = 1/\alpha < \Lambda_2,$$

and $J_F(0) = |b|^2(|c|^2 - 1) = 1$, thus $F(z)$ satisfies the hypothesis of Theorem 3.1, and

$$\lambda_F(0) = |b|(|c| - 1) = \alpha.$$

Applying the analogous proof of Lemma 2.5 in [20] (please also refer to example 2.1 in [3]), we may verify that if $\Lambda_1 \le \frac{\alpha}{3}$, then $F(z)$ is univalent on $\mathbb{D}$, and $F(\mathbb{D})$ contains a Schlicht disk $\mathbb{D}_{\sigma_0''}$, where

$$\sigma_0'' = \alpha - \Lambda_1.$$

If $\Lambda_1 > \frac{\alpha}{3}$, then $F(z)$ is univalent on $\mathbb{D}_{\rho_0''}$, and $F(\mathbb{D}_{\rho_0''})$ contains a Schlicht disk $\mathbb{D}_{\sigma_0''}$,, where

$$\rho_0'' = \sqrt{\frac{\alpha}{3\Lambda_1}}, \quad \sigma_0'' = \frac{2\alpha}{3}\sqrt{\frac{\alpha}{3\Lambda_1}},$$

and when $\arg c = \pi - \arg \frac{b}{a}$, the radii ρ_0'' and σ_0'' are sharp. □

Remark 3.1 For the biharmonic mapping $F(z)$ of the unit disk $\mathbb{D}$ with $J_F(0) = 1$ and $\Lambda_H(z) \le \Lambda_2$, it follows from Lemma 2.5 that $\Lambda_2 \ge 1$. Theorem 3.1 provides a sharp version of Landau-type theorem of biharmonic mappings for the case $J_F(0) = 1$, $\Lambda_1 \ge 0$ and $\Lambda_2 > 1$. If $J_F(0) = 1$, $\Lambda_1 \ge 0$ and $\Lambda_2 = 1$, then we prove Theorem 3.2 using Lemmas 2.3, 2.4 and 2.7, which is the sharp version of Landau-type theorem of biharmonic mappings and is also one of the main results in this paper.

Theorem 3.2 *Suppose that $\Lambda \ge 0$. Let $F(z) = |z|^2G(z) + H(z)$ be a biharmonic mapping of $\mathbb{D}$, where $G(z)$, $H(z)$ are harmonic on $\mathbb{D}$, satisfying $G(0) = H(0) = J_F(0) - 1 = 0$, $\Lambda_G(z) \le \Lambda$ and $\Lambda_H(z) \le 1$ or $|H(z)| \le 1$ for all $z \in \mathbb{D}$. Then F is univalent on the disk $\mathbb{D}_{\rho_1}$, and $F\left(\mathbb{D}_{\rho_1}\right)$ contains a Schlicht disk $\mathbb{D}_{\sigma_1}$, where ρ_1 and σ_1 are given by (1.16) and (1.17) respectively. This result is sharp.*

Proof Because $F(z) = |z|^2G(z) + H(z)$ satisfies the hypothesis of Theorem 3.2, where $G(z) = G_1(z) + \overline{G_2(z)}$ and $H(z) = H_1(z) + \overline{H_2(z)}$ with $G_1(z) =$

$\sum_{n=1}^{\infty} a_n z^n$, $G_2(z) = \sum_{n=1}^{\infty} b_n z^n$ and $H_1(z) = \sum_{n=1}^{\infty} c_n z^n$, $H_2(z) = \sum_{n=1}^{\infty} d_n z^n$ are analytic on $\mathbb{D}$. Then

$$J_H(0) = J_F(0) = |c_1|^2 - |d_1|^2 = 1.$$

By the hypothesis of Theorem 3.2 and Lemmas 2.3 and 2.4, we have

$$H(z) = c_1 z, \quad |c_1| = 1.$$

Now we prove that F is univalent in the disk $\mathbb{D}_{\rho_1}$, where

$$\rho_1 = \begin{cases} 1 & \text{if } 0 \leq \Lambda \leq \frac{1}{3}, \\ \frac{1}{\sqrt{3\Lambda}} & \text{if } \Lambda > \frac{1}{3}. \end{cases}$$

To this end, for any $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < \rho_1)$ with $z_1 \neq z_2$, by (3.4), we have

$$\begin{aligned} |F(z_1) - F(z_2)| &\geq \left| \int_{\overline{z_1 z_2}} H_z(z)dz + H_{\bar{z}}(z)d\bar{z} \right| - \int_{\overline{z_1 z_2}} 3\Lambda_1 r^2 |dz| \\ &= |z_1 - z_2| \left(|c_1| - 3\Lambda r^2 \right) \\ &= |z_1 - z_2| \left(1 - 3\Lambda r^2 \right) > 0. \end{aligned}$$

Then, we have $F(z_1) \neq F(z_2)$, which proves the univalence of F in the disk $\mathbb{D}_{\rho_1}$.

Noting that $F(0) = 0$, for any $z = \rho_1 e^{i\theta} \in \partial\mathbb{D}_{\rho_1}$, we have

$$\begin{aligned} |F(z) - F(0)| &= \left||z|^2 G(z) + H(z)\right| \geq |H(z)| - \rho_1^2 |G(z)| \\ &= \rho_1 |c_1| - \Lambda\rho_1^3 = \rho_1 - \Lambda\rho_1^3 = \sigma_1. \end{aligned}$$

Hence, $F\left(\mathbb{D}_{\rho_1}\right)$ contains a schlicht disk $\mathbb{D}_{\sigma_1}$.

Finally, for $F(z) = a_1\Lambda|z|^2 z + c_1 z$ with $|a_1| = |c_1| = 1$, we have $G(z) = a_1\Lambda z$, $H(z) = c_1 z$. Direct computation yields

$$G(0) = H(0) = 0,\ J_F(0) = |c_1| = 1,\ \Lambda_G(z) = |a_1\Lambda| \leq \Lambda.$$

and $|H(z)| = |c_1 z| \leq 1$ for all $z \in \mathbb{D}$. Applying Lemma 2.7, we obtain that the radii ρ_1 and σ_1 are sharp. This completes the proof. □

Next, we establish another new version of Landau-type theorem for biharmonic mappings $F(z)$ under the assumptions $G(0) = H(0) = J_F(0) - 1 = 0$, $\Lambda_G(z) \leq \Lambda$ and $|H(z)| \leq M$, $(M > 1)$ for all $z \in \mathbb{D}$.

Theorem 3.3 *Suppose that $\Lambda \geq 0$, $M > 1$. Let $F(z) = |z|^2 G(z) + H(z)$ be a biharmonic mapping of $\mathbb{D}$, where $G(z)$, $H(z)$ are harmonic on $\mathbb{D}$, satisfying $G(0) = H(0) = J_F(0) - 1 = 0$, $\Lambda_G(z) \leq \Lambda$ and $|H(z)| \leq M$ for all $z \in \mathbb{D}$. Then F*

is univalent on the disk $\mathbb{D}_{\rho_2}$*, where* ρ_2 *is the minimum positive root in* $(0, 1)$ *of the equation*

$$\lambda_0(M) - \lambda_0(M)\sqrt{M^4 - 1}\sqrt{\frac{4r^2 - 3r^4 + r^6}{\left(1 - r^2\right)^3}} - 3\Lambda r^2 = 0, \tag{3.10}$$

and $F\left(\mathbb{D}_{\rho_2}\right)$ *contains a Schlicht disk* $\mathbb{D}_{\sigma_2}$*, where*

$$\sigma_2 = \lambda_0(M)\rho_2 - \lambda_0(M)\sqrt{M^4 - 1}\frac{\rho_2^2}{\sqrt{1 - \rho_2^2}} - \rho_2^3\Lambda, \tag{3.11}$$

where $\lambda_0(M)$ *is given by* (1.8).

Proof By the hypothesis of Theorem 3.3, we can assume that

$$H(z) = \sum_{n=1}^{\infty} c_n z^n + \overline{\sum_{n=1}^{\infty} d_n z^n}, \; z \in \mathbb{D}.$$

Since $J_H(0) = J_F(0) = 1$ and $|H(z)| \leq M$, by Lemma 2.6, we have

$$\sqrt{\sum_{n=2}^{\infty} (|c_n| + |d_n|)^2} \leq \sqrt{M^4 - 1} \cdot \lambda_H(0),$$

and $\lambda_H(0) \geq \lambda_0(M)$, where $\lambda_0(M)$ is given by (1.8).

Now we prove that F is univalent in the disk $\mathbb{D}_{\rho_2}$. For all $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < \rho_2,$ $z_1 \neq z_2)$, we obtain from Lemmas 2.8 and 2.6 that

$$\begin{aligned}
&|F(z_1) - F(z_2)| \\
&\geq |z_1 - z_2| \left[\| c_1| - |d_1 \| - \sum_{n=2}^{\infty} (|c_n| + |d_n|)\, n r^{n-1} - 3\Lambda r^2 \right] \\
&\geq |z_1 - z_2| \left[\lambda_H(0) - \left(\sum_{n=2}^{\infty} (|c_n| + |d_n|)^2 \right)^{1/2} \left(\sum_{n=2}^{\infty} n^2 r^{2n-2} \right)^{1/2} - 3\Lambda r^2 \right] \\
&\geq |z_1 - z_2| \left(\lambda_H(0) - \lambda_H(0)\sqrt{M^4 - 1}\sqrt{\frac{4r^2 - 3r^4 + r^6}{\left(1 - r^2\right)^3}} - 3\Lambda r^2 \right) \\
&\geq |z_1 - z_2| \left(\lambda_0(M) - \lambda_0(M)\sqrt{M^4 - 1}\sqrt{\frac{4r^2 - 3r^4 + r^6}{\left(1 - r^2\right)^3}} - 3\Lambda r^2 \right) > 0.
\end{aligned}$$

Then, we have $F(z_1) \neq F(z_2)$, which proves the univalence of F in the disk $\mathbb{D}_{\rho_2}$.

Noting that $F(0)=0$, for any $z=\rho_2 e^{i\theta}\in\partial\mathbb{D}_{\rho_2}$, by (2.9), we have

$$
\begin{aligned}
|F(z)| &= \left||z|^2 G(z)+H(z)\right| \geq |H(z)|-\rho_2^2|G(z)| \\
&\geq \left\||c_1|-|d_1|\right\| \rho_2-\sum_{n=2}^{\infty}(|c_n|+|d_n|)\,\rho_2^n-\rho_2^3\Lambda \\
&\geq \lambda_H(0)\rho_2-\left(\sum_{n=2}^{\infty}(|c_n|+|d_n|)^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}\rho_2^{2n}\right)^{\frac{1}{2}}-\rho_2^3\Lambda \\
&\geq \lambda_H(0)\rho_2-\lambda_H(0)\sqrt{M^4-1}\frac{\rho_2^2}{\sqrt{1-\rho_2^2}}-\rho_2^3\Lambda \\
&\geq \lambda_0(M)\rho_2-\lambda_0(M)\sqrt{M^4-1}\frac{\rho_2^2}{\sqrt{1-\rho_2^2}}-\rho_2^3\Lambda=\sigma_2.
\end{aligned}
$$

Hence, $F\left(\mathbb{D}_{\rho_2}\right)$ contains a schlicht disk $\mathbb{D}_{\sigma_2}$. This completes the proof (Table 1). □

Now, we will consider the Landau-type theorem for the case $|G(z)|\leq M,\ \Lambda_H(z)<\Lambda$.

Theorem 3.4 *Suppose that $M\geq 0,\Lambda\geq 1$, $F(z)=|z|^2G(z)+H(z)$ is a biharmonic mapping on the unit disk $\mathbb{D}$, where $G(z)$, $H(z)$ are harmonic mappings on $\mathbb{D}$, satisfying $G(0)=H(0)=J_F(0)-1=0$ and $|G(z)|\leq M,\ \Lambda_H(z)<\Lambda$ for $z\in\mathbb{D}$.*

(i) If $M\geq 0,\Lambda>1$ or $M>0,\Lambda=1$, then $F(z)$ is univalent on $\mathbb{D}_{\rho_3}$, where ρ_3 is the minimum positive root in $(0,1)$ of the equation

$$\frac{\Lambda(1-\Lambda^2 r)}{\Lambda^2-r}-\frac{4M}{\pi}\frac{3r^2-2r^4}{1-r^2}=0, \tag{3.12}$$

and $F(\mathbb{D}_{\rho_3})$ contains a Schlicht disk $\mathbb{D}_{\sigma_3}$, where

$$\sigma_3=\Lambda^3\rho_3+\left(\Lambda^5-\Lambda\right)\ln\left(1-\frac{\rho_3}{\Lambda^2}\right)-\frac{4M}{\pi}\rho_3^3. \tag{3.13}$$

(ii) If $M=0,\Lambda=1$, then $F(z)$ is univalent on $\mathbb{D}$ and $F(\mathbb{D})=\mathbb{D}$.

Proof (i) Note that $J_H(0)=J_F(0)=1$, we split into two case to prove.

Table 1 The values of ρ_2,σ_2 are in Theorem 3.3

(Λ, M)	(0, 1.1)	(0.1, 1.1)	(0.5, 1.3)	(1, 1.6)	(1, 2)	(1.8, 2.3)	(2.5, 3.2)	(3, 3)
ρ_2	0.5128	0.4847	0.2735	0.1693	0.1145	0.0847	0.0458	0.0508
σ_2	0.2845	0.2650	0.1217	0.0612	0.0334	0.0214	0.0084	0.0099

Case 1. When $M \geq 0$, $\Lambda > 1$. We first prove that F is univalent on the disk $\mathbb{D}_{\rho_3}$. To this end, for all $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < \rho_3, z_1 \neq z_2)$, we obtain from Lemmas 2.1 and 2.10 that

$$\begin{aligned}|F(z_2) - F(z_1)| &= |(|z_2|^2 G(z_2) + H(z_2)) - (|z_1|^2 G(z_1) + H(z_1))| \\ &\geq |H(z_2) - H(z_1)| - ||z_2|^2 G(z_2) - |z_1|^2 G(z_1)| \\ &\geq |z_2 - z_1| \left(\Lambda \frac{\lambda_H(0) - \Lambda r}{\Lambda - \lambda_H(0) r} - \frac{4M}{\pi} \frac{3r^2 - 2r^4}{1 - r^2} \right).\end{aligned}$$

Since $\Lambda \frac{\lambda_H(0) - \Lambda r}{\Lambda - \lambda_H(0) r}$ is continuous and increasing about $\lambda_H(0)$ and by (3.3), we have

$$\begin{aligned}|F(z_2) - F(z_1)| &\geq |z_2 - z_1| \left(\Lambda \frac{\lambda_H(0) - \Lambda r}{\Lambda - \lambda_H(0) r} - \frac{4M}{\pi} \frac{3r^2 - 2r^4}{1 - r^2} \right) \\ &> \frac{\Lambda(1 - \Lambda^2 r)}{\Lambda^2 - r} - \frac{4M}{\pi} \frac{3r^2 - 2r^4}{1 - r^2} > 0.\end{aligned}$$

This shows that F is univalent on the disk $\mathbb{D}_{\rho_3}$.

Next, we prove $F(\mathbb{D}_{\rho_3}) \supset \mathbb{D}_{\sigma_3}$. For $z = \rho_3 e^{i\theta} \in \partial \mathbb{D}_{\rho_3}$, by Lemmas 2.2 and 2.11, we have

$$\begin{aligned}|F(z)| &\geq \Lambda \int_0^{\rho_3} \frac{\lambda_H(0) - \Lambda t}{\Lambda - \lambda_H(0) t} dt - \frac{4M}{\pi} \rho_3^3 \\ &\geq \Lambda \int_0^{\rho_3} \frac{1 - \Lambda^2 t}{\Lambda^2 - t} dt - \frac{4M}{\pi} \rho_3^3 \\ &= \Lambda^3 \rho_3 + \left(\Lambda^5 - \Lambda \right) \ln \left(1 - \frac{\rho_3}{\Lambda^2} \right) - \frac{4M}{\pi} \rho_3^3 = \sigma_3.\end{aligned}$$

Case 2. When $M > 0$, $\Lambda = 1$. Using Lemma 2.5, we have

$$H(z) = c_1 z, \ |c_1| = 1.$$

Similarly, we first prove that F is univalent on the disk $\mathbb{D}_{\rho_3}$. In fact, for all $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < \rho_3, z_1 \neq z_2)$, we have

$$\begin{aligned}|F(z_2) - F(z_1)| &\geq |H(z_2) - H(z_1)| - ||z_2|^2 G(z_2) - |z_1|^2 G(z_1)| \\ &\geq |z_2 - z_1| \left(1 - \frac{4M}{\pi} \frac{3r^2 - 2r^4}{1 - r^2} \right) > 0.\end{aligned}$$

This shows that F is univalent on the disk $\mathbb{D}_{\rho_3}$.

Next, we prove $F(\mathbb{D}_{\rho_3}) \supset \mathbb{D}_{\sigma_3}$. For $z = \rho_3 e^{i\theta} \in \partial \mathbb{D}_{\rho_3}$, by Lemma 2.11, we have

$$|F(z)| \geq |H(z)| - \rho_3^2 |G(z)| = \rho_3 - \frac{4M}{\pi} \rho_3^3 = \sigma_3.$$

Table 2 The values of ρ_3, σ_3 are in Theorem 3.4

(M, Λ)	(0.3, 1)	(0, 1.1)	(0.1, 1.1)	(0.5, 1.3)	(1, 1.6)	(1, 2)	(1.8, 2.3)	(2.5, 3.2)	(3, 3)
ρ_3	0.7680	0.8264	0.7260	0.4005	0.2514	0.1861	0.1353	0.0791	0.0842
σ_3	0.5950	0.6535	0.4498	0.1864	0.0910	0.0513	0.0325	0.0132	0.0152

Hence, $F\left(\mathbb{D}_{\rho_3}\right)$ contains a Schlicht disk $\mathbb{D}_{\sigma_3}$.

Now we prove (ii). If $M = 0$, $\Lambda = 1$, by Lemma 2.5, we have

$$F(z) = c_1 z, \ |c_1| = 1.$$

It's easy to verify that $F(z)$ is univalent on the unit disk $\mathbb{D}$, and $F(\mathbb{D}) = \mathbb{D}$. This completes the proof (Table 2). □

Finally, we improve Theorem D as follows.

Theorem 3.5 *Suppose that $M_1 \geq 0$, $M_2 \geq 1$, $F(z) = |z|^2 G(z) + H(z)$ is a biharmonic mapping on the unit disk $\mathbb{D}$, where $G(z)$, $H(z)$ are harmonic mappings on $\mathbb{D}$, satisfying $G(0) = H(0) = J_F(0) - 1 = 0$ and $|G(z)| \leq M_1$, $|H(z)| \leq M_2$ for $z \in \mathbb{D}$.*

(i) If $M_1 \geq 0$, $M_2 > 1$ or $M_1 > 0$, $M_2 = 1$, then $F(z)$ is univalent on $\mathbb{D}_{\rho_4}$, where ρ_4 is the minimum positive root in $(0, 1)$ of the equation

$$\lambda_0(M_2) - \lambda_0(M_2)\sqrt{M_2^4 - 1}\sqrt{\frac{4r^2 - 3r^4 + r^6}{\left(1 - r^2\right)^3}} - \frac{4M_1}{\pi}\frac{3r^2 - 2r^4}{1 - r^2} = 0, \quad (3.14)$$

and $F\left(\mathbb{D}_{\rho_4}\right)$ contains a Schlicht disk $\mathbb{D}_{\sigma_4}$, where

$$\sigma_4 = \lambda_0(M_2)\rho_4 - \lambda_0(M_2)\sqrt{M_2^4 - 1}\frac{\rho_4^2}{\sqrt{1 - \rho_4^2}} - \frac{4M_1}{\pi}\rho_4^3, \quad (3.15)$$

where $\lambda_0(M)$ is given by (1.8).

(ii) If $M_1 = 0$, $M_2 = 1$, then $F(z)$ is univalent on $\mathbb{D}$ and $F(\mathbb{D}) = \mathbb{D}$.

Proof By the hypothesis of Theorem 3.5, we can assume that

$$H(z) = \sum_{n=1}^{\infty} c_n z^n + \overline{\sum_{n=1}^{\infty} d_n z^n}, \ z \in \mathbb{D}.$$

Since $|H(z)| \leq M_2$ and $J_H(0) = J_F(0) = 1$, by Lemma 2.6, we have

$$\sqrt{\sum_{n=2}^{\infty} (|c_n| + |d_n|)^2} \leq \sqrt{M_2^4 - 1} \cdot \lambda_H(0),$$

and $\lambda_H(0) \geq \lambda_0(M_2)$, where $\lambda_0(M_2)$ is given by (1.8).

Now we prove that F is univalent in the disk $\mathbb{D}_{\rho_4}$. For all $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < \rho_4$, $z_1 \neq z_2)$, by Lemma 2.10, we have

$$
\begin{aligned}
&|F(z_1) - F(z_2)| \\
&\geq |H(z_2) - H(z_1)| - ||z_2|^2 G(z_2) - |z_1|^2 G(z_1)| \\
&\geq \left| \int_{\overline{z_1, z_2}} H_z(0)dz + H_{\overline{z}}(0)d\overline{z} \right| - \left| \int_{\overline{z_1, z_2}} (H_z(z) - H_z(0))dz + (H_{\overline{z}}(z) - H_{\overline{z}}(0))d\overline{z} \right| \\
&\quad - \left| |z_2|^2 G(z_2) - |z_1|^2 G(z_1) \right| \\
&\geq |z_1 - z_2| \left[\| c_1| - |d_1 \| - \sum_{n=2}^{\infty} (|c_n| + |d_n|)\, n r^{n-1} \right] - \left| |z_2|^2 G(z_2) - |z_1|^2 G(z_1) \right| \\
&\geq |z_1 - z_2| \left[\lambda_H(0) - \left(\sum_{n=2}^{\infty} (|c_n| + |d_n|)^2 \right)^{1/2} \left(\sum_{n=2}^{\infty} n^2 r^{2n-2} \right)^{1/2} \right] \\
&\quad - \left| |z_2|^2 G(z_2) - |z_1|^2 G(z_1) \right| \\
&\geq |z_1 - z_2| \left(\lambda_H(0) - \lambda_H(0) \sqrt{M_2^4 - 1} \sqrt{\frac{4r^2 - 3r^4 + r^6}{(1 - r^2)^3}} - \frac{4M_1}{\pi} \frac{3r^2 - 2r^4}{1 - r^2} \right) \\
&\geq |z_1 - z_2| \left(\lambda_0(M_2) - \lambda_0(M_2) \sqrt{M_2^4 - 1} \sqrt{\frac{4r^2 - 3r^4 + r^6}{(1 - r^2)^3}} - \frac{4M_1}{\pi} \frac{3r^2 - 2r^4}{1 - r^2} \right) \\
&> 0.
\end{aligned}
$$

Then, we have $F(z_1) \neq F(z_2)$, which proves the univalence of F in the disk $\mathbb{D}_{\rho_4}$.

Noting that $F(0) = 0$, for any $z = \rho_4 e^{i\theta} \in \partial\mathbb{D}_{\rho_4}$, by Lemmas 2.6 and 2.11, we have

$$
\begin{aligned}
|F(z)| &= ||z|^2 G(z) + H(z)| \geq |H(z)| - \rho_5^2 |G(z)| \\
&\geq \| c_1| - |d_1 \| \rho_4 - \sum_{n=2}^{\infty} (|c_n| + |d_n|)\, \rho_4^n - \frac{4M_1}{\pi} \rho_4^3 \\
&\geq \lambda_H(0)\rho_4 - \left(\sum_{n=2}^{\infty} (|c_n| + |d_n|)^2 \right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} \rho_4^{2n} \right)^{\frac{1}{2}} - \frac{4M_1}{\pi} \rho_4^3 \\
&\geq \lambda_H(0)\rho_4 - \lambda_H(0) \sqrt{M_2^4 - 1} \frac{\rho_4^2}{\sqrt{1 - \rho_4^2}} - \frac{4M_1}{\pi} \rho_4^3 \\
&\geq \lambda_0(M_2)\rho_4 - \lambda_0(M_2) \sqrt{M_2^4 - 1} \frac{\rho_4^2}{\sqrt{1 - \rho_4^2}} - \frac{4M_1}{\pi} \rho_4^3 = \sigma_4.
\end{aligned}
$$

Hence, $F\left(\mathbb{D}_{\rho_4}\right)$ contains a schlicht disk $\mathbb{D}_{\sigma_4}$.

Finally, if $M_1 = 0$, $M_2 = 1$, then by Lemma 2.3, we have

$$F(z) = c_1 z, |c_1| = 1.$$

It is evident that $F(z)$ is univalent on $\mathbb{D}$, and $F(\mathbb{D}) = \mathbb{D}$. This completes the proof. □

Remark 3.2 Note that for $r = \rho_4$, we have

$$\frac{4M_1}{\pi}\frac{3r^2 - 2r^4}{1 - r^2} = \frac{4M_1 r^2}{\pi(1 - r^2)} + \frac{8M_1 r^2}{\pi} < \frac{4M_1 r^2}{\pi(1 - r^2)} + 2M_1 r,$$

it is easy to verify that $\rho_4 > r_4$, $\sigma_4 > R_4$, where r_4, R_4 are given in Theorem D, please also see Table 3.

The Computer Algebra System Mathematica has calculated the numerical solutions to Eqs. (1.11) and (3.14). From Table 3 as follow, it is easy to see that the result of Theorem 3.5 is better than that of Theorem D. From Table 1, Table 2 and Table 3, it is easy to see that both of the result of Theorem 3.3 and that of Theorem 3.4 are better than that of Theorem 3.5.

4 The bi-Lipschitz theorems of biharmonic mappings

In this section, we will establish the Lipschitz characters of certain biharmonic mappings in their univalent disks.

Theorem 4.1 *Suppose $F(z)$ satisfies the hypothesis of Theorem 3.1. Then for each $r_0 \in (0, \rho_0)$, the biharmonic mapping $F(z)$ is bi-Lipschitz on $\overline{\mathbb{D}}_{r_0}$, where ρ_0 is given by* (3.1).

Proof Fixed $r_0 \in (0, \rho_0)$, set

$$l_0 = \Lambda_2 \frac{\lambda_H(0) - \Lambda_2 r_0}{\Lambda_2 - \lambda_H(0) r_0} - 3\Lambda_1 r_0^2.$$

Table 3 The values of r_4, R_4 and ρ_4, σ_4 are in Theorems D and 3.5 respectively

(M_1, M_2)	(0.3, 1)	(0, 1.1)	(0.1, 1.1)	(0.5, 1.3)	(1, 1.6)	(1, 2)	(1.8, 2.3)	(2.5, 3.2)	(3, 3)
r_4	0.7655	0.5128	0.4600	0.2103	0.1094	0.0761	0.0470	0.0243	0.0243
ρ_4	0.7680	0.5128	0.4744	0.2627	0.1629	0.1118	0.0824	0.0451	0.0497
R_4	0.5897	0.2845	0.2468	0.0895	0.0347	0.0197	0.0101	0.0038	0.0039
σ_4	0.5950	0.2845	0.2594	0.1167	0.0587	0.0325	0.0208	0.0082	0.0096

Then, for any $z_1, z_2 \in \overline{\mathbb{D}}_{r_0}$, it follows from the proof of Theorem 3.1 and Lemma 2.9 that $l_0 > g_0(\rho_0) = 0$, and

$$\begin{aligned} l_0 |z_1 - z_2| &= \left(\Lambda_2 \frac{\lambda_H(0) - \Lambda_2 r_0}{\Lambda_2 - \lambda_H(0) r_0} - 3\Lambda_1 r_0^2\right)|z_1 - z_2| \\ &\leq |F(z_1) - F(z_2)| \\ &\leq \left(\Lambda_H(z) + 3\Lambda_1 r_0^2\right)|z_1 - z_2| \\ &\leq \left(\Lambda_2 + 3\Lambda_1 r_0^2\right)|z_1 - z_2| . \end{aligned}$$

Hence f is bi-Lipschitz on $\overline{\mathbb{D}}_{r_0}$. □

By means of Theorems 3.2–3.5 and Lemmas 2.9 and 2.12, using the analogous proof of Theorem 4.1, we have the following four theorems.

Theorem 4.2 *Suppose $F(z)$ satisfies the hypothesis of Theorem 3.2. Then for each $r_1 \in (0, \rho_1)$, the biharmonic mapping $F(z)$ is bi-Lipschitz on $\overline{\mathbb{D}}_{r_1}$, i.e. for any $z_1, z_2 \in \overline{\mathbb{D}}_{r_1}$, there exists $l_1 = 1 - 3\Lambda r_1^2 > 0$ such that*

$$l_1 |z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq \left(\frac{4}{\pi(1 - r_1^2)} + 3\Lambda r_1^2\right)|z_1 - z_2| ,$$

where ρ_1 is given by (1.16).

Theorem 4.3 *Suppose $F(z)$ satisfies the hypothesis of Theorem 3.3. Then for each $r_2 \in (0, \rho_2)$, the biharmonic mapping $F(z)$ is bi-Lipschitz on $\overline{\mathbb{D}}_{r_2}$, i.e. for any $z_1, z_2 \in \overline{\mathbb{D}}_{r_2}$, there exists*

$$l_2 = \lambda_0(M) - \lambda_0(M)\sqrt{M^4 - 1}\sqrt{\frac{4r_2^2 - 3r_2^4 + r_2^6}{\left(1 - r_2^2\right)^3}} - 3\Lambda r_2^2 > 0$$

such that

$$l_2 |z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq \left(\frac{4M}{\pi(1 - r_2^2)} + 3\Lambda r_2^2\right)|z_1 - z_2| ,$$

where ρ_2 is given by (3.10) and $\lambda_0(M)$ is given by (1.8).

Theorem 4.4 *Suppose $F(z)$ satisfies the hypothesis of Theorem 3.4. Then for each $r_3 \in (0, \rho_3)$, the biharmonic mapping $F(z)$ is bi-Lipschitz on $\overline{\mathbb{D}}_{r_3}$, i.e. for any $z_1, z_2 \in \overline{\mathbb{D}}_{r_3}$, there exists*

$$l_3 = \frac{\Lambda(1 - \Lambda^2 r_3)}{\Lambda^2 - r_3} - \frac{4M}{\pi}\frac{3r_3^2 - 2r_3^4}{1 - r_3^2} > 0$$

such that

$$l_3 |z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq \left(\Lambda + \frac{12M}{\pi(1-r_3^2)} r_3^2 \right) |z_1 - z_2|,$$

where ρ_3 is given by (3.12).

Theorem 4.5 *Suppose $F(z)$ satisfies the hypothesis of Theorem 3.5. Then for each $r_4 \in (0, \rho_4)$, the biharmonic mapping $F(z)$ is bi-Lipschitz on $\overline{\mathbb{D}}_{r_4}$, i.e. for any $z_1, z_2 \in \overline{\mathbb{D}}_{r_4}$, there exists*

$$l_4 = \lambda_0(M_2) - \lambda_0(M_2)\sqrt{M_2^4 - 1}\sqrt{\frac{4r_4^2 - 3r_4^4 + r_4^6}{\left(1 - r_4^2\right)^3}} - \frac{4M_1}{\pi} \frac{3r_4^2 - 2r_4^4}{1 - r_4^2} > 0$$

such that

$$l_4 |z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq \left(\frac{4M_2}{\pi(1-r_4^2)} + \frac{12M_1}{\pi(1-r_4^2)} r_4^2 \right) |z_1 - z_2|,$$

where ρ_4 is given by (3.14) and $\lambda_0(M_2)$ is given by (1.8).

Acknowledgements This research of the second author is partly supported by Guangdong Natural Science Foundations (Grant No. 2018A030313508).

Data Availability Statement The authors declare that this research is purely theoretical and does not associate with any datas.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest, regarding the publication of this paper.

References

1. Abdulhadi, Z., Muhanna, Y.Abu, Khuri, S.: On univalent solutions of the Biharmonic equations. J. Inequal. Appl. **2005**(5), 469–478 (2005)
2. Abdulhadi, Z., Muhanna, Y.Abu: Landau's theorem for biharmonic mappings. J. Math. Anal. Appl. **338**, 705–709 (2008)
3. Aghalary, R., Mohammadian, A., Jahangiri, J.: Landau–Bloch theorems for bounded biharmonic mappings. Filomat **33**(14), 4593–4601 (2019)
4. Aleman, A., Constantin, A.: Harmonic maps and ideal fluid flows. Arch. Ration. Mech. Anal. **204**, 479–513 (2012)
5. Bai, X.X., Liu, M.S.: Landau-type theorems of polyharmonic mappings and log-p-harmonic mappings. Complex Anal. Oper. Theory **13**(2), 321–340 (2019)
6. Chen, H.H., Gauthier, P.M., Hengartner, W.: Bloch constants for planar harmonic mappings. Proc. Am. Math. Soc. **128**, 3231–3240 (2000)
7. Chen, Sh, Ponnusamy, S., Wang, X.: Landau's theorems for certain biharmonic mappings. Appl. Math. Comput. **208**(2), 427–433 (2009)

8. Chen, Sh, Ponnusamy, S., Wang, X.: Coecient estimates and Landau–Bloch's constant for planar harmonic mappings. Bull. Malays. Math. Sci. Soc. **34**(2), 255–265 (2011)
9. Chen, Sh, Ponnusamy, S., Wang, X.: Properties of some classes of planar harmonic and planar biharmonic mappings. Complex Anal. Oper. Theory **5**, 901–916 (2011)
10. Colonna, F.: The Bloch constant of bounded harmonic mappings. Indiana Univ. Math. J. **38**(4), 829–840 (1989)
11. Constantin, O., Martin, M.J.: A harmonic maps approach to fluid flows. Math. Ann. **369**, 1–16 (2017)
12. Graham, I., Kohr, G.: Geometric Function Theory in One and Higher Dimensions. Marcel Dekker Inc, New York (2003)
13. Knežević, M., Mateljević, M.: On the quasi-isometries of harmonic quasiconformal mappings. J. Math. Anal. Appl. **334**(1), 404–413 (2007)
14. Kalaj, D., Pavlović, M.: On quasiconformal self-mappings of the unit disk satisfying Poissons equation. Trans. Am. Math. Soc. **363**(8), 4043–4061 (2011)
15. Lewy, H.: On the non-vanishing of the Jacobian in certain one-to-one mappings. Bull. Am. Math. Soc. **42**, 689–692 (1936)
16. Liu, M.S.: Landau's theorems for biharmonic mappings. Complex Var. Elliptic Equ. **53**(9), 843–855 (2008)
17. Liu, M.S., Chen, H.H.: The Landau–Bloch type theorems for planar harmonic mappings with bounded dilation. J. Math. Anal. Appl. **468**(2), 1066–1081 (2018)
18. Liu, M.S., Liu, Z.W., Zhu, Y.C.: Landau's theorems for certain biharmonic mappings. Acta Math. Sin. Chin. Ser. **54**(1), 69–80 (2011). arXiv:1511.04642
19. Liu, M.S., Liu, Z.X.: Laudau-type theorems for p-harmonic mappings or log-p-harmonic mappings. Appl. Anal. **51**(1), 81–87 (2014)
20. Liu, M.S., Luo, L.F.: Landau-type theorems for certain bounded biharmonic mappings. Results Math., 74, Art. 170 (2019)
21. Liu, M.S., Luo, L.F., Luo, X.: Landau–Bloch type theorems for strongly bounded harmonic mappings. Monatshefte für Mathematik **191**(1), 175–185 (2020)
22. Liu, M.S., Liu, Z.X., Xu, J.F.: Landau-type theorems for certain biharmonic mappings. Abstr. Appl. Anal., Article ID 925947, 7 pages (2014). https://doi.org/10.1155/2014/925947
23. Liu, M.S., Xie, L., Yang, L.M.: Landau's theorems for biharmonic mappings(II). Math. Methods Appl. Sci. **40**, 2582–2595 (2017)
24. Luo, X., Liu, M.S.: Landau–Bloch type theorems for certain biharmonic mappings. Complex Var. Elliptic Equ. (2019). https://doi.org/10.1080/17476933.2019.1701665
25. Mateljević, M., Vuorinen, M.: On harmonic quasiconformal quasi-isometries. J. Inequal. Appl., Art. ID 178732 (2010)
26. Zhu, Y.C., Liu, M.S.: Landau-type theorems for certain planar harmonic mappings or biharmonic mappings. Complex Var. Elliptic Equ. **58**(12), 1667–1676 (2013)