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Inclusion modulo nonstationary

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Abstract

A classical theorem of Hechler asserts that the structure $(\omega^\omega, \leq^*)$ is universal in the sense that for any σ-directed poset $\mathbb{P}$ with no maximal element, there is a *ccc* forcing extension in which $(\omega^\omega, \leq^*)$ contains a cofinal order-isomorphic copy of $\mathbb{P}$. In this paper, we prove the following consistency result concerning the universality of the higher analogue $\left(\kappa^\kappa, \leq^S\right)$: assuming GCH, for every regular uncountable cardinal κ, there is a cofinality-preserving GCH-preserving forcing extension in which for every analytic quasi-order $\mathbb{Q}$ over κ^κ and every stationary subset S of κ, there is a Lipschitz map reducing $\mathbb{Q}$ to $(\kappa^\kappa, \leq^S)$.

Keywords Universal order · Nonstationary ideal · Diamond sharp · Local club condensation · Higher Baire space

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1 Introduction

Recall that a *quasi-order* is a binary relation which is reflexive and transitive. A well-studied quasi-order over the Baire space $\mathbb{N}^\mathbb{N}$ is the binary relation $\leq^*$ which is defined by letting, for any two elements $\eta : \mathbb{N} \to \mathbb{N}$ and $\xi : \mathbb{N} \to \mathbb{N}$,

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$$\eta \leq^* \xi \text{ iff } \{n \in \mathbb{N} \mid \eta(n) > \xi(n)\} \text{ is finite.}$$

This quasi-order is behind the definitions of cardinal invariants $\mathfrak{b}$ and $\mathfrak{d}$ (see [2, §2]), and serves as a key to the analysis of *oscillation of real numbers* which is known to have prolific applications to topology, graph theory, and forcing axioms (see [26]). By a classical theorem of Hechler [13], the structure $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ is universal in that sense that for any σ-directed poset $\mathbb{P}$ with no maximal element, there is a *ccc* forcing extension in which $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ contains a cofinal order-isomorphic copy of $\mathbb{P}$.

In this paper, we consider (a refinement of) the higher analogue of the relation $\leq^*$ to the realm of the generalized Baire space κ^κ (sometimes refered as the higher Baire space), where κ is a regular uncountable cardinal. This is done by simply replacing the ideal of finite sets with the ideal of nonstationary sets, as follows.[1]

Definition 1.1 Given a stationary subset $S \subseteq \kappa$, we define a quasi-order $\leq^S$ over κ^κ by letting, for any two elements $\eta : \kappa \to \kappa$ and $\xi : \kappa \to \kappa$,

$$\eta \leq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is nonstationary.}$$

Note that since the nonstationary ideal over S is σ-closed, the quasi-order $\leq^S$ is well-founded, meaning that we can assign a *rank* value $\|\eta\|$ to each element η of κ^κ. The utility of this approach is demonstrated in the celebrated work of Galvin and Hajnal [11] concerning the behavior of the power function over the singular cardinals, and, of course, plays an important role in Shelah's *pcf theory* (see [1, §4]). It was also demonstrated to be useful in the study of partition relations of singular cardinals of uncountable cofinality [24].

In this paper, we first address the question of how $\leq^S$ compares with $\leq^{S'}$ for various subsets S and S'. It is proved:

Theorem A *Suppose that κ is a regular uncountable cardinal and* GCH *holds. Then there exists a cofinality-preserving* GCH*-preserving forcing extension in which for all stationary subsets S, S' of κ, there exists a map $f : \kappa^{\leq\kappa} \to 2^{\leq\kappa}$ such that, for all $\eta, \xi \in \kappa^{\leq\kappa}$,*

- $\mathrm{dom}(f(\eta)) = \mathrm{dom}(\eta)$*;*
- *if $\eta \subseteq \xi$, then $f(\eta) \subseteq f(\xi)$;*
- *if* $\mathrm{dom}(\eta) = \mathrm{dom}(\xi) = \kappa$*, then $\eta \leq^S \xi$ iff $f(\eta) \leq^{S'} f(\xi)$.*

Note that as $\mathrm{Im}(f \restriction \kappa^\kappa) \subseteq 2^\kappa$, the above assertion is non-trivial even in the case $S = S' = \kappa$, and forms a contribution to the study of lossless encoding of substructures of $(\kappa^{\leq\kappa}, \ldots)$ as substructures of $(2^{\leq\kappa}, \ldots)$ (see, e.g., [3, Appendix]).

To formulate our next result—an optimal strengthening of Theorem A—let us recall a few basic notions from generalized descriptive set theory. *The generalized Baire space* is the set κ^κ endowed with the *bounded topology*, in which a basic open set takes the form $[\zeta] := \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$, with ζ, an element of $\kappa^{<\kappa}$. A subset $F \subseteq \kappa^\kappa$ is *closed* iff its complement is open iff there exists a tree $T \subseteq \kappa^{<\kappa}$ such that

[1] A comparison of the generalization considered here with the one obtained by replacing the ideal of finite sets with the ideal of bounded sets may be found in [4, §8].

$[T] := \{\eta \in \kappa^\kappa \mid \forall\alpha < \kappa(\eta \restriction \alpha \in T)\}$ is equal to F. A subset $A \subseteq \kappa^\kappa$ is *analytic* iff there is a closed subset F of the product space $\kappa^\kappa \times \kappa^\kappa$ such that its projection $\mathrm{pr}(F) := \{\eta \in \kappa^\kappa \mid \exists\xi \in \kappa^\kappa\ (\eta, \xi) \in F\}$ is equal to A. *The generalized Cantor space* is the subspace 2^κ of κ^κ endowed with the induced topology. The notions of open, closed and analytic subsets of 2^κ, $2^\kappa \times 2^\kappa$ and $\kappa^\kappa \times \kappa^\kappa$ are then defined in the obvious way.

Definition 1.2 The restriction of the quasi-order $\leq^S$ to 2^κ is denoted by $\subseteq^S$.

For all $\eta, \xi \in \kappa^\kappa$, denote $\Delta(\eta, \xi) := \min(\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cup \{\kappa\})$.

Definition 1.3 Let R_1 and R_2 be binary relations over $X_1, X_2 \in \{2^\kappa, \kappa^\kappa\}$, respectively. A function $f : X_1 \to X_2$ is said to be:

(a) a *reduction of* R_1 *to* R_2 iff, for all $\eta, \xi \in X_1$,

$$\eta\ R_1\ \xi \text{ iff } f(\eta)\ R_2\ f(\xi).$$

(b) *1-Lipschitz* iff for all $\eta, \xi \in X_1$,

$$\Delta(\eta, \xi) \leq \Delta(f(\eta), f(\xi)).$$

The existence of a function f satisfying (a) and (b) is denoted by $R_1 \hookrightarrow_1 R_2$.

In the above language, Theorem A provides a model in which, for all stationary subsets S, S' of κ, $\leq^S \hookrightarrow_1 \subseteq^{S'}$. As $\leq^S$ is an analytic quasi-order over κ^κ, it is natural to ask whether a stronger universality result is possible, namely, whether it is forceable that *any* analytic quasi-order over κ^κ admits a 1-Lipschitz reduction to $\subseteq^{S'}$ for some (or maybe even for all) stationary $S' \subseteq \kappa$. The answer turns out to be affirmative, hence the choice of the title of this paper.

Theorem B *Suppose that* κ *is a regular uncountable cardinal and* GCH *holds. Then there exists a cofinality-preserving* GCH*-preserving forcing extension in which, for every analytic quasi-order* Q *over* κ^κ *and every stationary* $S \subseteq \kappa$, $Q \hookrightarrow_1 \subseteq^S$.

Remark The universality statement under consideration is optimal, as $Q \hookrightarrow_1 \subseteq^S$ implies that Q is analytic.

The proof of the preceding goes through a new diamond-type principle for reflecting second-order formulas, introduced here and denoted by $\mathrm{Dl}^*_S(\Pi^1_2)$. This principle is a strengthening of Jensen's $\diamondsuit_S$ and a weakening of Devlin's $\diamondsuit^\sharp_S$. For κ a successor cardinal, we have $\mathrm{Dl}^*_S(\Pi^1_2) \Rightarrow \diamondsuit^*_S$ but not $\diamondsuit^*_S \Rightarrow \mathrm{Dl}^*_S(\Pi^1_2)$ (see Remark 4.3 below). Another crucial difference between the two is that, unlike $\diamondsuit^*_S$, the principle $\mathrm{Dl}^*_S(\Pi^1_2)$ is compatible with the set S being ineffable.

In Sect. 2, we establish the consistency of the new principle, in fact, proving that it follows from an abstract condensation principle that was introduced and studied in [9, 14]. It thus follows that it is possible to force $\mathrm{Dl}^*_S(\Pi^1_2)$ to hold over all stationary subsets S of a prescribed regular uncountable cardinal κ. It also follows that, in canonical models for Set Theory (including any $L[E]$ model with Jensen's λ-indexing which is sufficiently iterable and has no subcompact cardinals), $\mathrm{Dl}^*_S(\Pi^1_2)$ holds for every stationary subset S of every regular uncountable (including ineffable) cardinal κ.

Then, in Sect. 3, the core combinatorial component of our result is proved:

Theorem C *Suppose S is a stationary subset of a regular uncountable cardinal κ. If $\mathrm{Dl}^*_S(\Pi^1_2)$ holds, then, for every analytic quasi-order Q over κ^κ, $Q \hookrightarrow_1 \subseteq^S$.*

2 A Diamond reflecting second-order formulas

Devlin [5] introduced a strong form of the Jensen-Kunen principle $\diamondsuit^+_\kappa$, which he denoted by $\diamondsuit^\sharp_\kappa$, and proved:

Fact 2.1 *(Devlin* [5, Theorem 5]*) In L, for every regular uncountable cardinal κ that is not ineffable, $\diamondsuit^\sharp_\kappa$ holds.*

Remark 2.2 A subset S of a regular uncountable cardinal κ is said to be *ineffable* iff, for every sequence $\langle Z_\alpha \mid \alpha \in S\rangle$, there exists a subset $Z \subseteq \kappa$, for which $\{\alpha \in S \mid Z \cap \alpha = Z_\alpha \cap \alpha\}$ is stationary. Note that the collection of non-ineffable subsets of κ forms a normal ideal that contains $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) < \alpha\}$ as an element. Also note that if κ is ineffable, then κ is strongly inaccessible. Finally, we mention that by a theorem of Jensen and Kunen, for any ineffable set S, $\diamondsuit_S$ holds and $\diamondsuit^*_S$ fails.

As said before, in this paper, we consider a variation of Devlin's principle compatible with κ being ineffable. Devlin's principle as well as its variation provide us with Π^1_2-reflection over structures of the form $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle$. We now describe the relevant logic in detail.

A Π^1_2-sentence ϕ is a formula of the form $\forall X \exists Y \varphi$ where φ is a first-order sentence over a relational language $\mathcal{L}$ as follows:

- $\mathcal{L}$ has a predicate symbol ϵ of arity 2;
- $\mathcal{L}$ has a predicate symbol $\mathbb{X}$ of arity $m(\mathbb{X})$;
- $\mathcal{L}$ has a predicate symbol $\mathbb{Y}$ of arity $m(\mathbb{Y})$;
- $\mathcal{L}$ has infinitely many predicate symbols $(\mathbb{A}_n)_{n\in\omega}$, each $\mathbb{A}_n$ is of arity $m(\mathbb{A}_n)$.

Definition 2.3 For sets N and x, we say that N *sees* x iff N is transitive, p.r.-closed, and $x \cup \{x\} \subseteq N$.

Suppose that a set N sees an ordinal α, and that $\phi = \forall X \exists Y \varphi$ is a Π^1_2-sentence, where φ is a first-order sentence in the above-mentioned language $\mathcal{L}$. For every sequence $(A_n)_{n\in\omega}$ such that, for all $n \in \omega$, $A_n \subseteq \alpha^{m(\mathbb{A}_n)}$, we write

$$\langle \alpha, \in, (A_n)_{n\in\omega}\rangle \models_N \phi$$

to express that the two hold:

(1) $(A_n)_{n\in\omega} \in N$;
(2) $\langle N, \in\rangle \models (\forall X \subseteq \alpha^{m(\mathbb{X})})(\exists Y \subseteq \alpha^{m(\mathbb{Y})})[\langle \alpha, \in, X, Y, (A_n)_{n\in\omega}\rangle \models \varphi]$, where:

- $\in$ is the interpretation of ϵ;
- X is the interpretation of $\mathbb{X}$;
- Y is the interpretation of $\mathbb{Y}$, and
- for all $n \in \omega$, A_n is the interpretation of $\mathbb{A}_n$.

Convention 2.4 We write α^+ for $|\alpha|^+$, and write $\langle \alpha, \in, (A_n)_{n\in\omega}\rangle \models \phi$ for

$$\langle \alpha, \in, (A_n)_{n\in\omega}\rangle \models_{H_{\alpha^+}} \phi.$$

Definition 2.5 (*Devlin* [5]) Let κ be a regular and uncountable cardinal.

$\diamondsuit^{\sharp}_{\kappa}$ asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha < \kappa\rangle$ satisfying the following:

(1) for every infinite $\alpha < \kappa$, N_α is a set of cardinality $|\alpha|$ that sees α;
(2) for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C$, $C\cap\alpha$, $X\cap\alpha \in N_\alpha$;
(3) whenever $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, with ϕ a Π^1_2-sentence, there are stationarily many $\alpha < \kappa$ such that $\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi$.

Consider the following variation:

Definition 2.6 Let κ be a regular and uncountable cardinal, and $S \subseteq \kappa$ stationary.

$\mathrm{Dl}^*_S(\Pi^1_2)$ asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ satisfying the following:

(1) for every $\alpha \in S$, N_α is a set of cardinality $< \kappa$ that sees α;
(2) for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$;
(3) whenever $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, with ϕ a Π^1_2-sentence, there are stationarily many $\alpha \in S$ such that $|N_\alpha| = |\alpha|$ and $\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi$.

Remark 2.7 The choice of notation for the above principle is motivated by [23, Definition 2.10] and [25, Definition 45].

The goal of this section is to derive $\mathrm{Dl}^*_S(\Pi^1_2)$ from an abstract principle which is both forceable and a consequence of $V = L[E]$, for $L[E]$ an iterable extender model with Jensen λ-indexing without a subcompact cardinal (see [20,21]). Note that this covers all $L[E]$ models that can be built so far.

Convention 2.8 The class of ordinals is denoted by OR. The class of ordinals of cofinality μ is denoted by $\mathrm{cof}(\mu)$, and the class of ordinals of cofinality greater than μ is denoted by $\mathrm{cof}(>\mu)$. For a set of ordinals a, we write $\mathrm{acc}(a) := \{\alpha \in a \mid \sup(a \cap \alpha) = \alpha > 0\}$. ZF^- denotes ZF without the power-set axiom. The transitive closure of a set X is denoted by $\mathrm{trcl}(X)$, and the Mostowski collapse of a structure $\mathfrak{B}$ is denoted by $\mathrm{clps}(\mathfrak{B})$.

Definition 2.9 Suppose N is a transitive set. For a limit ordinal λ, we say that $\vec{M} = \langle M_\beta \mid \beta < \lambda\rangle$ is a *nice filtration* of N iff all of the following hold:

(1) $\bigcup_{\beta<\lambda} M_\beta = N$;

(2) $\vec{M}$ is $\in$-increasing, that is, $\alpha < \beta < \lambda \implies M_\alpha \in M_\beta$;
(3) $\vec{M}$ is continuous, that is, for every $\beta \in \mathrm{acc}(\lambda)$, $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$;
(4) for all $\beta < \lambda$, M_β is a transitive set with $M_\beta \cap \mathrm{OR} = \beta$ and $|M_\beta| \leq |\beta| + \aleph_0$.

Convention 2.10 Whenever λ is a limit ordinal, and $\vec{M} = \langle M_\beta \mid \beta < \lambda\rangle$ is a $\subseteq$-increasing, continuous sequence of sets, we denote its limit $\bigcup_{\beta<\lambda} M_\beta$ by M_λ.

Definition 2.11 (*Holy et al.* [14]) Let $\eta < \zeta$ be ordinals. We say that *local club condensation holds in* (η, ζ), and denote this by $\mathsf{LCC}(\eta, \zeta)$, iff there exist a limit ordinal $\lambda \geq \zeta$ and a sequence $\vec{M} = \langle M_\beta \mid \beta < \lambda\rangle$ such that all of the following hold:

(1) $\vec{M}$ is *nice filtration* of M_λ;
(2) $\langle M_\lambda, \in\rangle \models \mathsf{ZF}^-$;
(3) For every ordinal α in the open interval (η, ζ) and every sequence $\vec{\mathcal{F}} = \langle (F_n, k_n) \mid n \in \omega\rangle$ in M_λ such that, for all $n \in \omega, k_n \in \omega$ and $F_n \subseteq (M_\alpha)^{k_n}$, there is a sequence $\vec{\mathfrak{B}} = \langle \mathfrak{B}_\beta \mid \beta < |\alpha|\rangle$ in M_λ having the following properties:

   (a) for all $\beta < |\alpha|$, $\mathfrak{B}_\beta$ is of the form

$$\langle B_\beta, \in, \vec{M} \restriction (B_\beta \cap \mathrm{OR}), (F_n \cap (B_\beta)^{k_n})_{n\in\omega}\rangle;$$

   (b) for all $\beta < |\alpha|$, $\mathfrak{B}_\beta \prec \langle M_\alpha, \in, \vec{M} \restriction \alpha, (F_n)_{n\in\omega}\rangle$;
   (c) for all $\beta < |\alpha|$, $\beta \subseteq B_\beta$ and $|B_\beta| < |\alpha|$;
   (d) for all $\beta < |\alpha|$, there exists $\bar{\beta} < \lambda$ such that

$$\mathrm{clps}(\langle B_\beta, \in, \langle B_\delta \mid \delta \in B_\beta \cap \mathrm{OR}\rangle\rangle) = \langle M_{\bar{\beta}}, \in, \vec{M} \restriction \bar{\beta}\rangle;$$

   (e) $\langle B_\beta \mid \beta < |\alpha|\rangle$ is $\subseteq$-increasing, continuous and converging to M_α.

For $\vec{\mathfrak{B}}$ as in Clause (3) above we say that $\vec{\mathfrak{B}}$ *witnesses* LCC *at* α *with respect to* $\vec{M}$ *and* $\vec{\mathcal{F}}$.

Remark 2.12 There are first-order sentences $\psi_0(\dot{\eta}, \dot{\zeta})$ and $\psi_1(\dot{\eta})$ in the language $\mathcal{L}^* := \{\in, \vec{M}, \dot{\eta}, \dot{\zeta}\}$ of set theory augmented by a predicate for a nice filtration and two ordinals such that, for all $\eta < \zeta \leq \lambda$ and $\vec{M} = \langle M_\beta \mid \beta < \lambda\rangle$:

- $(\langle M_\lambda, \in, \vec{M}\rangle \models \psi_0(\eta, \zeta)) \iff (\vec{M}$ witnesses that $\mathsf{LCC}(\eta, \zeta)$ holds), and
- $(\langle M_\lambda, \in, \vec{M}\rangle \models \psi_1(\eta)) \iff (\vec{M}$ witnesses that $\mathsf{LCC}(\eta, \lambda)$ holds).

Therefore, we will later make an abuse of notation and write $\langle N, \in, \vec{M}\rangle \models \mathsf{LCC}(\eta, \zeta)$ to mean that $\vec{M}$ is a nice filtration of N witnessing that $\mathsf{LCC}(\eta, \zeta)$ holds.

Fact 2.13 *(Friedman–Holy, implicit in* [9]*) Assume* GCH*. For every inaccessible cardinal* κ*, there is a set-size cofinality-preserving notion of forcing* $\mathbb{P}$ *such that, in* $V^{\mathbb{P}}$*, the three hold:*

(1) GCH*;*
(2) *there is a nice filtration* $\vec{M} = \langle M_\beta \mid \beta < \kappa^+\rangle$ *of* H_{κ^+} *witnessing that* $\mathsf{LCC}(\omega_1, \kappa^+)$ *holds;*

(3) *there is a Δ_1-formula Θ and a parameter $a \subseteq \kappa$ such that the relation $<_\Theta$ defined by ($x <_\Theta y$ iff $H_{\kappa^+} \models \Theta(x, y, a)$) is a global well-ordering of H_{κ^+}.*

Fact 2.14 *(Holy et al.* [14, p. 1362 and §4]*) Assume* GCH. *For every regular cardinal κ, there is a set-size notion of forcing $\mathbb{P}$ which is $(<\kappa)$-directed-closed and has the κ^+-cc such that, in $V^{\mathbb{P}}$, the three hold:*

(1) GCH*;*
(2) *there is a nice filtration $\vec{M} = \langle M_\beta \mid \beta < \kappa^+\rangle$ of H_{κ^+} witnessing that* $\mathsf{LCC}(\kappa, \kappa^+)$ *holds;*
(3) *there is a Δ_1-formula Θ and a parameter $a \subseteq \kappa$ such that the relation $<_\Theta$ defined by ($x <_\Theta y$ iff $H_{\kappa^+} \models \Theta(x, y, a)$) is a global well-ordering of H_{κ^+}.*

The following is a improvement of [9, Theorem 8].

Fact 2.15 *(Fernandes* [7]*) Let $L[E]$ be an extender model with Jensen λ-indexing. Suppose that, for every $\alpha \in$* OR*, the premouse $L[E]||\alpha$ is weakly iterable.*[2] *Then, for every infinite cardinal κ, the following are equivalent:*

(a) *$\langle L_\beta[E] \mid \beta < \kappa^+\rangle$ witneses that* $\mathsf{LCC}(\kappa^+, \kappa^{++})$ *holds;*
(b) *$L[E] \models$ "κ is not a subcompact cardinal".*

In addition, for every infinite limit cardinal κ, $\langle L_\beta[E] \mid \beta < \kappa^+\rangle$ witnesses that $\mathsf{LCC}(\kappa, \kappa^+)$ *holds.*

Lemma 2.16 *Suppose that λ is a limit ordinal and that $\vec{M} = \langle M_\beta \mid \beta < \lambda\rangle$ is a nice filtration of H_λ. Then, for every infinite cardinal $\theta \leq \lambda$, $M_\theta \subseteq H_\theta$.*

Proof Let $\theta \leq \lambda$ be an infinite cardinal. By Clause (4) of Definition 2.9, for all $\beta < \theta$, the set M_β is transitive, $M_\beta \cap \mathrm{OR} = \beta$, and $|M_\beta| = |\beta| < \theta$. It thus follows that $M_\theta = \bigcup_{\beta<\theta} M_\beta \subseteq H_\theta$. □

Motivated by the property of acceptability that holds in extender models, we define the following property for nice filtrations:

Definition 2.17 Given a nice filtration $\vec{M} = \langle M_\beta \mid \beta < \kappa^+\rangle$ of H_{κ^+}, we say that $\vec{M}$ is *eventually slow at κ* iff there exists an infinite cardinal $\mu < \kappa$ such that, for every cardinal θ with $\mu < \theta \leq \kappa$, $M_\theta = H_\theta$.

Lemma 2.18 *Suppose that $\vec{M} = \langle M_\beta \mid \beta < \kappa^+\rangle$ is a nice filtration of H_{κ^+} that is eventually slow at κ. Then, for a tail of $\alpha < \kappa$, for every sequence $\vec{\mathcal{F}} = \langle (F_n, k_n) \mid n \in \omega\rangle$ such that, for all $n \in \omega$, $k_n \in \omega$ and $F_n \subseteq (M_{\alpha^+})^{k_n}$, there is $\mathfrak{B}$ that witnesses* LCC *at α^+ with respect to $\vec{M}$ and $\vec{\mathcal{F}}$.*

Proof Fix an infinite cardinal $\mu < \kappa$ such that, for every cardinal θ with $\mu < \theta \leq \kappa$, $M_\theta = H_\theta$. Let $\alpha \in (\mu, \kappa)$ be arbitrary. Now, given a sequence $\vec{\mathcal{F}}$ as in the statement of the lemma, build by recursion a $\subseteq$-increasing and continuous sequence $\langle \mathfrak{A}_\gamma \mid \gamma < \alpha^+\rangle$ of elementary submodels of $\langle M_{\alpha^+}, \in, \vec{M} \restriction \alpha^+, (F_n)_{n\in\omega}\rangle$, such that:

[2] Here, $L[E]||\alpha$ stands for $\langle J^E_\alpha, \in, E \restriction \omega\alpha, E_{\omega\alpha}\rangle$, following the notation from [27]. For the definition of *weakly iterable*, see [27, p. 311].

- for each $\gamma < \alpha^+$, $|A_\gamma| < \alpha^+$, and
- $\bigcup_{\gamma<\alpha^+} A_\gamma = H_{\alpha^+}$.

By a standard argument, $C := \{\gamma < \alpha^+ \mid A_\gamma = M_\gamma\}$ is a club in α^+. Let $\{\gamma_\beta \mid \beta < \alpha^+\}$ denote the increasing enumeration of C. Denote $\mathfrak{B}_\beta := \mathfrak{A}_{\gamma_\beta}$. Then $\vec{\mathfrak{B}} = \langle \mathfrak{B}_\beta \mid \beta < \alpha^+ \rangle$ is an $\in$-increasing and continuous sequence of elementary submodels of $\langle M_{\alpha^+}, \in, \vec{M} \restriction \alpha^+, (F_n)_{n\in\omega} \rangle$, such that, for all $\beta < \alpha^+$, $\operatorname{clps}(\mathfrak{B}_\beta) = \langle M_{\gamma_\beta}, \in, \ldots \rangle$. □

In the next two lemmas we find sufficient conditions for nice filtrations $\langle M_\beta \mid \beta < \kappa^+ \rangle$ to be eventually slow at κ.

Lemma 2.19 *Suppose that κ is a successor cardinal and that $\vec{M} = \langle M_\beta \mid \beta < \kappa^+ \rangle$ is a nice filtration of H_{κ^+} witnessing that* $\mathsf{LCC}(\kappa, \kappa^+)$ *holds. Then $\vec{M}$ is eventually slow at κ.*

Proof As κ is a successor cardinal, $\vec{M}$ is eventually slow at κ iff $M_\kappa = H_\kappa$. Thus, by Lemma 2.16, it suffices to verify that $H_\kappa \subseteq M_\kappa$. To this end, let $x \in H_\kappa$, and we will find $\beta < \kappa$ such that $x \in M_\beta$.

Set $\theta := |\operatorname{trcl}\{x\}|$ and fix a witnessing bijection $f : \theta \leftrightarrow \operatorname{trcl}\{x\}$. As $H_{\kappa^+} = M_{\kappa^+} = \bigcup_{\alpha<\kappa^+} M_\alpha$, we may fix $\alpha < \kappa^+$ such that $\{f, \theta, \operatorname{trcl}\{x\}\} \subseteq M_\alpha$. Let $\vec{\mathfrak{B}}$ witness $\mathsf{LCC}(\kappa, \kappa^+)$ at α with respect to $\vec{M}$ and $\vec{\mathcal{F}} := \langle (f, 2) \rangle$. Let $\beta < \kappa^+$ be such that $\operatorname{clps}(\mathfrak{B}_{\theta+1}) = \langle M_\beta, \in, \ldots \rangle$.

Claim 2.19.1 $\theta < \beta < \kappa$.

Proof By Definition 2.11(3)(c), $\theta + 1 \subseteq B_{\theta+1}$, so that, $\theta < \beta$. By Clause (4) of Definition 2.9 and by Definition 2.11(3)(c), $|\beta| = |M_\beta| = |B_{\theta+1}| < |\alpha| \leq \kappa$. □

Now, as

$$\mathfrak{B}_{\theta+1} \prec \langle H_{\kappa^+}, \in, \vec{M}, F_0 \rangle \models \exists y (\forall \alpha \forall \delta (F_0(\alpha, \delta) \leftrightarrow (\alpha, \delta) \in y)),$$

we have $f \in B_{\theta+1}$. Since $\operatorname{dom}(f) \subseteq B_{\theta+1}$, $\operatorname{Im}(f) \subseteq B_{\theta+1}$. But $\operatorname{Im}(f) = \operatorname{trcl}(\{x\})$ is a transitive set, so that the Mostowski collapsing map $\pi : B_{\theta+1} \to M_\beta$ is the identity over $\operatorname{trcl}(\{x\})$, meaning that $x \in \operatorname{trcl}(\{x\}) \subseteq M_\beta$. □

Lemma 2.20 *Suppose that κ is an inaccessible cardinal, $\mu < \kappa$ and $\vec{M} = \langle M_\beta \mid \beta < \kappa^+ \rangle$ witnesses that* $\mathsf{LCC}(\mu, \kappa^+)$ *holds. Then μ witnesses that $\vec{M}$ is eventually slow at κ.*

Proof Suppose not. It follows from Lemma 2.16 that we may fix an infinite cardinal θ with $\mu \leq \theta < \kappa$ along with $x \in H_{\theta^+} \setminus M_{\theta^+}$. Fix a surjection $f : \theta \to \operatorname{trcl}(\{x\})$. Let $\alpha < \kappa^+$ be the least ordinal such that $x \in M_\alpha$, so that $\mu < \theta^+ < \alpha < \kappa^+$. Let $\vec{\mathfrak{B}}$ witness $\mathsf{LCC}(\mu, \kappa^+)$ at α with respect to $\vec{M}$ and $\vec{\mathcal{F}} := \langle (f, 2) \rangle$. Let $\beta < \kappa^+$ be such that $\operatorname{clps}(\mathfrak{B}_{\theta+1}) = \langle M_\beta, \in, \ldots \rangle$.

Claim 2.20.1 $\beta < \alpha$.

Proof By Clause (4) of Definition 2.9 and by Definition 2.11(3)(c), $|\beta| = |M_\beta| = |B_{\theta+1}| < |\alpha|$. and hence $\beta < \alpha$. □

By the same argument used in the proof of Lemma 2.19, $x \in M_\beta$, contradicting the minimality of α. □

Question 2.21 Notice that if κ is an inaccessible cardinal and $\vec{M} = \langle M_\beta \mid \beta < \kappa^+\rangle$ is such that $\langle H_{\kappa^+}, \in, \vec{M}\rangle \models \mathsf{LCC}(\kappa, \kappa^+)$, then, for club many $\beta < \kappa$, $M_\beta = H_\beta$. We ask: is it consistent that κ is an inaccessible cardinal, $\vec{M} = \langle M_\beta \mid \beta < \kappa^+\rangle$ is such that $\langle H_{\kappa^+}, \in, \vec{M}\rangle \models \mathsf{LCC}(\kappa, \kappa^+)$, yet, for stationarily many $\beta < \kappa$, $M_{\beta^+} \subsetneq H_{\beta^+}$?

Lemma 2.22 *Suppose that $\vec{M} = \langle M_\beta \mid \beta < \kappa^+\rangle$ is a nice filtration of H_{κ^+}. Given a sequence $\vec{\mathcal{F}} = \langle (F_n, k_n) \mid n \in \omega\rangle$ such that, for all $n \in \omega$, $k_n \in \omega$ and $F_n \subseteq (H_{\kappa^+})^{k_n}$, there are club many $\delta < \kappa^+$ such that $\langle M_\delta, \in, \vec{M} \restriction \delta, (F_n \cap (M_\delta)^{k_n})_{n\in\omega}\rangle \prec \langle M_{\kappa^+}, \in, \vec{M}, (F_n)_{n\in\omega}\rangle$.*

Proof Build by recursion an $\in$-increasing continuous sequence $\vec{\mathfrak{B}} = \langle \mathfrak{B}_\beta \mid \beta < \kappa^+\rangle$ of elementary submodels of $\langle M_{\kappa^+}, \in, \vec{M}, (F_n)_{n\in\omega}\rangle$, such that:

- for each $\beta < \kappa^+$, $|B_\beta| < \kappa^+$, and
- $\bigcup_{\beta<\kappa^+} B_\beta = H_{\kappa^+}$.

By a standard back-and-forth argument, utilizing the continuity of $\vec{\mathfrak{B}}$ and $\vec{M}$, $\{\delta < \kappa^+ \mid B_\delta = M_\delta\}$ is a club in κ^+. □

Definition 2.23 Suppose $\vec{M} = \langle M_\beta \mid \beta < \lambda\rangle$ is a nice filtration of M_λ for some limit ordinal $\lambda > 0$. Given $\alpha < \lambda$ and $\vec{\mathcal{F}} = \langle (F_n, k_n) \mid n \in \omega\rangle$ in M_λ such that, for each $n \in \omega$, $k_n \in \omega$ and $F_n \subseteq (M_\alpha)^{k_n}$, for every sequence $\vec{\mathfrak{B}} = \langle \mathfrak{B}_\beta \mid \beta < |\alpha|\rangle$ in M_λ and every letter $l \in \{a, b, c, d, e\}$, we let $\psi_l(\vec{\mathfrak{B}}, \vec{\mathcal{F}}, \alpha, \vec{M} \restriction (\alpha + 1))$ be some formula expressing that Clause (3)(l) of Definition 2.11 holds.

The following forms the main result of this section.

Theorem 2.24 *Suppose that κ is a regular uncountable cardinal, and $\vec{M} = \langle M_\beta \mid \beta < \kappa^+\rangle$ is a nice filtration of H_{κ^+} that is eventually slow at κ, and witnesses that $\mathsf{LCC}(\kappa, \kappa^+)$ holds. Suppose further that there is a subset $a \subseteq \kappa$ and a formula $\Theta \in \Sigma_\omega$ which defines a well-order $<_\Theta$ in H_{κ^+} via $x <_\Theta y$ iff $H_{\kappa^+} \models \Theta(x, y, a)$. Then, for every stationary $S \subseteq \kappa$, $\mathrm{Dl}^*_S(\Pi^1_2)$ holds.*

Proof Let $S' \subseteq \kappa$ be stationary. We shall prove that $\mathrm{Dl}^*_{S'}(\Pi^1_2)$ holds by adjusting Devlin's proof of Fact 2.1.

As a first step, we identify a subset S of S' of interest.

Claim 2.24.1 *There exists a stationary non-ineffable subset $S \subseteq S' \setminus \omega$ such that, for every $\alpha \in S' \setminus S$, $|H_{\alpha^+}| < \kappa$.*

Proof If S' is non-ineffable, then let $S := S' \setminus \omega$, so that $H_{\alpha^+} = H_\omega$ for all $\alpha \in S' \setminus S$. From now on, suppose that S' is ineffable. In particular, κ is strongly inaccessible and $|H_{\alpha^+}| < \kappa$ for every $\alpha < \kappa$. Let $S := S' \setminus (\omega \cup T)$, where

$$T := \{\alpha \in \kappa \cap \mathrm{cof}(>\omega) \mid S' \cap \alpha \text{ is stationary in } \alpha\}.$$

To see that S is stationary, let E be an arbitrary club in κ.

- If $S' \cap \mathrm{cof}(\omega)$ is stationary, then since $S' \cap \mathrm{cof}(\omega) \subseteq S$, we infer that $S \cap E \neq \emptyset$.
- If $S' \cap \mathrm{cof}(\omega)$ is non-stationary, then fix a club $C \subseteq E$ disjoint from $S' \cap \mathrm{cof}(\omega)$, and let $\alpha := \min(\mathrm{acc}(C) \cap S')$. Then $\mathrm{cf}(\alpha) > \omega$ and $C \cap \alpha$ is a club in α disjoint from S', so that $\alpha \notin T$. Altogether, $\alpha \in S \cap E$.

To see that S is non-ineffable, we define a sequence $\langle Z_\alpha \mid \alpha \in S\rangle$, as follows. For every $\alpha \in S$, fix a closed and cofinal subset Z_α of α with $\mathrm{otp}(Z_\alpha) = \mathrm{cf}(\alpha)$ such that, if $\mathrm{cf}(\alpha) > \omega$, then the club Z_α is disjoint from $S' \cap \alpha$. Towards a contradiction, suppose that $Z \subseteq \kappa$ is a set for which $\{\alpha \in S \mid Z \cap \alpha = Z_\alpha\}$ is stationary. Clearly, Z is closed and cofinal in κ, so that $Z \cap S'$ is stationary, $\mathrm{otp}(Z \cap S') = \kappa$ and hence $D := \{\alpha < \kappa \mid \mathrm{otp}(Z \cap S' \cap \alpha) = \alpha > \omega\}$ is a club. Pick $\alpha \in D \cap S$ such that $Z \cap \alpha = Z_\alpha$. As

$$\mathrm{cf}(\alpha) = \mathrm{otp}(Z_\alpha) = \mathrm{otp}(Z \cap \alpha) \geq \mathrm{otp}(Z \cap S' \cap \alpha) = \alpha > \omega,$$

it must be the case that Z_α is a club disjoint from $S' \cap \alpha$, while $Z_\alpha = Z \cap \alpha$ and $Z \cap S' \cap \alpha \neq \emptyset$. This is a contradiction. □

Let S be given by the preceding claim. We shall focus on constructing a sequence $\langle N_\alpha \mid \alpha \in S\rangle$ witnessing $\mathrm{Dl}^*_S(\Pi^1_2)$ such that, in addition, $|N_\alpha| = |\alpha|$ for every $\alpha \in S$. It will then immediately follow that the sequence $\langle N'_\alpha \mid \alpha \in S'\rangle$ defined by letting $N'_\alpha := N_\alpha$ for $\alpha \in S$, and $N'_\alpha := H_{\alpha^+}$ for $\alpha \in S' \setminus S$ will witness the validity of $\mathrm{Dl}^*_{S'}(\Pi^1_2)$. As $\vec{M}$ is eventually slow at κ, we may also assume that, for every $\alpha \in S$, $M_{\alpha^+} = H_{\alpha^+}$ and the conclusion of Lemma 2.18 holds true.[3] If κ is a successor cardinal, we may moreover assume that, for every $\alpha \in S$, $M_{\alpha^+} = H_\kappa$.

Here we go. As S is non-ineffable, fix a sequence $\vec{Z} = \langle Z_\alpha \mid \alpha \in S\rangle$ with $Z_\alpha \subseteq \alpha$ for all $\alpha \in S$, such that, for every $Z \subseteq \kappa$, $\{\alpha \in S \mid Z \cap \alpha = Z_\alpha\}$ is nonstationary. In the course of the rest of the proof, we shall occasionally take witnesses to LCC at some ordinal α with respect to $\vec{M}$ and a finite sequence $\vec{\mathcal{F}} = \langle (F_n, k_n) \mid n \in 4\rangle$; for this, we introduce the following piece of notation for any positive $m < \omega$, $X \subseteq (\kappa^+)^m$ and $\alpha < \kappa^+$:

$$\vec{\mathcal{F}}_{X,\alpha} := \langle (X \cap \alpha^m, m), (a \cap \alpha, 1), (S \cap \alpha, 1), (\vec{Z} \restriction \alpha, 2)\rangle.$$

Next, for each $\alpha \in S$, we define S_α to be the set of all $\beta \in \alpha^+$ satisfying the following list of conditions:

(i) $\langle M_\beta, \in, \vec{M} \restriction \beta\rangle \models \mathsf{LCC}(\alpha, \beta)$,[4]
(ii) $\langle M_\beta, \in\rangle \models \mathsf{ZF}^-$ & α is the largest cardinal,[5]
(iii) $\langle M_\beta, \in\rangle \models \alpha$ is regular & $S \cap \alpha$ is stationary,
(iv) $\langle M_\beta, \in\rangle \models \Theta(x, y, a \cap \alpha)$ defines a global well-order,
(v) $\vec{Z} \restriction (\alpha + 1) \notin M_\beta$.

[3] For all the small $\alpha \in S' \setminus S$ such that $M_{\alpha^+} \neq H_{\alpha^+}$, simply let $N'_\alpha := N_{\min(S)}$.

[4] Note that β is not needed to define $\mathsf{LCC}(\alpha, \beta)$ in the structure $\langle M_\beta, \in, \vec{M} \restriction \beta\rangle$. Indeed, by $\mathsf{LCC}(\alpha, \beta)$ we mean $\psi_1(\alpha)$ as in Remark 2.12.

[5] In particular, $\langle M_\beta, \in\rangle \models \alpha$ is uncountable.

Then, we consider the set

$$D := \{\alpha \in S \mid S_\alpha \neq \emptyset \ \&\ S_\alpha \text{ has no largest element}\}.$$

Define a function $f : S \to \kappa$ as follow. For every $\alpha \in D$, let $f(\alpha) := \sup(S_\alpha)$; for every $\alpha \in S \setminus D$, let $f(\alpha)$ be the least $\beta < \kappa$ such that M_β sees α, and $\vec{Z} \restriction (\alpha+1) \in M_\beta$.

Claim 2.24.2 *f is well-defined. Furthermore, for all $\alpha \in S$, $\alpha < f(\alpha) < \alpha^+$.*

Proof Let $\alpha \in S$ be arbitrary. The analysis splits into two cases:

► Suppose $\alpha \in D$. As $\alpha \in S$, we have $\bigcup_{\beta<\alpha^+} M_\beta = M_{\alpha^+} = H_{\alpha^+}$, and hence we may find some $\beta < \alpha^+$ such that $\vec{Z} \restriction (\alpha + 1) \in M_\beta$. Then, condition (v) in the definition of S_α implies that $\alpha < f(\alpha) \leq \beta < \alpha^+$.

► Suppose $\alpha \notin D$. As $\alpha \in S$, let us fix $\langle \mathfrak{B}_\beta \mid \beta < \alpha^+\rangle$ that witnesses LCC at α^+ with respect to $\vec{M}$ and $\vec{\mathcal{F}}_{\emptyset,\alpha^+}$. Set $\beta := \alpha + 2$ and fix $\bar{\beta} < \kappa^+$ such that $\mathrm{clps}(\mathfrak{B}_\beta) = \langle M_{\bar{\beta}}, \ldots\rangle$. As $\beta \subseteq B_\beta$ and $|B_\beta| < \alpha^+$, by Clause (4) of Definition 2.9, $\beta \leq \bar{\beta} < \alpha^+$. In addition, $\vec{Z} \restriction (\alpha+1) \in M_{\bar{\beta}}$ and there exists an elementary embedding from $\langle M_{\bar{\beta}}, \in\rangle$ to $\langle H_{\alpha^+}, \in\rangle$, so that $M_{\bar{\beta}}$ sees α. Altogether, $\alpha < f(\alpha) \leq \bar{\beta} < \alpha^+$. □

Define $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ by letting $N_\alpha := M_{f(\alpha)}$ for all $\alpha \in S$. It follows from Definition 2.9(4) and the preceding claim that $|N_\alpha| = |\alpha|$ for all $\alpha \in S$.

Claim 2.24.3 *Let $X \subseteq \kappa$. Then there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$.*

Proof By Lemma 2.22, we now fix $\delta < \kappa^+$ such that $\kappa, S, a \in M_\delta$ and $\langle M_\delta, \in, \vec{M} \restriction \delta\rangle \prec \langle M_{\kappa^+}, \in, \vec{M}\rangle$. Note that $|\delta| = \kappa$. Let $\vec{\mathfrak{B}} = \langle \mathfrak{B}_\alpha \mid \alpha < \kappa\rangle$ witness LCC at δ with respect to $\vec{M}$ and $\vec{\mathcal{F}}_{X,\kappa}$.

Subclaim 2.24.3.1 $C := \{\alpha < \kappa \mid B_\alpha \cap \kappa = \alpha\}$ is a club in κ.

Proof To see that C is closed in κ, fix an arbitrary $\alpha < \kappa$ with $\sup(C \cap \alpha) = \alpha > 0$. As $\langle B_\beta \mid \beta < \kappa\rangle$ is $\subseteq$-increasing and continuous, we have

$$\alpha = \bigcup_{\beta \in (C\cap\alpha)} \beta = \bigcup_{\beta \in (C\cap\alpha)} (B_\beta \cap \kappa) = \bigcup_{\beta<\alpha} (B_\beta \cap \kappa) = B_\alpha \cap \kappa.$$

To see that C is unbounded in κ, fix an arbitrary $\varepsilon < \kappa$, and we shall find $\alpha \in C$ above ε. Recall that, by Clause (3)(c) of Definition 2.11, for each $\beta < \kappa$, $\beta \subseteq B_\beta$ and $|B_\beta| < \kappa$. It follows that we may recursively construct an increasing sequence of ordinals $\langle \alpha_n \mid n < \omega\rangle$ such that:

- $\alpha_0 := \sup(B_\varepsilon \cap \kappa)$, and, for all $n < \omega$:
- $\sup(B_{\alpha_n} \cap \kappa) < \alpha_{n+1} < \kappa$.

In particular, $\sup(B_{\alpha_n} \cap \kappa) \in \alpha_{n+1}$ for all $n < \omega$. Consequently, for $\alpha := \sup_{n<\omega} \alpha_n$, we have that $\alpha < \kappa$, and

$$B_\alpha \cap \kappa = \bigcup_{n<\omega} (B_{\alpha_n} \cap \kappa) \leq \bigcup_{n<\omega} \alpha_{n+1} \leq \bigcup_{n<\omega} (B_{\alpha_{n+2}} \cap \kappa) = \alpha,$$

so that $\alpha \in C \setminus (\varepsilon + 1)$. □

To see that the club C is as sought, let $\alpha \in C \cap S$ be arbitrary, and we shall verify that $X \cap \alpha \in N_\alpha$. Let $\beta(\alpha)$ be such that $\text{clps}(\mathfrak{B}_\alpha) = \langle M_{\beta(\alpha)}, \in, \ldots \rangle$, and let $j_\alpha : M_{\beta(\alpha)} \to B_\alpha$ denote the inverse of the collapsing map. As $\alpha \in C$, $j_\alpha(\alpha) = \kappa$, and $j_\alpha^{-1}(Y) = Y \cap \alpha$ for all $Y \in B_\alpha \cap \mathcal{P}(\kappa)$.

Subclaim 2.24.3.2 For every $\beta < \kappa^+$ such that $\vec{Z} \restriction (\alpha + 1) \in M_\beta$, $\beta > \beta(\alpha)$.

Proof Suppose not, so that $\vec{Z} \restriction (\alpha + 1) \in M_{\beta(\alpha)}$. As $\langle M_\delta, \in \rangle \prec \langle M_{\kappa^+}, \in \rangle$, we infer that

$$\langle M_\delta, \in \rangle \models \forall Z \subseteq \kappa \; \exists E \text{ club in } \kappa \; (\forall \gamma \in E \cap S \to Z \cap \gamma \neq Z_\gamma),$$

and hence

$$\langle M_{\beta(\alpha)}, \in \rangle \models \forall Z \subseteq \alpha \; \exists E \text{ club in } \alpha \; (\forall \gamma \in E \cap S \to Z \cap \gamma \neq Z_\gamma).$$

In particular, using $Z := Z_\alpha$, we find some E such that

$$\langle M_{\beta(\alpha)}, \in \rangle \models (E \text{ is a club in } \alpha) \wedge (\forall \gamma \in E \cap S \to Z_\alpha \cap \gamma \neq Z_\gamma).$$

Pushing forward with $E^* := j_\alpha(E)$ and $Z^* := j_\alpha(Z_\alpha)$, we infer that

$$\langle M_\delta, \in \rangle \models (E^* \text{ is a club in } \kappa) \wedge (\forall \gamma \in E^* \cap S \to Z^* \cap \gamma \neq Z_\gamma).$$

Then $Z^* \cap \alpha = j_\alpha(Z_\alpha) \cap \alpha = Z_\alpha$, and hence $\alpha \notin E^*$ (recall that $\alpha \in S$). Likewise $E^* \cap \alpha = j_\alpha(E) \cap \alpha = E$, and hence $\alpha \in \text{acc}(E^*) \subseteq E^*$. This is a contradiction. □

Now, since $\vec{\mathfrak{B}}$ witnesses LCC at δ with respect to $\vec{M}$ and $\vec{\mathcal{F}}_{X,\kappa}$, for each Y in $\{X, a, S\}$, we have that

$$\langle B_\alpha, \in, Y \cap B_\alpha \rangle \prec \langle M_{\kappa^+}, \in, Y \rangle \models \exists y \forall z ((z \in y) \leftrightarrow (z \in \kappa \wedge Y(z))),$$

therefore each of X, a, S is a definable element of $\mathfrak{B}_\alpha$. So, as, for all $Y \in B_\alpha \cap \mathcal{P}(\kappa)$, $j_\alpha^{-1}(Y) = Y \cap \alpha$, we infer that $X \cap \alpha$, $a \cap \alpha$, and $S \cap \alpha$ are all in $M_{\beta(\alpha)}$. We will show that $\beta(\alpha) < f(\alpha)$, from which it will follow that $X \cap \alpha \in N_\alpha$.

Subclaim 2.24.3.3 $\beta(\alpha) < f(\alpha)$.

Proof Naturally, the analysis splits into two cases:

► Suppose $\alpha \notin D$. By definition of $f(\alpha)$ and by Subclaim 2.24.3.2, $\beta(\alpha) < f(\alpha)$.

► Suppose $\alpha \in D$. As $\mathfrak{B}_\alpha \prec \langle M_\delta, \in, \vec{M} \restriction \delta, X, a, S, \vec{Z} \rangle$ and $\text{Im}(j_\alpha) = B_\alpha$, we infer that $j_\alpha : M_{\beta(\alpha)} \to M_\delta$ forms an elementary embedding from $\langle M_{\beta(\alpha)}, \in, \ldots \rangle$ to $\langle M_\delta, \in, \vec{M} \restriction \delta, X, a, S, \vec{Z} \rangle$ with $j_\alpha(\alpha) = \kappa$. As $\kappa, S, a \in M_\delta$ and $\langle M_\delta, \in, M \restriction \delta \rangle \prec \langle M_\kappa, \in, \vec{M} \rangle$, we have:

(I) $\langle M_\delta, \in, \vec{M} \restriction \delta\rangle \models \mathsf{LCC}(\kappa, \delta)$,
(II) $\langle M_\delta, \in\rangle \models \mathsf{ZF}^- \ \&\ \kappa$ is the largest cardinal,
(III) $\langle M_\delta, \in\rangle \models \kappa$ is regular & $S \cap \kappa$ is stationary,
(IV) $\langle M_\delta, \in\rangle \models \Theta(x, y, a \cap \kappa)$ defines a global well-order.

It now follows that $\beta(\alpha)$ satisfies clauses (i),(ii),(iii) and (iv) of the definition of S_α. Together with Subclaim 2.24.3.2, then, $\beta(\alpha) \in S_\alpha$. So, by definitions of f and D, $\beta(\alpha) < f(\alpha)$. □

This completes the proof of Claim 2.24.3. □

We are left with addressing Clause (3) of Definition 2.6.

Claim 2.24.4 *The sequence $\langle N_\alpha \mid \alpha \in S\rangle$ reflects Π^1_2-sentences.*

Proof We need to show that whenever $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, with $\phi = \forall X \exists Y \varphi$ a Π^1_2-sentence, for every club $E \subseteq \kappa$, there is $\alpha \in E \cap S$, such that

$$\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi.$$

But by adding E to the list $(A_n)_{n\in\omega}$ of predicates, and by slightly extending the first-order formula φ to also assert that E is unbounded, we would get that any ordinal α satisfying the above will also satisfy that α is an accumulation point of the closed set E, so that $\alpha \in E$. It follows that if any Π^1_2-sentence valid in a structure of the form $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle$ reflects to some ordinal $\alpha' \in S$, then any Π^1_2-sentence valid in a structure of the form $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle$ reflects stationarily often in S.

Consider a Π^1_2-formula $\forall X \exists Y \varphi$, with integers p, q such that X is a p-ary second-order variable and Y is a q-ary second-order variable. Suppose $\vec{A} = (A_n)_{n\in\omega}$ is a sequence of finitary predicates on κ, and $\langle \kappa, \in, \vec{A}\rangle \models \forall X \exists Y \varphi$. By the reduction established in the proof of Proposition 3.1 below, we may assume that $\vec{A}$ consists of a single predicate A_0 of arity, say, m_0. Recalling Convention 2.4 and since $M_{\kappa^+} = H_{\kappa^+}$, this altogether means that

$$\langle \kappa, \in, A_0\rangle \models_{M_{\kappa^+}} \forall X \exists Y \varphi.$$

Let γ be the least ordinal such that $\vec{Z}, A_0, S \in M_\gamma$. Note that $\kappa < \gamma < \kappa^+$. Let Δ denote the set of all $\delta \le \kappa^+$ such that:

(a) $\langle M_\delta, \in, \vec{M} \restriction \delta\rangle \models \mathsf{LCC}(\kappa, \delta)$,[6]
(b) $\langle M_\delta, \in\rangle \models \mathsf{ZF}^- \ \&\ \kappa$ is the largest cardinal,
(c) $\langle M_\delta, \in\rangle \models \kappa$ is regular & S is stationary in κ,
(d) $\langle M_\delta, \in\rangle \models \Theta(x, y, a)$ defines a global well-order,
(e) $\langle \kappa, \in, A_0\rangle \models_{M_\delta} \forall X \exists Y \varphi$,
(f) $\langle M_\delta, \in\rangle \models \vec{Z}$ witness that S is not ineffable, and
(g) $\delta > \gamma$.

[6] In particular, $\delta > \kappa$.

As $\kappa^+ \in \Delta$, it follows from Lemma 2.22 and elementarity that $\mathrm{otp}(\Delta \cap \kappa^+) = \kappa^+$. Let $\{\delta_n \mid n < \omega\}$ denote the increasing enumeration of the first ω many elements of Δ.

Definition 2.24.4.1 Let $T(\vec{M}, \kappa, S, a, A_0, \vec{Z}, \gamma)$ denote the theory consisting of the following axioms:

(A) $\vec{M}$ witness $\mathsf{LCC}(\kappa, \kappa^+)$,
(B) ZF^- & κ is the largest cardinal,
(C) κ is regular & S is stationary in κ,
(D) $\Theta(x, y, a)$ defines a global well-order,
(E) $\langle \kappa, \in, A_0 \rangle \models \forall X \exists Y \varphi$,
(F) $\vec{Z}$ witness that S is not ineffable,
(G) γ is the least ordinal such that $\{\vec{Z}, A_0, S\} \in \vec{M}(\gamma)$.

Let $n < \omega$. Since M_{δ_n} is transitive, standard facts (cf. [6, Chapter 3, §5]) yield the existence of a formula Ψ in the language $\{\dot{\vec{M}}, \in\}$ which is $\Delta_1^{\mathsf{ZF}^-}$, and for all $\delta \in (\gamma, \delta_n)$,

$$\begin{gathered}\langle M_\delta, \in, \vec{M} \restriction \delta \rangle \models T(\vec{M} \restriction \delta, \kappa, S, a, A_0, \vec{Z}, \gamma) \\ \Longleftrightarrow \\ \Psi(\vec{M} \restriction \delta, \kappa, S, a, A_0, \vec{Z}, \gamma) \\ \Longleftrightarrow \\ \langle M_{\delta_n}, \in, \vec{M} \restriction \delta_n \rangle \models \Psi(\vec{M} \restriction \delta, \kappa, S, a, A_0, \vec{Z}, \gamma).\end{gathered} \tag{$\star_1$}$$

Since $\{\delta_k \mid k < \omega\}$ enumerates the first ω many elements of Δ, M_{δ_n} believes that there are exactly n ordinals δ such that Clauses (a)–(g) hold for M_δ. In fact,

$$\langle M_{\delta_n}, \in, \vec{M} \restriction \delta_n \rangle \models \{\delta \mid \Psi(\vec{M} \restriction \delta, \kappa, S, a, A_0, \vec{Z}, \gamma)\} = \{\delta_k \mid k < n\}. \tag{$\star_2$}$$

Next, for every $n < \omega$, as $\langle M_{\delta_{n+1}}, \in \rangle \models |\delta_n| = \kappa$, we may fix in $M_{\delta_{n+1}}$ a sequence $\vec{\mathfrak{B}}_n = \langle \mathfrak{B}_{n,\alpha} \mid \alpha < \kappa \rangle$ witnessing LCC at δ_n with respect to $\vec{M} \restriction \delta_{n+1}$ and $\vec{\mathcal{F}}_{A_0,\kappa}$ such that, moreover,

$$\langle M_{\delta_{n+1}}, \in, \vec{M} \restriction \delta_{n+1} \rangle \models \text{“}\vec{\mathfrak{B}}_n \text{ is the } <_\Theta\text{-least such witness”.}$$ [7]

For every $n < \omega$, consider the club $C_n := \{\alpha < \kappa \mid B_{n,\alpha} \cap \kappa = \alpha\}$, and then let

$$\alpha' := \min\left(\left(\bigcap\nolimits_{n \in \omega} C_n\right) \cap S\right).$$

For every $n < \omega$, let β_n be such that $\mathrm{clps}(\mathfrak{B}_{n,\alpha'}) = \langle M_{\beta_n}, \in, \ldots \rangle$, and let $j_n : M_{\beta_n} \to B_{n,\alpha'}$ denote the inverse of the Mostowski collapse.

[7] Recalling Definition 2.23, this means that $\langle M_{\delta_{n+1}}, \in, \vec{M} \restriction \delta_{n+1} \rangle \models$ “$\vec{\mathfrak{B}}_n$ is the $<_\Theta$-least $\vec{\mathfrak{B}}$ such that $(\psi_a \wedge \psi_b \wedge \psi_c \wedge \psi_d \wedge \psi_e)(\vec{\mathfrak{B}}, \vec{\mathcal{F}}_{A_0,\kappa}, \delta_n, \vec{M} \restriction (\delta_n + 1))$”.

Subclaim 2.24.4.1 Let $n \in \omega$. Then $j_n^{-1}(\gamma) = j_0^{-1}(\gamma)$.

Proof Since $j_n^{-1}(\vec{Z}) = \vec{Z} \restriction \alpha'$, $j_n^{-1}(A_0) = A_0 \cap (\alpha')^{m_0}$ and $j_n^{-1}(S) = S \cap \alpha'$, it follows from

$$\langle M_{\delta_n}, \in, \vec{M} \restriction \delta_n \rangle \models \gamma \text{ is the least ordinal with } \{\vec{Z}, A_0, S\} \subseteq M_\gamma,$$

that
$\langle M_{\beta_n}, \in, \vec{M} \restriction \beta_n \rangle \models j_n^{-1}(\gamma)$ is the least ordinal with $\{\vec{Z} \restriction \alpha', A_0 \cap (\alpha')^{m_0}, S \cap \alpha'\} \subseteq M_\gamma$.

Now, let $\bar{\gamma}$ be such that

$$\langle M_{\beta_0}, \in, \vec{M} \restriction \beta_0 \rangle \models \bar{\gamma} \text{ is the least ordinal such that } \{\vec{Z} \restriction \alpha', A_0 \cap (\alpha')^{m_0}, S \cap \alpha'\} \subseteq M_{\bar{\gamma}}.$$

Since $\vec{M}$ is continuous, it follows that $\bar{\gamma}$ is a successor ordinal, that is, $\bar{\gamma} = \sup(\bar{\gamma})+1$. So $\langle M_{\beta_0}, \in, \vec{M} \restriction \beta_0 \rangle$ satisfies the conjunction of the two:

- $\{\vec{Z} \restriction \alpha', A_0 \cap (\alpha')^{m_0}, S \cap \alpha'\} \subseteq M_{\bar{\gamma}}$, and
- $\{\vec{Z} \restriction \alpha', A_0 \cap (\alpha')^{m_0}, S \cap \alpha'\} \nsubseteq M_{\sup(\bar{\gamma})}$.

But the two are Δ_0-formulas in the parameters $\vec{Z} \restriction \alpha'$, $A_0 \cap (\alpha')^{m_0}$, $S \cap \alpha'$, $M_{\bar{\gamma}}$ and $M_{\sup(\bar{\gamma})}$, which are all elements of M_{β_0}. Therefore,

$$\langle M_{\beta_n}, \in, \vec{M} \restriction \beta_n \rangle \models \bar{\gamma} \text{ is the least ordinal such that } \{\vec{Z} \restriction \alpha', A_0 \cap (\alpha')^{m_0}, S \cap \alpha'\} \subseteq M_\gamma,$$

so that $j_n^{-1}(\gamma) = \bar{\gamma} = j_0^{-1}(\gamma)$. ◻

Denote $\bar{\gamma} := j_0^{-1}(\gamma)$. Let Ψ be the same formula used in statement ($\star_1$). For all $n < \omega$ and $\bar{\beta} \in (\bar{\gamma}, \beta_n)$, setting $\beta := j_n(\bar{\beta})$, by elementarity of j_n:

$$\begin{gathered}\langle M_{\beta_n}, \in, \vec{M} \restriction \beta_n \rangle \models \Psi(\vec{M} \restriction \bar{\beta}, \alpha', S \cap \alpha', a \cap \alpha', A_0 \cap (\alpha')^{m_0}, \vec{Z} \restriction \alpha', \bar{\gamma}) \\ \Longleftrightarrow \\ \langle M_{\delta_n}, \in, \vec{M} \restriction \delta_n \rangle \models \Psi(\vec{M} \restriction \beta, \kappa, S, a, A_0, \vec{Z}, \gamma).\end{gathered} \tag{$\star_3$}$$

Hence, for all $n < \omega$, by statements ($\star_2$) and ($\star_3$), it follows that

$$\begin{aligned}\langle M_{\beta_n}, \in, \vec{M} \restriction \beta_n \rangle \models\ & \{\beta \mid \Psi(\vec{M} \restriction \beta, \alpha', S \cap \alpha', a \cap \alpha', A_0 \cap (\alpha')^{m_0}, \vec{Z} \restriction \alpha', \bar{\gamma})\} \\ &= \{j_n^{-1}(\delta_k) \mid k < n\},\end{aligned}$$

and that, for each $k < n$, $j_n(\beta_k) = \delta_k$.

Subclaim 2.24.4.2 $\beta' := \sup_{n \in \omega} \beta_n$ is equal to $\sup(S_{\alpha'})$.

Proof For each $n < \omega$, as $\mathrm{clps}(\mathfrak{B}_{n,\alpha'}) = \langle M_{\beta_n}, \in, \ldots \rangle$, the proof of Subclaim 2.24.3.3, establishing that $\beta(\alpha) \in S_\alpha$, makes clear that $\beta_n \in S_{\alpha'}$.

We first argue that $\beta' \notin S_{\alpha'}$ by showing that $\langle M_{\beta'}, \in \rangle \not\models \mathsf{ZF}^-$, and then we will argue that no $\beta > \beta'$ is in $S_{\alpha'}$. Note that $\{\beta_n \mid n < \omega\}$ is a definable subset of

β' since it can be defined as the first ω ordinals to satisfy Clauses (a)–(g), replacing $\vec{M}\restriction\delta, \kappa, S, a, A_0, \vec{Z}, \gamma$ by $\vec{M}\restriction\beta, \alpha', S\cap\alpha', a\cap\alpha', A_0\cap(\alpha')^{m_0}, \vec{Z}\restriction\alpha', \bar{\gamma}$, respectively. So if $\langle M_{\beta'}, \in\rangle$ were to model ZF^-, we would have get that $\sup_{n<\omega}\beta_n$ is in $M_{\beta'}$, contradicting the fact that $M_{\beta'}\cap\mathrm{OR}=\beta'$.

Now, towards a contradiction, suppose that there exists $\beta>\beta'$ in $S_{\alpha'}$, and let β be the least such ordinal. In particular, $\langle M_\beta, \in\rangle\models\mathsf{ZF}^-$, and $\langle\beta_n \mid n<\omega\rangle\in M_\beta$, so that $\langle M_{\beta_n}\mid n\in\omega\rangle\in M_\beta$. We will reach a contradiction to Clause (iii) of the definition of $S_{\alpha'}$, asserting, in particular, that $S\cap\alpha'$ is stationary in $\langle M_\beta, \in\rangle$.

For each $n<\omega$, we have that $\langle M_{\delta_{n+1}}, \in, \vec{M}\restriction\delta_{n+1}\rangle\models\Phi(C_n, \delta_n, \vec{\mathfrak{B}}_n, \kappa)$, where $\Phi(C_n, \delta_n, \vec{\mathfrak{B}}_{\mathfrak{n}}, \kappa)$ is the conjunction of the following two formulas:

- $C_n=\{\alpha<\kappa \mid B_{n,\alpha}\cap\kappa=\alpha\}$, and
- $\vec{\mathfrak{B}}_n$ is the $<_\Theta$-least witness to LCC at δ_n with respect to $\vec{M}\restriction\delta_{n+1}$ and $\mathcal{F}_{A_0,\kappa}$.

Therefore, for $\overline{C_n}:=j_{n+1}^{-1}(C_n)$ and $\overline{\mathfrak{B}_n}:=j_{n+1}^{-1}(\vec{\mathfrak{B}}_n)$, we have

$$\langle M_{\beta_{n+1}}, \in, \vec{M}\restriction\beta_{n+1}\rangle\models\Phi(\overline{C_n}, \beta_n, \overline{\mathfrak{B}_n}, \alpha').$$

In particular, $\overline{C_n}=j_{n+1}^{-1}(C_n)=C_n\cap\alpha'$. Recalling that $\alpha'=\min((\bigcap_{n\in\omega}C_n)\cap S)$, we infer that $\bigcap_{n<\omega}\overline{C_n}$ is disjoint from $S\cap\alpha'$. Thus, to establish that $S\cap\alpha'$ is nonstationary, it suffices to verify the two:

(1) $\langle\overline{C_n}\mid n<\omega\rangle$ belongs to M_β, and
(2) for every $n<\omega$, $\langle M_\beta, \in\rangle\models\overline{C_n}$ is a club in α'.

As $\langle M_{\beta_n}\mid n\in\omega\rangle\in M_\beta$, we can define $\langle\overline{\mathfrak{B}}_n\mid n\in\omega\rangle$ using that, for all $n\in\omega$,

$$\langle M_{\beta_{n+1}}, \in, \vec{M}\restriction\beta_{n+1}\rangle\models\text{“}\overline{\mathfrak{B}}_n\text{ is the }<_\Theta\text{-least witness to LCC at }\alpha'\text{ w.r.t. }\vec{M}\restriction\beta_{n+1}\text{ and }\mathcal{F}_{A_0,\alpha'}\text{”}.$$

This takes care of Clause (1), and shows that $\langle M_{\beta_{n+1}}, \in\rangle\models\overline{C_n}$ is a club in α'. Since M_β is transitive and the formula expressing that $\overline{C_n}$ is a club is Δ_0, we have also taken care of Clause (2). □

It follows that $\alpha'\in D$ and $f(\alpha')=\sup(S_{\alpha'})=\beta'$.[8] Finally, as, for every $n<\omega$, we have

$$\langle\alpha', \in, A_0\cap(\alpha')^{m_0}\rangle\models_{M_{\beta_n}}\forall X\exists Y\varphi,$$

we infer that $N_{\alpha'}=M_{f(\alpha')}=M_{\beta'}=\bigcup_{n\in\omega}M_{\beta_n}$ is such that

$$\langle\alpha', \in, A_0\cap(\alpha')^{m_0}\rangle\models_{N_{\alpha'}}\forall X\exists Y\varphi.$$

Indeed, otherwise there is $X_0\in[\alpha']^p\cap N_{\alpha'}$ such that, for all $Y\in[\alpha']^q\cap N_{\alpha'}$, $N_{\alpha'}\models[\langle\alpha', \in, A_0\cap(\alpha')^{m_0}\rangle\models\neg\varphi(X_0, Y)]$. Find a large enough $n<\omega$ such that

[8] Notice that the argument of this claim also showed that D is stationary.

$X_0 \in M_{\beta_n}$. Now, since "$\langle\alpha', \in, A_0 \cap (\alpha')^{m_0}\rangle \models \neg\varphi(X_0, Y)$" is a $\Delta_1^{\mathsf{ZF}^-}$ formula on the parameters $\langle\alpha', \in, A_0 \cap (\alpha')^{m_0}\rangle$, φ, and since M_{β_n} is transitive subset of $N_{\alpha'}$ it follows that, for all $Y \in [\alpha']^q \cap M_{\beta_n}$, $M_{\beta_n} \models [\langle\alpha', \in, A_0 \cap (\alpha')^{m_0}\rangle \models \neg\varphi(X_0, Y)]$, which is a contradiction. □

This completes the proof of Theorem 2.24. □

As a corollary we have found a strong combinatorial axiom that holds everywhere (including at ineffable sets) in canonical models of Set Theory (including Gödel's constructible universe).

Corollary 2.25 *Suppose that:*

- *$L[E]$ is an extender model with Jensen λ-indexing;*
- *$L[E] \models$ "there are no subcompact cardinals";*
- *for every $\alpha \in \mathrm{OR}$, the premouse $L[E]||\alpha$ is weakly iterable.*

Then, in $L[E]$, for every regular uncountable cardinal κ, for every stationary $S \subseteq \kappa$, $\mathrm{Dl}^*_S(\Pi^1_2)$ holds.

Proof Work in $L[E]$. Let κ be any regular and uncountable cardinal. By Fact 2.15, $\vec{M} = \langle L_\beta[E] \mid \beta < \kappa^+\rangle$ witnesses that $\mathsf{LCC}(\kappa, \kappa^+)$ holds. Since $L_{\kappa^+}[E]$ is an acceptable J-structure,[9] $\vec{M}$ is a nice filtration of $L_{\kappa^+}[E]$ that is eventually slow at κ. In addition (cf. [22, Lemma 1.11]), there is a Σ_1-formula Θ for which

$$x <_\Theta y \text{ iff } L[E]|\kappa^+ \models \Theta(x, y)$$

defines a well-ordering of $L_{\kappa^+}[E]$. Finally, acceptability implies that $L_{\kappa^+}[E] = H_{\kappa^+}$. Now, appeal to Theorem 2.24. □

3 Universality of inclusion modulo nonstationary

Throughout this section, κ denotes a regular uncountable cardinal satisfying $\kappa^{<\kappa} = \kappa$. Here, we will be proving Theorems B and C. Before we can do that, we shall need to establish a *transversal lemma*, as well as fix some notation and coding that will be useful when working with structures of the form $\langle\kappa, \in, (A_n)_{n\in\omega}\rangle$.

Proposition 3.1 (Transversal lemma) *Suppose that $\langle N_\alpha \mid \alpha \in S\rangle$ is a $\mathrm{Dl}^*_S(\Pi^1_2)$-sequence, for a given stationary $S \subseteq \kappa$. For every Π^1_2-sentence ϕ, there exists a transversal $\langle\eta_\alpha \mid \alpha \in S\rangle \in \prod_{\alpha\in S} N_\alpha$ satisfying the following.*

For every $\eta \in \kappa^\kappa$, whenever $\langle\kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, there are stationarily many $\alpha \in S$ such that

(i) *$\eta_\alpha = \eta \restriction \alpha$, and*
(ii) *$\langle\alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi$.*

[9] For the definition of acceptable J-structure, see [27, p. 4].

Proof Let $c : \kappa \times \kappa \leftrightarrow \kappa$ be some primitive-recursive pairing function. For each $\alpha \in S$, fix a surjection $f_\alpha : \kappa \to N_\alpha$ such that $f_\alpha[\alpha] = N_\alpha$ whenever $|N_\alpha| = |\alpha|$. Then, for all $i < \kappa$, as $f_\alpha(i) \in N_\alpha$, we may define a set η^i_α in N_α by letting

$$\eta^i_\alpha := \begin{cases} \{(\beta, \gamma) \in \alpha \times \alpha \mid c(i, c(\beta, \gamma)) \in f_\alpha(i)\}, & \text{if } i < \alpha; \\ \emptyset, & \text{otherwise.} \end{cases}$$

We claim that for every Π^1_2-sentence ϕ, there exists $i(\phi) < \kappa$ for which $\langle \eta^{i(\phi)}_\alpha \mid \alpha \in S\rangle$ satisfies the conclusion of our proposition. Before we prove this, let us make a few reductions.

First of all, it is clear that for every Π^1_2-sentence $\phi = \forall X \exists Y \varphi$, there exists a large enough $n' < \omega$ such that all predicates mentioned in φ are in $\{\epsilon, \mathbb{X}, \mathbb{Y}, \mathbb{A}_n \mid n < n'\}$. So the only structures of interest for ϕ are in fact $\langle \alpha, \in, (A_n)_{n<n'}\rangle$, where $\alpha \leq \kappa$. Let $m' := \max\{m(\mathbb{A}_n) \mid n < n'\}$. Then, by a trivial manipulation of φ, we may assume that the only structures of interest for ϕ are in fact $\langle \alpha, \in, A_0\rangle$, where $\omega \leq \alpha \leq \kappa$ and $m(\mathbb{A}_0) = m' + 1$.

Having the above reductions in hand, we now fix a Π^1_2-sentence $\phi = \forall X \exists Y \varphi$ and positive integers m and k such that the only predicates mentioned in φ are in $\{\epsilon, \mathbb{X}, \mathbb{Y}, \mathbb{A}_0\}$, $m(\mathbb{A}_0) = m$ and $m(\mathbb{Y}) = k$.

Claim 3.1.1 *There exists $i < \kappa$ satisfying the following. For all $\eta \in \kappa^\kappa$ and $A \subseteq \kappa^m$, whenever $\langle \kappa, \in, A\rangle \models \phi$, there are stationarily many $\alpha \in S$ such that*

(i) $\eta^i_\alpha = \eta \restriction \alpha$, *and*
(ii) $\langle \alpha, \in, A \cap (\alpha^m)\rangle \models_{N_\alpha} \phi$.

Proof Suppose not. Then, for every $i < \kappa$, we may fix $\eta_i \in \kappa^\kappa$, $A_i \subseteq \kappa^m$ and a club $C_i \subseteq \kappa$ such that $\langle \kappa, \in, A_i\rangle \models \phi$, but, for all $\alpha \in C_i \cap S$, one of the two fails:

(i) $\eta^i_\alpha = \eta_i \restriction \alpha$, or
(ii) $\langle \alpha, \in, A_i \cap (\alpha^m)\rangle \models_{N_\alpha} \phi$.

Let

- $Z := \{c(i, c(\beta, \gamma)) \mid i < \kappa, (\beta, \gamma) \in \eta_i\}$,
- $A := \{(i, \delta_1, \ldots, \delta_m) \mid i < \kappa, (\delta_1, \ldots, \delta_m) \in A_i\}$, and
- $C := \triangle_{i<\kappa}\{\alpha \in C_i \mid \eta_i[\alpha] \subseteq \alpha\}$.

Fix a variable i that does not occur in φ. Define a first-order sentence ψ mentioning only the predicates in $\{\epsilon, \mathbb{X}, \mathbb{Y}, \mathbb{A}_1\}$ with $m(\mathbb{A}_1) = 1 + m$ and $m(\mathbb{Y}) = 1 + k$ by replacing all occurrences of the form $\mathbb{A}_0(x_1, \ldots, x_m)$ and $\mathbb{Y}(y_1, \ldots, y_k)$ in φ by $\mathbb{A}_1(i, x_1, \ldots, x_m)$ and $\mathbb{Y}(i, y_1, \ldots, y_k)$, respectively. Then, let $\varphi' := \forall i(\psi)$, and finally let $\phi' := \forall X \exists Y \varphi'$, so that ϕ' is a Π^1_2-sentence.

A moment reflection makes it clear that $\langle \kappa, \in, A\rangle \models \phi'$. Thus, let S' denote the set of all $\alpha \in S$ such that all of the following hold:

(1) $\alpha \in C$;
(2) $c[\alpha \times \alpha] = \alpha$;
(3) $Z \cap \alpha \in N_\alpha$;

(4) $|N_\alpha| = |\alpha|$;
(5) $\langle \alpha, \in, A \cap (\alpha^{m+1}) \rangle \models_{N_\alpha} \phi'$.

By hypothesis, S' is stationary. For all $\alpha \in S'$, by Clauses (3) and (4), we have $Z \cap \alpha \in N_\alpha = f_\alpha[\alpha]$, so, by Fodor's lemma, there exists some $i < \kappa$ and a stationary $S'' \subseteq S' \setminus (i+1)$ such that, for all $\alpha \in S''$:

(3') $Z \cap \alpha = f_\alpha(i)$.

Let $\alpha \in S''$. By Clause (5), we in particular have

(5') $\langle \alpha, \in, A_i \cap (\alpha^m) \rangle \models_{N_\alpha} \phi$.

Also, by Clause (1), we have $\alpha \in C_i$, and so we must conclude that $\eta_i \restriction \alpha \neq \eta^i_\alpha$. However, $\eta_i[\alpha] \subseteq \alpha$, and $Z \cap \alpha = f_\alpha(i)$, so that, by Clause (2),

$$\eta_i \restriction \alpha = \eta_i \cap (\alpha \times \alpha) = \{(\beta, \gamma) \in \alpha \times \alpha \mid c(i, c(\beta, \gamma)) \in f_\alpha(i)\} = \eta^i_\alpha.$$

This is a contradiction. □

This completes the proof of Proposition 3.1. □

Lemma 3.2 *There is a first-order sentence ψ_{fnc} in the language with binary predicate symbols ϵ and $\mathbb{X}$ such that, for every ordinal α and every $X \subseteq \alpha \times \alpha$,*

$$(X \text{ is a function from } \alpha \text{ to } \alpha) \text{ iff } (\langle \alpha, \in, X \rangle \models \psi_{\text{fnc}}).$$

Proof Let $\psi_{\text{fnc}} := \forall \beta \exists \gamma (\mathbb{X}(\beta, \gamma) \wedge (\forall \delta (\mathbb{X}(\beta, \delta) \to \delta = \gamma)))$. □

Lemma 3.3 *Let α be an ordinal. Suppose that ϕ is a Σ^1_1-sentence involving a predicate symbol $\mathbb{A}$ and two binary predicate symbols $\mathbb{X}_0, \mathbb{X}_1$. Denote $R_\phi := \{(X_0, X_1) \mid \langle \alpha, \in, A, X_0, X_1 \rangle \models \phi\}$. Then there are Π^1_2-sentences $\psi_{\text{Reflexive}}$ and $\psi_{\text{Transitive}}$ such that:*

(1) *$(R_\phi \supseteq \{(\eta, \eta) \mid \eta \in {}^\alpha\alpha\})$ iff $(\langle \alpha, \in, A \rangle \models \psi_{\text{Reflexive}})$;*
(2) *$(R_\phi$ is transitive$)$ iff $(\langle \alpha, \in, A \rangle \models \psi_{\text{Transitive}})$.*

Proof (1) Fix a first-order sentence ψ_{fnc} such that $(X_0 \in {}^\alpha\alpha)$ iff $(\langle \alpha, \in, X_0 \rangle \models \psi_{\text{fnc}})$. Now, let $\psi_{\text{Reflexive}}$ be $\forall X_0 \forall X_1 ((\psi_{\text{fnc}} \wedge (X_1 = X_0)) \to \phi)$.
(2) Fix a Σ^1_1-sentence ϕ' involving predicate symbols $\mathbb{A}, \mathbb{X}_1, \mathbb{X}_2$ and a Σ^1_1-sentence ϕ'' involving binary symbols $\mathbb{A}, \mathbb{X}_0, \mathbb{X}_2$ such that

$$\begin{aligned}&\{(X_1, X_2) \mid \langle \alpha, \in, A, X_1, X_2 \rangle \models \phi'\}\\ &\quad = R_\phi = \{(X_0, X_2) \mid \langle \alpha, \in, A, X_0, X_2 \rangle \models \phi''\}\end{aligned}$$

Now, let $\psi_{\text{Transitive}} := \forall X_0 \forall X_1 \forall X_2 ((\phi \wedge \phi') \to \phi'')$.

□

Definition 3.4 Denote by $\mathrm{Lev}_3(\kappa)$ the set of level sequences in $\kappa^{<\kappa}$ of length 3:

$$\mathrm{Lev}_3(\kappa) := \bigcup_{\tau<\kappa} \kappa^\tau \times \kappa^\tau \times \kappa^\tau.$$

Fix an injective enumeration $\{\ell_\delta \mid \delta < \kappa\}$ of $\mathrm{Lev}_3(\kappa)$. For each $\delta < \kappa$, we denote $\ell_\delta = (\ell^0_\delta, \ell^1_\delta, \ell^2_\delta)$. We then encode each $T \subseteq \mathrm{Lev}_3(\kappa)$ as a subset of κ^5 via:

$$T_\ell := \{(\delta, \beta, \ell^0_\delta(\beta), \ell^1_\delta(\beta), \ell^2_\delta(\beta)) \mid \delta < \kappa, \ell_\delta \in T, \beta \in \mathrm{dom}(\ell^0_\delta)\}.$$

We now prove Theorem C.

Theorem 3.5 *Suppose* $\mathrm{Dl}^*_S(\Pi^1_2)$ *holds for a given stationary* $S \subseteq \kappa$.

For every analytic quasi-order Q *over* κ^κ, *there is a* 1*-Lipschitz map* $f : \kappa^\kappa \to 2^\kappa$ *reducing* Q *to* $\subseteq^S$.

Proof Let Q be an analytic quasi-order over κ^κ. Fix a tree T on $\kappa^{<\kappa} \times \kappa^{<\kappa} \times \kappa^{<\kappa}$ such that $Q = \mathrm{pr}([T])$, that is,

$$(\eta, \xi) \in Q \iff \exists \zeta \in \kappa^\kappa\ \forall \tau < \kappa\ (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau) \in T.$$

We shall be working with a first-order language having a 5-ary predicate symbol $\mathbb{A}$ and binary predicate symbols $\mathbb{X}_0, \mathbb{X}_1, \mathbb{X}_2$ and ϵ. By Lemma 3.2, for each $i < 3$, let us fix a sentence ψ^i_{fnc} concerning the binary predicate symbol $\mathbb{X}_i$ instead of $\mathbb{X}$, so that

$$(X_i \in \kappa^\kappa) \text{ iff } (\langle \kappa, \in, A, X_0, X_1, X_2\rangle \models \psi^i_{\mathrm{fnc}}).$$

Define a sentence φ_Q to be the conjunction of four sentences: ψ^0_{fnc}, ψ^1_{fnc}, ψ^2_{fnc}, and

$$\begin{aligned}&\forall\tau\exists\delta\forall\beta[\epsilon(\beta,\tau) \to \\ &\quad \exists\gamma_0\exists\gamma_1\exists\gamma_2(\mathbb{X}_0(\beta,\gamma_0) \wedge \mathbb{X}_1(\beta,\gamma_1) \wedge \mathbb{X}_2(\beta,\gamma_2) \wedge \mathbb{A}(\delta,\beta,\gamma_0,\gamma_1,\gamma_2))].\end{aligned}$$

Set $A := T_\ell$ as in Definition 3.4. Evidently, for all $\eta, \xi, \zeta \in \mathcal{P}(\kappa \times \kappa)$, we get that

$$\langle \kappa, \in, A, \eta, \xi, \zeta\rangle \models \varphi_Q$$

iff the two hold:

(1) $\eta, \xi, \zeta \in \kappa^\kappa$, and
(2) for every $\tau < \kappa$, there exists $\delta < \kappa$, such that $\ell_\delta = (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau)$ is in T.

Let $\phi_Q := \exists X_2(\varphi_Q)$. Then ϕ_Q is a Σ^1_1-sentence involving predicate symbols $\mathbb{A}, \mathbb{X}_0, \mathbb{X}_1$ and ϵ for which the induced binary relation

$$R_{\phi_Q} := \{(\eta, \xi) \in (\mathcal{P}(\kappa \times \kappa))^2 \mid \langle \kappa, \in, A, \eta, \xi\rangle \models \phi_Q\}$$

coincides with the quasi-order Q. Now, appeal to Lemma 3.3 with ϕ_Q to receive the corresponding Π^1_2-sentences $\psi_{\mathrm{Reflexive}}$ and $\psi_{\mathrm{Transitive}}$. Then, consider the following two Π^1_2-sentences:

- $\psi_Q^0 := \psi_{\text{Reflexive}} \wedge \psi_{\text{Transitive}} \wedge \phi_Q$, and
- $\psi_Q^1 := \psi_{\text{Reflexive}} \wedge \psi_{\text{Transitive}} \wedge \neg(\phi_Q)$.

Let $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ be a $\mathrm{Dl}^*_S(\Pi^1_2)$-sequence. Appeal to Proposition 3.1 with the Π^1_2-sentence ψ_Q^1 to obtain a corresponding transversal $\langle \eta_\alpha \mid \alpha \in S\rangle \in \prod_{\alpha\in S} N_\alpha$. Note that we may assume that, for all $\alpha \in S$, $\eta_\alpha \in {}^\alpha\alpha$, as this does not harm the key feature of the chosen transversal.[10]

For each $\eta \in \kappa^\kappa$, let

$$Z_\eta := \{\alpha \in S \mid A \cap \alpha^5 \text{ and } \eta \restriction \alpha \text{ are in } N_\alpha\}.$$

Claim 3.5.1 *Suppose $\eta \in \kappa^\kappa$. Then $S \backslash Z_\eta$ is nonstationary.*

Proof Fix primitive-recursive bijections $c : \kappa^2 \leftrightarrow \kappa$ and $d : \kappa^5 \leftrightarrow \kappa$. Given $\eta \in \kappa^\kappa$, consider the club D_0 of all $\alpha < \kappa$ such that:

- $\eta[\alpha] \subseteq \alpha$;
- $c[\alpha \times \alpha] = \alpha$;
- $d[\alpha \times \alpha \times \alpha \times \alpha \times \alpha] = \alpha$.

Now, as $c[\eta]$ is a subset of κ, by the choice $\vec{N}$, we may find a club $D_1 \subseteq \kappa$ such that, for all $\alpha \in D_1 \cap S$, $c[\eta] \cap \alpha \in N_\alpha$. Likewise, we may find a club $D_2 \subseteq \kappa$ such that, for all $\alpha \in D_2 \cap S$, $d[A] \cap \alpha \in N_\alpha$.

For all $\alpha \in S \cap D_0 \cap D_1 \cap D_2$, we have

- $c[\eta \restriction \alpha] = c[\eta \cap (\alpha \times \alpha)] = c[\eta] \cap c[\alpha \times \alpha] = c[\eta] \cap \alpha \in N_\alpha$, and
- $d[A \cap \alpha^5] = d[A] \cap d[\alpha^5] = d[A] \cap \alpha \in N_\alpha$.

As N_α is p.r.-closed, it then follows that $\eta \restriction \alpha$ and $A \cap \alpha^5$ are in N_α. Thus, we have shown that $S \backslash Z_\eta$ is disjoint from the club $D_0 \cap D_1 \cap D_2$. □

For all $\eta \in \kappa^\kappa$ and $\alpha \in Z_\eta$, let:

$$\mathcal{P}_{\eta,\alpha} := \{p \in \alpha^\alpha \cap N_\alpha \mid \langle \alpha, \in, A \cap \alpha^5, p, \eta \restriction \alpha\rangle \models_{N_\alpha} \psi_Q^0\}.$$

Finally, define a function $f : \kappa^\kappa \to 2^\kappa$ by letting, for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$,

$$f(\eta)(\alpha) := \begin{cases} 1, & \text{if } \alpha \in Z_\eta \text{ and } \eta_\alpha \in \mathcal{P}_{\eta,\alpha}; \\ 0, & \text{otherwise.} \end{cases}$$

Claim 3.5.2 *f is 1-Lipschitz.*

Proof Let η, ξ be two distinct elements of κ^κ. Let $\alpha \leq \Delta(\eta, \xi)$ be arbitrary.

As $\eta \restriction \alpha = \xi \restriction \alpha$, we have $\alpha \in Z_\eta$ iff $\alpha \in Z_\xi$. In addition, as $\eta \restriction \alpha = \xi \restriction \alpha$, $\mathcal{P}_{\eta,\alpha} = \mathcal{P}_{\xi,\alpha}$ whenever $\alpha \in Z_\eta$. Thus, altogether, $f(\eta)(\alpha) = 1$ iff $f(\xi)(\alpha) = 1$. □

[10] For any α such that η_α is not a function from α to α, simply replace η_α by the constant function from α to $\{0\}$.

Claim 3.5.3 *Suppose* $(\eta, \xi) \in Q$. *Then* $f(\eta) \subseteq^S f(\xi)$.

Proof As $(\eta, \xi) \in Q$, let us fix $\zeta \in \kappa^\kappa$ such that, for all $\tau < \kappa$, $(\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau) \in T$. Define a function $g : \kappa \to \kappa$ by letting, for all $\tau < \kappa$,

$$g(\tau) := \min\{\delta < \kappa \mid \ell_\delta = (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau)\}.$$

As $(S \backslash Z_\eta)$, $(S \backslash Z_\xi)$ and $(S \backslash Z_\zeta)$ are nonstationary, let us fix a club $C \subseteq \kappa$ such that $C \cap S \subseteq Z_\eta \cap Z_\xi \cap Z_\zeta$. Consider the club $D := \{\alpha \in C \mid g[\alpha] \subseteq \alpha\}$. We shall show that, for every $\alpha \in D \cap S$, if $f(\eta)(\alpha) = 1$ then $f(\xi)(\alpha) = 1$.

Fix an arbitrary $\alpha \in D \cap S$ satisfying $f(\eta)(\alpha) = 1$. In effect, the following three conditions are satisfied:

(1) $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Reflexive}}$,
(2) $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Transitive}}$, and
(3) $\langle \alpha, \in, A \cap \alpha^5, \eta_\alpha, \eta \restriction \alpha \rangle \models_{N_\alpha} \phi_Q$.

In addition, since α is a closure point of g, by definition of φ_Q, we have

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha \rangle \models \varphi_Q.$$

As $\alpha \in S$ and φ_Q is first-order,[11]

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha \rangle \models_{N_\alpha} \varphi_Q,$$

so that, by definition of ϕ_Q,

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha \rangle \models_{N_\alpha} \phi_Q.$$

By combining the preceding with clauses (2) and (3) above, we infer that the following holds, as well:

(4) $\langle \alpha, \in, A \cap \alpha^5, \eta_\alpha, \xi \restriction \alpha \rangle \models_{N_\alpha} \phi_Q$.

Altogether, $f(\xi)(\alpha) = 1$, as sought. □

Claim 3.5.4 *Suppose* $(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \backslash Q$. *Then* $f(\eta) \not\subseteq^S f(\xi)$.

Proof As $(S \backslash Z_\eta)$ and $(S \backslash Z_\xi)$ are nonstationary, let us fix a club $C \subseteq \kappa$ such that $C \cap S \subseteq Z_\eta \cap Z_\xi$. As Q is a quasi-order and $(\eta, \xi) \notin Q$, we have:

(1) $\langle \kappa, \in, A \rangle \models \psi_{\text{Reflexive}}$,
(2) $\langle \kappa, \in, A \rangle \models \psi_{\text{Transitive}}$, and
(3) $\langle \kappa, \in, A, \eta, \xi \rangle \models \neg(\phi_Q)$.

[11] N_α is transitive and rud-closed (in fact, p.r.-closed), so that $N_\alpha \models$ GJ (see [18, §Other remarks on GJ]). Now, by [18, §The cure in GJ, proposition 10.31], **Sat** is Δ_1^{GJ}.

so that, altogether,

$$\langle \kappa, \in, A, \eta, \xi \rangle \models \psi^1_Q.$$

Then, by the choice of the transversal $\langle \eta_\alpha \mid \alpha \in S \rangle$, there is a stationary subset $S' \subseteq S \cap C$ such that, for all $\alpha \in S'$:

(1') $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Reflexive}}$,
(2') $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Transitive}}$,
(3') $\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha \rangle \models_{N_\alpha} \neg(\phi_Q)$, and
(4') $\eta_\alpha = \eta \restriction \alpha$.

By Clauses (3') and (4'), we have that $\eta_\alpha \notin \mathcal{P}_{\xi,\alpha}$, so that $f(\xi)(\alpha) = 0$.
By Clauses (1'), (2') and (4'), we have that $\eta_\alpha \in \mathcal{P}_{\eta,\alpha}$, so that $f(\eta)(\alpha) = 1$.
Altogether, $\{\alpha \in S \mid f(\eta)(\alpha) > f(\xi)(\alpha)\}$ covers the stationary set S', so that $f(\eta) \not\subseteq^S f(\xi)$. □

This completes the proof of Theorem 3.5 □
Theorem B now follows as a corollary.

Corollary 3.6 *Suppose that κ is a regular uncountable cardinal and* GCH *holds. Then there is a set-size cofinality-preserving* GCH*-preserving notion of forcing $\mathbb{P}$, such that, in $V^{\mathbb{P}}$, for every analytic quasi-order Q over κ^κ and every stationary $S \subseteq \kappa$, $Q \hookrightarrow_1 \subseteq^S$.*

Proof This follows from Theorems 2.24 and 3.5, and one of the following:
- ▶ If κ is inaccessible, then we use Fact 2.13 and Lemma 2.20.
- ▶ If κ is a successor cardinal, then we use Fact 2.14 and Lemma 2.19.[12] □

Remark 3.7 By combining the proof of the preceding with a result of Lücke [17, Theorem 1.5], we arrive at following conclusion. Suppose that κ is an infinite successor cardinal and GCH holds. For every binary relation R over κ^κ, there is a set-size GCH-preserving ($<\kappa$)-closed, κ^+-cc notion of forcing $\mathbb{P}_R$ such that, in $V^{\mathbb{P}_R}$, the conclusion of Corollary 3.6 holds, and, in addition, R is analytic.

Remark 3.8 A quasi-order $\trianglelefteq$ over a space $X \in \{2^\kappa, \kappa^\kappa\}$ is said to be *Σ^1_1-complete* iff it is analytic and, for every analytic quasi-order Q over X, there exists a κ-Borel function $f : X \to X$ reducing Q to $\trianglelefteq$. As Lipschitz $\Longrightarrow$ continuous $\Longrightarrow$ κ-Borel, the conclusion of Corollary 3.6 gives that each $\subseteq^S$ is a Σ^1_1-complete quasi-order. Such a consistency was previously only known for S's of one of two specific forms, and the witnessing maps were not Lipschitz.

4 Concluding remarks

Remark 4.1 The referee asked whether the conclusions of the main theorems are also known to be false. This is indeed the case, as witnessed by the model of [10, §4], in

12 Note that in this case, $\mathbb{P}$ is moreover ($<\kappa$)-directed-closed and has the κ^+-cc.

which for any $i, j < 2$ with $i + j = 1$ there are no Borel reductions from $\subseteq^{\aleph_2 \cap \mathrm{cof}(\aleph_i)}$ to $\subseteq^{\aleph_2 \cap \mathrm{cof}(\aleph_j)}$. In a recent paper [8], we slightly improved this to get no Baire measurable reductions from $\subseteq^{\aleph_2 \cap \mathrm{cof}(\aleph_i)}$ to $\leq^{\aleph_2 \cap \mathrm{cof}(\aleph_j)}$.

Remark 4.2 By [15, Corollary 4.5], in L, for every successor cardinal κ and every theory (not necessarily complete) T over a countable relational language, the corresponding equivalence relation $\cong_T$ over 2^κ is either Δ^1_1 or Σ^1_1-complete. This dissatisfying dichotomy suggests that L is a singular universe, unsuitable for studying the correspondence between generalized descriptive set theory and model-theoretic complexities. However, using Theorem 3.5, it can be verified that the above dichotomy holds as soon as κ is a successor of an uncountable cardinal $\lambda = \lambda^{<\lambda}$ in which $\mathrm{Dl}^*_S(\Pi^1_2)$ holds for both $S := \kappa \cap \mathrm{cof}(\omega)$ and $S := \kappa \cap \mathrm{cof}(\lambda)$. This means that the dichotomy is in fact not limited to L and can be forced to hold starting with any ground model.

Remark 4.3 Let $=^S$ denote the symmetric version of $\subseteq^S$. It is well known that, in the special case $S := \kappa \cap \mathrm{cof}(\omega)$, $=^S$ is a κ-Borel* equivalence relation [19, §6]. It thus follows from Theorem 3.5 that if $\mathrm{Dl}^*_S(\Pi^1_2)$ holds for $S := \kappa \cap \mathrm{cof}(\omega)$, then the class of Σ^1_1 sets coincides with the class of κ-Borel* sets. Now, as the proof of [16, Theorem 3.1] establishes that the failure of the preceding is consistent with, e.g., $\kappa = \aleph_2 = 2^{2^{\aleph_0}}$, which in turn, by [12, Lemma 2.1], implies that $\diamondsuit^*_S$ holds, we infer that the hypothesis $\mathrm{Dl}^*_S(\Pi^1_2)$ of Theorem 3.5 cannot be replaced by $\diamondsuit^*_S$. We thus feel that we have identified the correct combinatorial principle behind a line of results that were previously obtained under the heavy hypothesis of "$V = L$".

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