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On the Cauchy problem for a modified Camassa–Holm equation

Zhaonan Luo[1] · Zhijun Qiao[2] · Zhaoyang Yin[1,3]

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Abstract
In this paper, we first study the local well-posedness for the Cauchy problem of a modified Camassa–Holm equation in nonhomogeneous Besov spaces. Then we obtain a blow-up criteria and present a blow-up result for the equation. Finally, with proving the norm inflation we show the ill-posedness occurs to the equation in critical Besov spaces.

Keywords A modified Camassa–Holm equation · Bseov spaces · Local well-posedness · Blow up · Ill-posedness

Mathematics Subject Classification 35Q53 · 35A01 · 35B44 · 35B65

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✉ Zhaoyang Yin
mcsyzy@mail.sysu.edu.cn

Zhaonan Luo
1411919168@qq.com

Zhijun Qiao
zhijun.qiao@utrgv.edu

1 Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China

2 School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Edinburg, TX 78539, USA

3 Faculty of Information Technology, Macau University of Science and Technology, Macau, China

1 Introduction

In this paper, we consider the Cauchy problem for the following modified Camassa–Holm equation:

$$\begin{cases} \gamma_t = \lambda\left(v_x - \gamma - \frac{1}{\lambda}v\gamma\right)_x, & t > 0,\ x \in \mathbb{R}, \\ v_{xx} - v = \gamma_x + \frac{\gamma^2}{2\lambda}, & t \geq 0,\ x \in \mathbb{R}, \\ \gamma(0,x) = \gamma_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

which was called by Gorka and Reyes [19]. Let $G = \partial_x^2 - 1$, $m = Gv$. Then, Eq. (1.1) can be rewritten as

$$\begin{cases} \gamma_t + G^{-1}m\gamma_x = \frac{\gamma^2}{2} + \lambda G^{-1}m - \gamma G^{-1}m_x, & t > 0,\ x \in \mathbb{R}, \\ m = \gamma_x + \frac{\gamma^2}{2\lambda}, & t \geq 0,\ x \in \mathbb{R}, \\ \gamma(0,x) = \gamma_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

The Eq. (1.1) was first studied through the geometric approach in [14,27]. Pseudo-potentials, conservation laws and the existence and uniqueness of weak solutions to the modified Camassa–Holm equation were presented in [19]. We observe that if we solve (1.2), then m will formally satisfy the following physical form of the Camassa–Holm equation [10]:

$$m_t = -2vm_x - mv_x + \lambda v_x.$$

If $\lambda = 0$, it is known as the well-known Camassa–Holm(CH) equation. It was derived as a model for shallow water waves [10,11]. The CH equation is completely integrable [6,10]and has a bi-Hamiltonian structure [4,17]. It admits peakon solitons of the form $ce^{-|x-ct|}$ with $c > 0$, which are orbitally stable [13]. The local well-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces was proved in [7,8,15,26]. It was shown that there exist finite time blow-up strong solutions and global strong solutions to the CH equation [5,7–9]. Recently, norm inflation and ill-posedness for the CH eqution in the critical Sobolev Space and Besov spaces was proved in [15,16,18]. The existence and uniqueness of global weak solutions were presented in [12,29]. The global conservative, dissipative, and algebro-geometric solutions were studied in [2,3,25].

In this paper, we investigate the local well-posedness for the Cauchy problem of a modified Camassa-Holm Eq. (1.2) in Besov spaces, present a blow-up result to (1.2) and prove norm inflation and hence ill-posedness for the equation in critical Besov spaces. This paper is organized as follows. In Sect. 2, we introduce some preliminaries which will be used in sequel. In Sect. 3, we prove the local well-posedness of (1.2) in $B^s_{p,r}$ with $s > \max(\frac{1}{2}, \frac{1}{p})$ or ($s = \frac{1}{p}$, $1 \leq p \leq 2$, $r = 1$) in the sense of Hadamard (i.e. (1.2) has a unique local solution in $B^s_{p,r}$ with continuity with respect to the initial data). The main approach is based on the Littlewood–Paley theory and transport equations theory. In Sect. 4, we present a blow-up result of the Eq. (1.2) and then prove that (1.2) is ill-posed in $H^{\frac{1}{2}}$ and in $B^{\frac{1}{2}}_{2,r}$, $1 < r \leq \infty$ by a contradiction argument.

2 Preliminaries

In this section, we will recall some propositions about the Littlewood–Paley decomposition and Besov spaces.

Proposition 2.1 [1] *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. There exist radial functions χ and φ, valued in the interval $[0, 1]$, belonging respectively to $\mathcal{D}(B(0, \frac{4}{3}))$ and $\mathcal{D}(\mathcal{C})$, and such that*

$$
\begin{aligned}
&\forall \xi \in \mathbb{R}^d,\ \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1,\\
&\forall \xi \in \mathbb{R}^d\backslash\{0\},\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1,\\
&|j-j'| \geq 2 \Rightarrow \mathrm{Supp}\ \varphi(2^{-j}\cdot) \cap \mathrm{Supp}\ \varphi(2^{-j'}\cdot) = \emptyset,\\
&j \geq 1 \Rightarrow \mathrm{Supp}\ \chi(\cdot) \cap \mathrm{Supp}\ \varphi(2^{-j}\cdot) = \emptyset.
\end{aligned}
$$

The set $\widetilde{\mathcal{C}} = B(0, \frac{2}{3}) + \mathcal{C}$ is an annulus, and we have

$$
|j-j'| \geq 5 \Rightarrow 2^j\mathcal{C} \cap 2^{j'}\widetilde{\mathcal{C}} = \emptyset.
$$

Further, we have

$$
\begin{aligned}
&\forall \xi \in \mathbb{R}^d,\ \frac{1}{2} \leq \chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1,\\
&\forall \xi \in \mathbb{R}^d\backslash\{0\},\ \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1.
\end{aligned}
$$

$\mathcal{F}$ represents the Fourier transform and its inverse is denoted by $\mathcal{F}^{-1}$. Let u be a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$. For all $j \in \mathbb{Z}$, define

$$
\begin{aligned}
\Delta_j u &= 0 \ \text{if}\ j \leq -2,\\
\Delta_{-1} u &= \mathcal{F}^{-1}(\chi \mathcal{F}u),\\
\Delta_j u &= \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}u) \ \text{if}\ j \geq 0,\\
S_j u &= \sum_{j'<j} \Delta_{j'} u.
\end{aligned}
$$

Then the Littlewood–Paley decomposition is given as follows:

$$
u = \sum_{j\in\mathbb{Z}} \Delta_j u \quad \text{in}\ \mathcal{S}'(\mathbb{R}^d).
$$

Let $s \in \mathbb{R},\ 1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ is defined by

$$B^s_{p,r} = B^s_{p,r}(\mathbb{R}^d) = \{u \in S'(\mathbb{R}^d) : \|u\|_{B^s_{p,r}(\mathbb{R}^d)} = \left\|(2^{js}\|\Delta_j u\|_{L^p})_j\right\|_{l^r(\mathbb{Z})} < \infty\}.$$

There are some properties about Besov spaces.

Proposition 2.2 [1,20] *Let $s \in \mathbb{R},\ 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$.*

(1) *$B^s_{p,r}$ is a Banach space, and is continuously embedded in $\mathcal{S}'$.*
(2) *If $r < \infty$, then $\lim_{j\to\infty}\|S_j u - u\|_{B^s_{p,r}} = 0$. If $p, r < \infty$, then C_0^∞ is dense in $B^s_{p,r}$.*
(3) *If $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B^s_{p_1,r_1} \hookrightarrow B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$. If $s_1 < s_2$, then the embedding $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ is locally compact.*
(4) *$B^s_{p,r} \hookrightarrow L^\infty \Leftrightarrow s > \frac{d}{p}$ or $s = \frac{d}{p},\ r = 1$.*
(5) *Fatou property: if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $B^s_{p,r}$, then an element $u \in B^s_{p,r}$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ exist such that*

$$\lim_{k\to\infty} u_{n_k} = u \text{ in } \mathcal{S}' \quad \text{and} \quad \|u\|_{B^s_{p,r}} \leq C \liminf_{k\to\infty} \|u_{n_k}\|_{B^s_{p,r}}.$$

(6) *Let $m \in \mathbb{R}$ and f be a S^m-mutiplier (i.e. f is a smooth function and satisfies that $\forall \alpha \in \mathbb{N}^d$, $\exists C = C(\alpha)$, such that $|\partial^\alpha f(\xi)| \leq C(1+|\xi|)^{m-|\alpha|}$, $\forall \xi \in \mathbb{R}^d$). Then the operator $f(D) = \mathcal{F}^{-1}(f\mathcal{F})$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

We introduce two useful interpolation inequalities.

Proposition 2.3 [1,20] *(1) If $s_1 < s_2$, $\theta \in (0,1)$, and (p, r) is in $[1,\infty]^2$, then we have*

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|^\theta_{B^{s_1}_{p,r}} \|u\|^{1-\theta}_{B^{s_2}_{p,r}}.$$

(2) If $s \in \mathbb{R},\ 1 \leq p \leq \infty,\ \varepsilon > 0$, a constant $C = C(\varepsilon)$ exists such that

$$\|u\|_{B^s_{p,1}} \leq C\|u\|_{B^s_{p,\infty}} \ln\Big(e + \frac{\|u\|_{B^{s+\varepsilon}_{p,\infty}}}{\|u\|_{B^s_{p,\infty}}}\Big).$$

Proposition 2.4 [1] *Let $s \in \mathbb{R},\ 1 \leq p, r \leq \infty$.*

$$\begin{cases} B^s_{p,r} \times B^{-s}_{p',r'} \longrightarrow \mathbb{R}, \\ (u, \phi) \longmapsto \sum\limits_{|j-j'|\leq 1} \langle \Delta_j u, \Delta_{j'}\phi\rangle, \end{cases}$$

defines a continuous bilinear functional on $B^s_{p,r} \times B^{-s}_{p',r'}$. Denote by $Q^{-s}_{p',r'}$ the set of functions ϕ in $\mathcal{S}'$ such that $\|\phi\|_{B^{-s}_{p',r'}} \leq 1$. If u is in $\mathcal{S}'$, then we have

$$\|u\|_{B^s_{p,r}} \leq C \sup_{\phi \in Q^{-s}_{p',r'}} \langle u, \phi\rangle.$$

We then have the following product laws:

Lemma 2.5 [1,20] (1) *For any* $s > 0$ *and any* (p, r) *in* $[1, \infty]^2$*, the space* $L^\infty \cap B^s_{p,r}$ *is an algebra, and a constant* $C = C(s, d)$ *exists such that*

$$\|uv\|_{B^s_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|u\|_{B^s_{p,r}}\|v\|_{L^\infty}).$$

(2) *If* $1 \leq p, r \leq \infty$, $s_1 \leq s_2$, $s_2 > \frac{d}{p}$ $(s_2 \geq \frac{d}{p}$ *if* $r = 1)$ *and* $s_1+s_2 > \max(0, \frac{2d}{p}-d)$*, there exists* $C = C(s_1, s_2, p, r, d)$ *such that*

$$\|uv\|_{B^{s_1}_{p,r}} \leq C\|u\|_{B^{s_1}_{p,r}}\|v\|_{B^{s_2}_{p,r}}.$$

(3) *If* $1 \leq p \leq 2$*, there exists* $C = C(p, d)$ *such that*

$$\|uv\|_{B^{\frac{d}{p}-d}_{p,\infty}} \leq C\|u\|_{B^{\frac{d}{p}-d}_{p,\infty}}\|v\|_{B^{\frac{d}{p}}_{p,1}}.$$

We now state the so-called Osgood lemma, a generalization of the Gronwall lemma.

Lemma 2.6 [1] *Let* ρ *be a measurable function from* $[t_0, T]$ *to* $[0, a]$*,* γ *a locally integrable function from* $[t_0, T]$ *to* $\mathbb{R}^+$*, and* μ *a continuous and nondecreasing function from* $[0, a]$ *to* $\mathbb{R}^+$*. Assume that for some* $c \geq 0$*, the function* ρ *satisfies*

$$\rho(t) \leq c + \int_{t_0}^{t} \gamma(t')\mu(\rho(t'))dt' \ \text{ for a.e. } t \in [t_0, T].$$

If $c > 0$*, then for a.e.* $t \in [t_0, T]$*,*

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(c) \leq \int_{t_0}^{t} \gamma(t')dt' \ \text{ with } \ \mathcal{M}(x) = \int_x^a \frac{dr}{\mu(r)}.$$

If $c = 0$*, and* μ *satisfies* $\int_0^a \frac{dr}{\mu(r)} = \infty$*, then* $\rho = 0$*, a.e.*

Remark 2.7 [21] For example, when $\mu(r) = r(1-\ln r)$, $r \in [0, 1]$, we have $\mathcal{M}(x) = \ln(1 - \ln x)$, and $\rho(t) \leq ec^{\exp - \int_{t_0}^{t} \gamma(t')dt'}$, if $c > 0$. We will use this result later.

Now we state some useful estimates in the study of transport equations, which are crucial to the proofs of our main theorem later.

$$\begin{cases} f_t + v \cdot \nabla f = g, \ x \in \mathbb{R}^d, \ t > 0, \\ f(0, x) = f_0(x). \end{cases} \tag{2.1}$$

Lemma 2.8 [1,23] *Let* $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$*. There exists a constant* C *such that for all solutions* $f \in L^\infty([0, T]; B^s_{p,r})$ *of (2.1) in one dimension with initial data* f_0 *in* $B^s_{p,r}$*, and* g *in* $L^1([0, T]; B^s_{p,r})$*, we have, for a.e.* $t \in [0, T]$*,*

$$\|f(t)\|_{B^s_{p,r}} \leq e^{CV(t)}\Big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(t')}\|g(t')\|_{B^s_{p,r}}dt'\Big)$$

with

$$V'(t) = \begin{cases} \|\nabla v\|_{B^{s+1}_{p,r}}, \text{ if } s > \max(-\frac{1}{2}, \frac{1}{p}-1), \\ \|\nabla v\|_{B^s_{p,r}}, \text{ if } s > \frac{1}{p} \text{ or } (s = \frac{1}{p},\ p < \infty,\ r = 1), \\ \|\nabla v\|_{B^{\frac{1}{p}}_{p,1}}, \text{ if } s = \frac{1}{p} - 1, 1 \leq p \leq 2, r = \infty, \end{cases}$$

and when $s = \frac{1}{p} - 1,\ 1 \leq p \leq 2,\ r = \infty,$ *and* $V'(t) = \|\nabla v\|_{B^{\frac{1}{p}}_{p,1}}$.

Lemma 2.9 [24] *Let* $s > 0,\ 1 \leq p, r \leq \infty$. *Define* $R_j = [v \cdot \nabla, \Delta_j]f$. *There exists a constant* C *such that*

$$\Big\|(2^{js}\|R_j\|_{L^p})_j\Big\|_{l^r(\mathbb{Z})} \leq C(\|\nabla v\|_{L^\infty}\|f\|_{B^s_{p,r}} + \|\nabla v\|_{B^s_{p,r}}\|f\|_{L^\infty}).$$

Hence, if f *solves the equation (2.1), we have*

$$\|f(t)\|_{B^s_{p,r}} \leq \|f_0\|_{B^s_{p,r}} + C\int_0^t (\|\nabla v\|_{L^\infty}\|f\|_{B^s_{p,r}} + \|\nabla v\|_{B^s_{p,r}}\|f\|_{L^\infty} + \|g\|_{B^s_{p,r}}dt').$$

Lemma 2.10 [1,23] *Let* $1 \leq p \leq p_1 \leq \infty,\ 1 \leq r \leq \infty,\ s > -d\min(\frac{1}{p_1}, \frac{1}{p'})$. *Let* $f_0 \in B^s_{p,r}$, $g \in L^1([0,T]; B^s_{p,r})$, *and let* v *be a time-dependent vector field such that* $v \in L^\rho([0,T]; B^{-M}_{\infty,\infty})$ *for some* $\rho > 1$ *and* $M > 0$, *and*

$$\begin{array}{ll} \nabla v \in L^1([0,T]; B^{\frac{d}{p_1}}_{p_1,\infty} \bigcap L^\infty), & \text{if } s < 1 + \frac{d}{p_1}, \\ \nabla v \in L^1([0,T]; B^{1+\frac{d}{p}}_{p,r}), & \text{if } s = 1 + \frac{d}{p},\ r > 1, \\ \nabla v \in L^1([0,T]; B^{s-1}_{p_1,r}), & \text{if } s > 1 + \frac{d}{p_1} \text{ or } (s = 1 + \frac{d}{p_1} \text{ and } r = 1). \end{array}$$

Then the equation (2.1) has a unique solution f *in*
-the space $C([0,T]; B^s_{p,r})$, *if* $r < \infty$,
-the space $\Big(\bigcap_{s'<s} C([0,T]; B^{s'}_{p,\infty})\Big) \bigcap C_w([0,T]; B^s_{p,\infty})$, *if* $r = \infty$.

Lemma 2.11 [22] *Let* $1 \leq p \leq \infty,\ 1 \leq r < \infty,\ s > \frac{d}{p}$ *(or* $s = \frac{d}{p},\ p < \infty,\ r = 1$*). Denote* $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. *Let* $(v^n)_{n\in\bar{\mathbb{N}}} \in C([0,T]; B^{s+1}_{p,r})$. *Assume that* $(f^n)_{n\in\bar{\mathbb{N}}}$ *in* $C([0,T]; B^s_{p,r})$ *is the solution to*

$$\begin{cases} f^n_t + v^n \cdot \nabla f^n = g,\ x \in \mathbb{R}^d,\ t > 0, \\ f^n(0,x) = f_0(x) \end{cases} \tag{2.2}$$

with initial data $f_0 \in B^s_{p,r}$, $g \in L^1([0,T]; B^s_{p,r})$ *and that for some* $\alpha \in L^1([0,T])$, $\sup_{n\in\bar{\mathbb{N}}} \|v^n(t)\|_{B^{s+1}_{p,r}} \le \alpha(t)$. *If* $v^n \to v^\infty$ *in* $L^1([0,T]; B^s_{p,r})$, *then* $f^n \to f^\infty$ *in* $C([0,T]; B^s_{p,r})$.

3 Local well-posedness

In this section, we will investigate the local well-posedness for (1.2) in Besov spaces.We introduce the following function spaces.

Definition 3.1 Let $T > 0$, $s \in \mathbb{R}$, and $1 \le p, r \le \infty$. Set

$$E^s_{p,r}(T) \triangleq \begin{cases} C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}), & \text{if } r < \infty, \\ C_w([0,T]; B^s_{p,\infty}) \cap C^{0,1}([0,T]; B^{s-1}_{p,\infty}), & \text{if } r = \infty. \end{cases}$$

In this section, our main theorem is stated as follows.

Theorem 3.2 *Let* $1 \le p, r \le \infty$, $s \in \mathbb{R}$ *and let* (s, p, r) *satisfy the condition* $s > \max(\frac{1}{2}, \frac{1}{p})$ *or* $(s = \frac{1}{p},\ 1 \le p \le 2,\ r = 1)$. *Assume that* $\gamma_0 \in B^s_{p,r}$. *Then there exists a time* $T > 0$ *such that (1.2) has a unique solution* γ *in* $E^s_{p,r}(T)$. *Moreover the solution depends continuously on the initial data.*

We divide it into six steps to prove Theorem 3.2.

Step one: Constructing approximate solutions.
We starts from $\gamma^0 \triangleq 0$, and define a sequence $(\gamma^n)_{n\in\mathbb{N}}$ of smooth functions by solving the following linear transport equations:

$$\begin{cases} \gamma^{n+1}_t + G^{-1}m^n\gamma^{n+1}_x = \frac{(\gamma^n)^2}{2} + \lambda G^{-1}m^n - \gamma G^{-1}m^n_x, \\ m^n = \gamma^n_x + \frac{(\gamma^n)^2}{2\lambda}, \\ \gamma^{n+1}(0,x) = S_{n+1}\gamma_0. \end{cases} \tag{3.1}$$

Define $G^n = G^{-1}m^n$, $F^n = \frac{(\gamma^n)^2}{2} + \lambda G^{-1}m^n - \gamma G^{-1}m^n_x$. Assume that $\gamma_n \in L^\infty([0,T]; B^s_{p,r})$ for all $T > 0$. Note that under the assumptions on (s, p, r), $B^s_{p,r}$ is an algebra. We have

$$\begin{aligned} \|G^n_x\|_{B^s_{p,r}} &\le C\|m^n\|_{B^{s-1}_{p,r}} \\ &\le C\left(\|\gamma^n\|_{B^s_{p,r}} + \|\gamma^n\|^2_{B^s_{p,r}}\right). \end{aligned} \tag{3.2}$$

$$\begin{aligned} \|F^n\|_{B^s_{p,r}} &\le C\left(\|\gamma^n\|^2_{B^s_{p,r}} + \|G^{-1}m^n\|_{B^s_{p,r}} + \|\gamma^n\|_{B^s_{p,r}}\|G^{-1}m^n_x\|_{B^s_{p,r}}\right) \\ &\le C\left(\|\gamma^n\|_{B^s_{p,r}} + \|\gamma^n\|^2_{B^s_{p,r}} + \|\gamma^n\|^3_{B^s_{p,r}}\right). \end{aligned} \tag{3.3}$$

Therefore G^n_x, $F^n \in L^\infty([0,T]; B^s_{p,r})$. Hence, applying Lemma 2.10 ensures that (3.1) has a global solution γ^{n+1} which belongs to $E^s_{p,r}(T)$ for all $T > 0$.

Step two: Uniform bounds.
Define $R_n = \|\gamma^n(t)\|_{B^s_{p,r}}$.Using Lemma 2.8 together with (3.2) and (3.3), we have

$$R_{n+1} \le e^{C\int_0^t \|G^n_x\|_{B^s_{p,r}}dt'}\Big(\|S_{n+1}\gamma_0\|_{B^s_{p,r}} + \int_0^t e^{-C\int_0^{t'}\|G^n_x\|_{B^s_{p,r}}dt''}\|F^n\|_{B^s_{p,r}}dt'\Big)$$
$$\le Ce^{C\int_0^t R_n+R_n^2 dt'}\Big(\|\gamma_0\|_{B^s_{p,r}} + \int_0^t e^{-C\int_0^{t'} R_n+R_n^2 dt''}(R_n+R_n^2+R_n^3)dt'\Big). \tag{3.4}$$

The case where $\|\gamma_0\|_{B^s_{p,r}} = 0$ is trivial, we start with the case where $\|\gamma_0\|_{B^s_{p,r}} \neq 0$. We have known that $R_0 = 0$. Fix a $T > 0$ such that $4C^3T\|\gamma_0\|^2_{B^s_{p,r}} < 1$ and suppose that

$$\forall t \in [0,T],\ \ R_n \le \frac{C\|\gamma_0\|_{B^s_{p,r}}}{2(1-4C^3t\|\gamma_0\|^2_{B^s_{p,r}})^{\frac{1}{2}}}. \tag{3.5}$$

Pluge (3.5) into (3.4) and choose $C \ge 2\|\gamma_0\|_{B^s_{p,r}}$. After a simple calculation we derive

$$R_{n+1} \le C\|\gamma_0\|_{B^s_{p,r}}(1-4C^3t\|\gamma_0\|^2_{B^s_{p,r}})^{-\frac{1}{4}}\Big(1 + C^3\|\gamma_0\|^2_{B^s_{p,r}}\int_0^t (1$$
$$-4C^3t'\|\gamma_0\|^2_{B^s_{p,r}})^{-\frac{5}{4}}dt'\Big)$$
$$\le \frac{C\|\gamma_0\|_{B^s_{p,r}}}{2(1-4C^3t\|\gamma_0\|^2_{B^s_{p,r}})^{\frac{1}{2}}}.$$

Therefore, $(\gamma^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T];B^s_{p,r})$.

Step three: Cauchy sequence.
When $s > \max(\frac{1}{2},\frac{1}{p})$ or $(s = \frac{1}{p},\ 1 \le p \le 2,\ r = 1)$, some estimates we need are a little different, so we have to discuss separately.

Case 1 $s > \max(\frac{1}{2},\frac{1}{p})$.

We are going to prove that $(\gamma^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty([0,T];B^{s-1}_{p,r})$. For that purpose, for all $(n,k) \in \mathbb{N}^2$, we have

$$(\gamma^{n+k+1}-\gamma^{n+1})_t + G^{n+k}(\gamma^{n+k+1}-\gamma^{n+1})_x = \Big(G^n - G^{n+k}\Big)\gamma^{n+1}_x + F^{n+k} - F^n,$$

where

$$G^n - G^{n+k} = G^{-1}(\gamma^n-\gamma^{n+k})_x - \frac{1}{2\lambda}G^{-1}\Big((\gamma^n)^2 - \Big(\gamma^{n+k}\Big)^2\Big),$$
$$F^{n+k}-F^n = \frac{1}{2}\Big(\gamma^{n+k}-\gamma^n\Big)\Big(\gamma^{n+k}+\gamma^n\Big) + \lambda G^{-1}\Big(\gamma^{n+k}-\gamma^n\Big) + \frac{1}{2}G^{-1}\Big((\gamma^n)^2$$
$$-\Big(\gamma^{n+k}\Big)^2\Big) - \gamma^{n+k}\Big(\partial_x G^{-1}m^{n+k} - \partial_x G^{-1}m^n\Big) - (\gamma^{n+k}-\gamma^n)\partial_x G^{-1}m^n.$$

Applying Lemma 2.8, for any t in $[0, T]$, we get

$$\|(\gamma^{n+k+1}-\gamma^{n+1})(t)\|_{B^{s-1}_{p,r}} \le e^{C\int_0^t \|G^{n+k}_x\|_{B^s_{p,r}}dt'}\Big(\|S_{n+k+1}\gamma_0 - S_{n+1}\gamma_0\|_{B^{s-1}_{p,r}} + \int_0^t e^{-C\int_0^{t'}\|G^{n+k}_x\|_{B^s_{p,r}}dt''}\Big(\|\left(G^n - G^{n+k}\right)m^{n+1}_x\|_{B^{s-1}_{p,r}} + \|F^{n+k}-F^n\|_{B^{s-1}_{p,r}}\Big)dt'\Big). \tag{3.6}$$

Using the fact $B^s_{p,r}$ is an algebra and applying Lemma 2.5 (2), we have

$$\|\left(G^n - G^{n+k}\right)\gamma^{n+1}_x\|_{B^s_{p,r}} \le C\|\gamma^{n+1}\|_{B^s_{p,r}}\left(1+\|\gamma^n\|_{B^s_{p,r}}\|\gamma^{n+k}\|_{B^s_{p,r}}\right)\|\gamma^{n+k}-\gamma^n\|_{B^{s-1}_{p,r}}, \tag{3.7}$$

and

$$\|F^{n+k}-F^n\|_{B^{s-1}_{p,r}} \le C\left(1+\|\gamma^{n+k}\|_{B^s_{p,r}} + \|\gamma^{n+k}\|_{B^s_{p,r}}\|\gamma^n\|_{B^s_{p,r}} + \|\gamma^{n+k}\|^2_{B^s_{p,r}} + \|\gamma^n\|_{B^s_{p,r}} + \|\gamma^n\|^2_{B^s_{p,r}}\right)\|\gamma^{n+k}-\gamma^n\|_{B^{s-1}_{p,r}}. \tag{3.8}$$

Since $(\gamma^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T]; B^s_{p,r})$ for all t in $[0, T]$, we finally get

$$\|\left(\gamma^{n+k+1}-\gamma^{n+1}\right)(t)\|_{B^{s-1}_{p,r}} \le C\Big(\|S_{n+k+1}\gamma_0 - S_{n+1}\gamma_0\|_{B^{s-1}_{p,r}} + \int_0^t \|\gamma^{n+k}-\gamma^n\|_{B^{s-1}_{p,r}}dt'\Big).$$

Taking an upper bound on $[0, t]$, we have

$$\|\gamma^{n+k+1}-\gamma^{n+1}\|_{L^\infty_t(B^{s-1}_{p,r})} \le C\Big(\|S_{n+k+1}\gamma_0 - S_{n+1}\gamma_0\|_{B^{s-1}_{p,r}} + \int_0^t \|\gamma^{n+k}-\gamma^n\|_{L^\infty_{t'}(B^{s-1}_{p,r})}dt'\Big). \tag{3.9}$$

Let $g_n(t) = \sup_k \|\gamma^{n+k}-\gamma^n\|_{L^\infty_t(B^{s-1}_{p,r})}$. Then (3.9) becomes

$$g_{n+1}(t) \le C\Big(\sup_k \|S_{n+k+1}\gamma_0 - S_{n+1}\gamma_0\|_{B^{s-1}_{p,r}} + \int_0^t g_n(t')dt'\Big).$$

Since $(S_n\gamma^0)_{n\in\mathbb{N}}$ is a Cauchy sequence in $B^{s-1}_{p,r}$, applying Fatou's lemma, we have

$$g(t) \triangleq \limsup_{n\to\infty} g_{n+1}(t) \le C\int_0^t g(t')dt'.$$

The Gronwall lemma implies that $g(t) = 0$ for all $t \in [0, T]$. Therefore $(\gamma^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; B^{s-1}_{p,r})$ and converges to some limit function $\gamma \in C([0,T]; B^{s-1}_{p,r})$.

Case 2 $s = \frac{1}{p},\ 1 \le p \le 2,\ r = 1.$

From Lemma 2.8, comparing with Case 1, we do not have the estimate for the norm $B^{s-1}_{p,1}$ but $B^{s-1}_{p,\infty}$. In fact, we only have

$$\|(\gamma^{n+k+1} - \gamma^{n+1})(t)\|_{B^{\frac{1}{p}-1}_{p,\infty}} \le e^{C\int_0^t \|G_x^{n+k}\|_{B^{\frac{1}{p}}_{p,1}} dt'} \Big(\|S_{n+k+1}\gamma_0 - S_{n+1}\gamma_0\|_{B^{\frac{1}{p}-1}_{p,\infty}}$$
$$+ \int_0^t e^{-C\int_0^{t'} \|G_x^{n+k}\|_{B^{\frac{1}{p}}_{p,1}} dt''} (\|(G^n - G^{n+k})\gamma_x^{n+1}\|_{B^{\frac{1}{p}-1}_{p,\infty}} + \|F^{n+k} - F^n\|_{B^{\frac{1}{p}-1}_{p,\infty}}) dt'\Big). \tag{3.10}$$

Applying Lemma 2.5 (3), we deduce that

$$\|(G^n - G^{n+k})\gamma_x^{n+1}\|_{B^{\frac{1}{p}-1}_{p,\infty}} \le C\|\gamma^{n+1}\|_{B^{\frac{1}{p}}_{p,1}} (1 + \|\gamma^n\|_{B^{\frac{1}{p}}_{p,1}} + \|\gamma^{n+k}\|_{B^{\frac{1}{p}}_{p,1}})\|\gamma^{n+k} - \gamma^n\|_{B^{\frac{1}{p}-1}_{p,1}}. \tag{3.11}$$

$$\|F^{n+k} - F^n\|_{B^{\frac{1}{p}-1}_{p,\infty}} \le C(1 + \|\gamma^n\|_{B^{\frac{1}{p}}_{p,1}} + \|\gamma^{n+k}\|_{B^{\frac{1}{p}}_{p,1}} + \|\gamma^{n+k}\|^2_{B^{\frac{1}{p}}_{p,1}}$$
$$+ \|\gamma^n\|^2_{B^{\frac{1}{p}}_{p,1}} + \|\gamma^n\|_{B^{\frac{1}{p}}_{p,1}} \|\gamma^{n+k}\|_{B^{\frac{1}{p}}_{p,1}})\|\gamma^{n+k} - \gamma^n\|_{B^{\frac{1}{p}-1}_{p,1}}. \tag{3.12}$$

Plugging (3.11), (3.12) into (3.10), and using uniform bounds of $(\gamma^n)_{n\in\mathbb{N}}$, we have

$$\|(\gamma^{n+k+1} - \gamma^{n+1})(t)\|_{B^{\frac{1}{p}-1}_{p,\infty}}$$
$$\le C\Big(\|S_{n+k+1}\gamma_0 - S_{n+1}\gamma_0\|_{B^{\frac{1}{p}}_{p,1}} + \int_0^t \|\gamma^{n+k} - \gamma^n\|_{B^{\frac{1}{p}-1}_{p,1}} dt'\Big). \tag{3.13}$$

Applying Proposition 2.3 (2), we find

$$\|\gamma^{n+k} - \gamma^n\|_{B^{\frac{1}{p}-1}_{p,1}}$$
$$\le C\|\gamma^{n+k} - \gamma^n\|_{B^{\frac{1}{p}-1}_{p,\infty}} \ln\Big(e + \frac{\|\gamma^{n+k} - \gamma^n\|_{B^{\frac{1}{p}}_{p,1}}}{\|\gamma^{n+k} - \gamma^n\|_{B^{\frac{1}{p}-1}_{p,\infty}}}\Big). \tag{3.14}$$

Since the function $x\ln(e+\frac{C}{x})$ is nondecreasing in $(0,\infty)$, from (3.13) and (3.14), we have

$$\|\gamma^{n+k+1}-\gamma^{n+1}\|_{L_t^\infty(B_{p,\infty}^{\frac{1}{p}-1})}\leq C\Big(\|S_{n+k+1}\gamma_0-S_{n+1}\gamma_0\|_{B_{p,1}^{\frac{1}{p}-1}}$$
$$+\int_0^t\|\gamma^{n+k}-\gamma^n\|_{L_{t'}^\infty(B_{p,\infty}^{\frac{1}{p}-1})}\ln\Big(e+\frac{C}{\|\gamma^{n+k}-\gamma^n\|_{L_{t'}^\infty(B_{p,\infty}^{\frac{1}{p}-1})}}\Big)dt'\Big).$$

Let $g(t)\triangleq\limsup_{n\to\infty}\sup_k\|\gamma^{n+k}-\gamma^n\|_{L_t^\infty(B_{p,\infty}^{\frac{1}{p}-1})}$. The above inequality can be written as

$$g(t)\leq C\int_0^t g(t')\ln\left(e+\frac{C}{g(t')}\right)dt'.$$

Hence Lemma 2.6 implies that $g(t)\equiv 0$, and $(\gamma^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B_{p,\infty}^{\frac{1}{p}-1})$ and converges to some limit function γ in $C([0,T];B_{p,\infty}^{\frac{1}{p}-1})$.

Step four Convergence.

We have to prove that γ belongs to $E_{p,r}^s(T)$ and satisfies (1.2). Since $(\gamma^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T];B_{p,r}^s)$, we can apply the Fatou property for the Besov spaces to show that γ also belongs to $L^\infty([0,T];B_{p,r}^s)$. Now, applying interpolation inequalities implies that $(\gamma^n)_{n\in\mathbb{N}}$ converges to γ in $C([0,T];B_{p,r}^{s'})$ for any $s'<s$. Then it is easy to pass to the limit in (3.1) and to conclude that γ is indeed a solution of (1.2) in the sense of distributions.

Finally, since γ belongs to $L^\infty([0,T];B_{p,r}^s)$, the right-hand side of (1.2) also belongs to $L^\infty([0,T];B_{p,r}^s)$. According to Lemma 2.10, we can deduce that γ belongs to $C([0,T];B_{p,r}^s)$ (resp., $C_w([0,T];B_{p,r}^s)$) if $r<\infty$ (resp., $r=\infty$). Again using the equation (1.2), we prove that γ_t is in $C([0,T];B_{p,r}^{s-1})$ if r is finite, and in $L^\infty([0,T];B_{p,r}^{s-1})$ otherwise. Hence, γ belongs to $E_{p,r}^s(T)$.

Step five Uniqueness.

Then, we will prove the uniqueness of solutions to (1.2). The proof follows almost exactly the proofs which we use in Step 3. Suppose that γ_1,γ_2 are two solutions of (1.2). We obtain

$$\partial_t(\gamma_1-\gamma_2)+G_1\partial_x(\gamma_1-\gamma_2)=(G_2-G_1)\partial_x\gamma_2+F_1-F_2,$$

where for $i=1,2$,

$$m_i=\partial_x\gamma_i+\frac{\gamma_i^2}{2\lambda},\ G_i=G^{-1}m_i,\ F_i=\frac{1}{2}\gamma_i^2+\lambda G^{-1}m_i-\gamma_i\partial_xG^{-1}m_i.$$

Case 1 $s>\max(\frac{1}{2},\frac{1}{p})$.

Applying Lemma 2.8, we get

$$\|(\gamma_1-\gamma_2)(t)\|_{B^{s-1}_{p,r}} \le e^{C\int_0^t \|\partial_x G_1\|_{B^s_{p,r}}dt'}\Big(\|(\gamma_1-\gamma_2)(0)\|_{B^{s-1}_{p,r}} + \int_0^t e^{-C\int_0^{t'} \|\partial_x G_1\|_{B^s_{p,r}}dt''}(\|(G_2-G_1)\partial_x\gamma_2\|_{B^{s-1}_{p,r}}+\|F_1-F_2\|_{B^{s-1}_{p,r}})dt'\Big). \tag{3.15}$$

After a similar calculation as in Step 3, we have

$$\|(G_2-G_1)\partial_x\gamma_2\|_{B^{s-1}_{p,r}} \le C\|\gamma_1-\gamma_2\|_{B^{s-1}_{p,r}}(1+\|\gamma_1\|^2_{B^s_{p,r}}+\|\gamma_2\|^2_{B^s_{p,r}}), \tag{3.16}$$

$$\|F_1-F_2\|_{B^{s-1}_{p,r}} \le C\|\gamma_1-\gamma_2\|_{B^{s-1}_{p,r}}(1+\|\gamma_1\|^2_{B^s_{p,r}}+\|\gamma_2\|^2_{B^s_{p,r}}). \tag{3.17}$$

Plugging (3.16), (3.17) into (3.15) yields that

$$\|(\gamma_1-\gamma_2)(t)\|_{B^{s-1}_{p,r}} \le e^{C\int_0^t (\|\gamma_1\|^2_{B^s_{p,r}}+1)dt'}\Big(\|(\gamma_1-\gamma_2)(0)\|_{B^{s-1}_{p,r}} + C\int_0^t e^{-C\int_0^{t'} (\|\gamma_1\|^2_{B^s_{p,r}}+1)dt''}(1+\|\gamma_1\|^2_{B^s_{p,r}}+\|\gamma_2\|^2_{B^s_{p,r}})\|\gamma_1-\gamma_2\|_{B^{s-1}_{p,r}}dt'\Big). \tag{3.18}$$

Appling Gronwall's inequality, we finally get

$$\|\gamma_1(t)-\gamma_2(t)\|_{B^{s-1}_{p,r}} \le \|\gamma_1(0)-\gamma_2(0)\|_{B^{s-1}_{p,r}} e^{C\int_0^t (1+\|\gamma_1\|^2_{B^s_{p,r}}+\|\gamma_2\|^2_{B^s_{p,r}})dt'}. \tag{3.19}$$

Case 2 $s=\frac{1}{p},\ 1\le p\le 2,\ r=1.$
According to Lemma 2.8, we get

$$\|(\gamma_1-\gamma_2)(t)\|_{B^{\frac{1}{p}-1}_{p,\infty}} \le e^{C\int_0^t \|\partial_x G_1\|_{B^{\frac{1}{p}}_{p,1}}dt'}\Big(\|(\gamma_1-\gamma_2)(0)\|_{B^{\frac{1}{p}-1}_{p,\infty}} + \int_0^t e^{-C\int_0^{t'} \|\partial_x G_1\|_{B^{\frac{1}{p}}_{p,1}}dt''}(\|(G_2-G_1)\partial_x\gamma_2\|_{B^{\frac{1}{p}-1}_{p,\infty}}+\|F_1-F_2\|_{B^{\frac{1}{p}-1}_{p,\infty}})dt'\Big). \tag{3.20}$$

Similarly, we deduce that

$$\|(G_2-G_1)\partial_x\gamma_2\|_{B^{\frac{1}{p}-1}_{p,\infty}} \le C(1+\|\gamma_1\|^2_{B^{\frac{1}{p}}_{p,\infty}}+\|\gamma_2\|^2_{B^{\frac{1}{p}}_{p,\infty}})\|\gamma_1-\gamma_2\|_{B^{\frac{1}{p}-1}_{p,1}}, \tag{3.21}$$

$$\|F_1-F_2\|_{B^{\frac{1}{p}-1}_{p,\infty}} \le C(1+\|\gamma_1\|^2_{B^{\frac{1}{p}}_{p,\infty}}+\|\gamma_2\|^2_{B^{\frac{1}{p}}_{p,\infty}})\|\gamma_1-\gamma_2\|_{B^{\frac{1}{p}-1}_{p,\infty}}. \tag{3.22}$$

Plugging (3.21), (3.22) into (3.20), and using the uniform bounds of γ_i, we have

$$\|(\gamma_1-\gamma_2)(t)\|_{B^{\frac{1}{p}-1}_{p,\infty}} \le C\Big(\|(\gamma_1-\gamma_2)(0)\|_{B^{\frac{1}{p}}_{p,\infty}} + \int_0^t \|\gamma_1-\gamma_2\|_{B^{\frac{1}{p}-1}_{p,1}} dt'\Big). \quad (3.23)$$

Applying Proposition 2.3 (2), it follows that

$$\begin{aligned}\|(\gamma_1-\gamma_2)(t)\|_{B^{\frac{1}{p}-1}_{p,\infty}} \le C\Big(&\|(\gamma_1-\gamma_2)(0)\|_{B^{\frac{1}{p}-1}_{p,\infty}}\\ &+\int_0^t \|\gamma_1-\gamma_2\|_{B^{\frac{1}{p}-1}_{p,\infty}} \ln\Big(e+\frac{C}{\|\gamma_1-\gamma_2\|_{B^{\frac{1}{p}-1}_{p,\infty}}}\Big)dt'\Big).\end{aligned} \quad (3.24)$$

Now let $h(t)=\|(\gamma_1-\gamma_2)(t)\|_{B^{\frac{1}{p}-1}_{p,\infty}}$. From above, h satisfies

$$\begin{aligned}h(t) &\le C\Big(h(0)+\int_0^t h(t')\ln\Big(e+\frac{C}{h(t')}\Big)dt'\Big)\\ &\le C\Big(h(0)+\int_0^t h(t')\Big(1-\ln h(t')\Big)dt'\Big).\end{aligned}$$

By virtue of Remark 2.7, we finally get

$$\|\gamma_1(t)-\gamma_2(t)\|_{B^{\frac{1}{p}-1}_{p,\infty}} \le C\|\gamma_1(0)-\gamma_2(0)\|^{e^{-Ct}}_{B^{\frac{1}{p}-1}_{p,\infty}}. \quad (3.25)$$

Therefore, the uniqueness is obvious in view of (3.19) and (3.25). Moreover, an interpolation argument ensures that the continuity with respect to the initial data holds for the norm $C([0,T];B^{s'}_{p,r})$ whenever $s'<s$.

Step six Continuity with respect to the initial data.
Finally, we end up with a proposition about continuity until the exponent s.

Proposition 3.3 *Let (s,p,r) be the statement of Theorem 3.2. Denote $\bar{\mathbb{N}}=\mathbb{N}\cup\{\infty\}$. Suppose that $(\gamma^n)_{n\in\bar{\mathbb{N}}}$ is the corresponding solution to (1.2) given by Theorem 3.2 with the initial data $\gamma_0^n\in B^s_{p,r}$. If $\gamma_0^n\to\gamma_0^\infty$ in $B^s_{p,r}$, then $\gamma^n\to\gamma^\infty$ in $C([0,T];B^s_{p,r})$ (resp., $C_w([0,T];B^s_{p,r})$) if $r<\infty$ (resp., $r=\infty$) with $4C^3T\sup_{n\in\bar{\mathbb{N}}}\|\gamma_0^n\|^2_{B^s_{p,r}}<1$.*

Proof According to the proof of the existence, we find for all $n\in\bar{\mathbb{N}},\ t\in[0,T]$,

$$\|\gamma^n(t)\|_{B^s_{p,r}} \le \frac{C\|\gamma_0^n\|_{B^s_{p,r}}}{2\Big(1-4C^3t\|\gamma_0^n\|^2_{B^s_{p,r}}\Big)^{\frac{1}{4}}}.$$

Then $(\gamma^n)_{n\in\bar{\mathbb{N}}}$ is bounded in $L^\infty([0,T];B^s_{p,r})$. We split $\gamma^n=y^n+z^n$ with (y^n,z^n) satisfying

$$\begin{cases} y^n_t+G^ny^n_x=F^\infty, \\ y^n|_{t=0}=\gamma_0^\infty, \end{cases} \quad \text{and} \quad \begin{cases} z^n_t+G^nz^n_x=F^n-F^\infty, \\ z^n|_{t=0}=\gamma_0^n-\gamma_0^\infty. \end{cases}$$

Obviously we have

$$\|G^n\|_{B^{s+1}_{p,r}}\le C(\|\gamma^n\|_{B^s_{p,r}}+\|\gamma^n\|^2_{B^s_{p,r}}), \tag{3.26}$$

$$\|G^n-G^\infty\|_{B^s_{p,r}}\le C(\|\gamma^n\|_{B^s_{p,r}}+\|\gamma^\infty\|_{B^s_{p,r}}+1)\|\gamma^n-\gamma^\infty\|_{B^{s-1}_{p,r}}. \tag{3.27}$$

We have already known $\gamma^n\to\gamma^\infty$ in $L^\infty([0,T];B^{s-1}_{p,r})$. At the same time, according to (3.27), G^n satisfy the condition of Lemma 2.11. Then we deduce that $y^n\to y^\infty$ in $C([0,T];B^s_{p,r})$ if $r<\infty$.

According to Lemma 2.8, we have for all $n\in\mathbb{N}$ and $t\in[0,T]$,

$$\|z^n(t)\|_{B^s_{p,r}}\le e^{C\int_0^t\|G^n_x\|_{B^s_{p,r}}dt'}\Big(\|\gamma_0^n-\gamma_0^\infty\|_{B^s_{p,r}} + \int_0^t e^{-C\int_0^{t'}\|G^n_x\|_{B^s_{p,r}}dt''}\|F^n-F^\infty\|_{B^s_{p,r}}dt'\Big). \tag{3.28}$$

We get

$$\|G^n_x\|_{B^s_{p,r}}\le C(\|\gamma^n\|^2_{B^s_{p,r}}+\|\gamma^n\|_{B^s_{p,r}}),$$

$$\|F^n-F^\infty\|_{B^s_{p,r}}\le C(\|\gamma^n\|^2_{B^s_{p,r}}+\|\gamma^\infty\|^2_{B^s_{p,r}}+1)\|\gamma^n-\gamma^\infty\|_{B^s_{p,r}}. \tag{3.29}$$

Plugging (3.29) into (3.28), and using the uniform bounds of γ^n, we obtain

$$\|z^n(t)\|_{B^s_{p,r}}\le C\Big(\|\gamma_0^n-\gamma_0^\infty\|_{B^s_{p,r}}+\int_0^t\|\gamma^n-\gamma^\infty\|_{B^s_{p,r}}dt'\Big).$$

Observing that $y^\infty=\gamma^\infty,\ z^\infty=0$, we can easily deduce that

$$\|z^n(t)\|_{B^s_{p,r}}\le C\Big(\|\gamma_0^n-\gamma_0^\infty\|_{B^s_{p,r}}+\int_0^t(\|y^n-y^\infty\|_{B^s_{p,r}}+\|z^n\|_{B^s_{p,r}})dt'\Big).$$

Appling Gronwall's inequality yields that

$$\|z^n(t)\|_{B^s_{p,r}}\le e^{Ct}\Big(\|\gamma_0^n-\gamma_0^\infty\|_{B^s_{p,r}}+\int_0^t e^{-Ct}\|y^n-y^\infty\|_{B^s_{p,r}}dt'\Big).$$

Therefore when $r < \infty$, $z^n \to 0$ in $C([0,T]; B^s_{p,r})$, and hence $\gamma^n \to \gamma^\infty$ in $C([0,T]; B^s_{p,r})$.

Considering the case $r = \infty$, we have weak continuity. In fact, for fixed $\phi \in B^{-s}_{p',1}$, we write

$$\langle \gamma^n(t) - \gamma^\infty(t), \phi\rangle = \langle \gamma^n(t) - \gamma^\infty(t), S_j\phi\rangle + \langle \gamma^n(t) - \gamma^\infty(t), \phi - S_j\phi\rangle.$$

According to the duality, we have

$$\begin{aligned}|\langle \gamma^n(t) - \gamma^\infty(t), \phi\rangle| \le \|\gamma^n(t) - \gamma^\infty(t)\|_{B^{s-1}_{p,\infty}} \|S_j\phi\|_{B^{1-s}_{p',1}} + \|\gamma^n(t)\\ -\gamma^\infty(t)\|_{B^s_{p,\infty}} \|\phi - S_j\phi\|_{B^{-s}_{p',1}}.\end{aligned}$$

Using the fact that $\gamma^n \to \gamma^\infty$ in $L^\infty([0,T]; B^{s-1}_{p,\infty})$, and $S_j\phi \to \phi$ in $B^{-s}_{p',1}$ and $(\gamma^n)_{n\in\bar{\mathbb{N}}}$ is bounded in $L^\infty([0,T]; B^s_{p,r})$, it is easy to conclude that $\langle\gamma^n(t) - \gamma^\infty(t), \phi\rangle \to 0$ uniformly on $[0,T]$.

□

4 Blow-up and ill-posedness

First we prove a conservation law for (1.2).

Lemma 4.1 *Let $\gamma_0 \in H^s$, $s > \frac{1}{2}$ and let T^* be the the maximal existence time of the corresponding solution γ to (1.2). For any $t \in [0, T^*)$, then we have*

$$\|\gamma(t)\|_{L^2} = \|\gamma_0\|_{L^2}.$$

Proof Arguing by density, it suffices to consider the case where $\gamma \in C_0^\infty(\mathbb{R})$. The Eq. (1.2) can be rewritten as a conservation law

$$\gamma_t = [(v_x - \gamma)\lambda - \gamma v]_x. \tag{4.1}$$

Using the fact that $v_{xx} - v = \gamma_x + \frac{\gamma^2}{2\lambda}$ and then multiplying (4.1) with γ and integrating by parts, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \gamma^2 dx &= \int_{\mathbb{R}} \lambda\gamma(v_{xx} - \gamma_x) - \gamma\gamma_x v - \gamma^2 v_x dx\\ &= \int_{\mathbb{R}} -\lambda\gamma_x v_x - \gamma^2 v_x - \gamma\gamma_x v dx\\ &= \int_{\mathbb{R}} v v_x - v_x v_{xx} - \frac{1}{2}\gamma^2 v_x - \gamma\gamma_x v dx\\ &= 0.\end{aligned}$$

□

Next we state a blow-up criterion for (1.2).

Lemma 4.2 *Let $\gamma_0 \in B^s_{p,r}$ with (s, p, r) being as in Theorem 3.2, and let T^* be the maximal existence time of the corresponding solution γ to (1.2). Then γ blows up in finite time $T^* < \infty$ if and only if*

$$\int_0^{T^*} \|\gamma(t')\|^2_{L^\infty} dt' = \infty.$$

Proof Applying Lemma 2.9,

$$\|\gamma(t)\|_{B^s_{p,r}} \le C(\|\gamma_0\|_{B^s_{p,r}} + \int_0^t \|G^1_x\|_{L^\infty}\|\gamma\|_{B^s_{p,r}} + \|G^1_x\|_{B^s_{p,r}}\|\gamma\|_{L^\infty} + \|F\|_{B^s_{p,r}} dt'), \tag{4.2}$$

where $G^1 = G^{-1}m,\ F = \frac{1}{2}\gamma^2 + \lambda G^{-1}m - \gamma G^{-1}m_x,\ m = G^{-1}(\gamma_x + \frac{\gamma^2}{2\lambda})$.

Note that the operator G^{-1} coincides with the convolution by the function $x \mapsto \frac{-1}{2}e^{-|x|}$, which implies that $\|G^{-1}\gamma\|_{L^\infty}$, $\|G^{-1}\gamma_x\|_{L^\infty}$ and $\|G^{-1}\gamma_{xx}\|_{L^\infty}$ can be bounded by $\|\gamma\|_{L^\infty}$. Then

$$\|G^1_x\|_{L^\infty} \le C(\|\gamma\|_{L^\infty} + \|\gamma\|^2_{L^\infty}). \tag{4.3}$$

As $s > 0$, by Lemma 2.5, we have

$$\|G^1_x\|_{B^s_{p,r}} \le C\|\gamma\|_{B^s_{p,r}}(1 + \|\gamma\|_{L^\infty}), \tag{4.4}$$

and

$$\begin{aligned}\|F\|_{B^s_{p,r}} &\le C(\|\gamma\|_{B^s_{p,r}}\|\gamma\|_{L^\infty} + \|\gamma\|_{B^s_{p,r}}(1 + \|\gamma\|_{L^\infty}) + \|G^1_x\|_{B^s_{p,r}}\|\gamma\|_{L^\infty} \\ &\quad + \|G^1_x\|_{L^\infty}\|\gamma\|_{B^s_{p,r}}) \\ &\le C\|\gamma\|_{B^s_{p,r}}(1 + \|\gamma\|_{L^\infty} + \|\gamma\|^2_{L^\infty}).\end{aligned} \tag{4.5}$$

Plugging (4.3), (4.4) and (4.5) into (4.2), we get

$$\|\gamma(t)\|_{B^s_{p,r}} \le C(\|\gamma_0\|_{B^s_{p,r}} + \int_0^t (1 + \|\gamma\|_{L^\infty} + \|\gamma\|^2_{L^\infty})\|\gamma\|_{B^s_{p,r}} dt'). \tag{4.6}$$

Appling Gronwall's inequality yields that

$$\|\gamma(t)\|_{B^s_{p,r}} \le \|\gamma_0\|_{B^s_{p,r}} e^{C\int_0^t (1+\|\gamma\|_{L^\infty}+\|\gamma\|^2_{L^\infty})dt'}.$$

If T^* is finite, and $\int_0^{T^*} \|\gamma\|^2_{L^\infty} dt' < \infty$, then $\gamma \in L^\infty([0, T^*); B^s_{p,r})$, which contradicts the assumption that T^* is the maximal existence time.

On the other hand, by Theorem 3.2 and the fact that $B^s_{p,r} \hookrightarrow L^\infty$, if $\int_0^{T^*} \|\gamma\|^2_{L^\infty} dt' = \infty$, then γ must blow up in finite time. □

Let us consider the ordinary differential equation:

$$\begin{cases} q_t(t,x) = G^{-1}\left(\gamma_x + \frac{\gamma^2}{2\lambda}\right)(t, q(t,x)), & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{4.7}$$

If $\gamma \in B^s_{p,r}$ with (s,p,r) being as in Theorem 3.2, then $G^{-1}(\gamma_x + \frac{\gamma^2}{2\lambda}) \in C([0,T); C^{0,1})$. According to the classical results in the theory of ordinary differential equations, we can easily infer that (4.7) have a unique solution $q \in C^1([0,T) \times \mathbb{R}; \mathbb{R})$ such that the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(t,x) = \exp\Big(\int_0^t G^{-1}(\gamma_x + \frac{\gamma^2}{2\lambda})(t', q(t',x))\Big)dt' > 0, \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

We prove the following theorem which shows that the corresponding solution of (1.2) will blow up by giving negative condition for the initial data.

Theorem 4.3 *Let $\gamma_0 \in H^s$, $s > \frac{1}{2}$. Assume $\gamma_0(x_0) < -2\sqrt{d}$, with $d = C(\|\gamma_0\|_{L^2} + \frac{1}{2|\lambda|}\|\gamma_0\|^2_{L^2})(C\|\gamma_0\|_{L^2} + C\frac{1}{2|\lambda|}\|\gamma_0\|^2_{L^2} + |\lambda|)$. Then the corresponding solution γ of (1.2) blows up in finite time.*

Proof Arguing by density, now we assume $s > \frac{3}{2}$. Then applying Lemma 4.1 and Young's inequality, we get

$$\|G^{-1}\gamma(t)\|_{L^\infty}, \|G^{-1}\gamma_x(t)\|_{L^\infty} \le \|\gamma(t)\|_{L^2} = \|\gamma_0\|_{L^2},$$

and

$$\|G^{-1}\gamma^2(t)\|_{L^\infty}, \|G^{-1}(\gamma^2)_x(t)\|_{L^\infty} \le \|\gamma^2(t)\|_{L^1} = \|\gamma(t)\|^2_{L^2} = \|\gamma_0\|^2_{L^2}.$$

Denote $c = C(\|\gamma_0\|_{L^2} + \frac{1}{2|\lambda|}\|\gamma_0\|^2_{L^2})$. Then we have:

$$\begin{aligned} \gamma_t(t, q(t,x_0)) &= -\frac{1}{2}\gamma^2 - \gamma\left(G^{-1}\gamma + \frac{1}{2\lambda}\partial_x G^{-1}\gamma^2\right) + \lambda G^{-1}\left(\gamma_x + \frac{\gamma^2}{2\lambda}\right) \\ &\le -\frac{1}{2}\gamma^2 + c|\gamma| + |\lambda|c \\ &\le -\frac{1}{4}\gamma^2 + d. \end{aligned}$$

Denote $f(t) = \frac{-\sqrt{d} - \frac{1}{2}\gamma_0(x_0)}{\sqrt{d} - \frac{1}{2}\gamma_0(x_0)} e^{dt}$. Solving the above inequality, we finally get

$$\gamma(t, q(t,x_0)) \le \frac{2\sqrt{d}f(t) + 2\sqrt{d}}{f(t) - 1}. \tag{4.8}$$

As $\gamma_0(x_0) < -2\sqrt{d}$, we get

$$\gamma_t(t, q(t, x_0)) < 0. \tag{4.9}$$

Then we can deduce that $\gamma(t)$ decreases monotonly and is less than zero at the point $q(t, x_0)$ along the flow. Finally, we prove that the solution $\gamma(t)$ blows up in finite time. It is obvious that $0 < f(0) < 1$ *and* $f(\infty) = \infty$, which implies that exists $T > 0$, $f(T) = 1$. Denote the maximal time of the solution by T^*. So we can easily deduce that $T^* \leq T = \frac{1}{d}\ln(\frac{\sqrt{d}-\frac{1}{2}\gamma_0(x_0)}{-\sqrt{d}-\frac{1}{2}\gamma_0(x_0)})$. Therefore, from (4.8) we know $\gamma(t, q(t, x_0)) \to -\infty$ as $t \to T^*$. Applying Lemma (4.2), the solution γ must blow up in finite time. □

Lemma 4.4 *Assume $\gamma \in H^1$ to (1.2). We have*

$$\|\gamma\|_{L^\infty} \leq C(\|\gamma\|_{B^0_{\infty,\infty}} \cdot \log_2(2 + \|\gamma\|_{H^1}) + 1)$$

Proof Fixing an integer $N > 0$, we get

$$\begin{aligned}\|\gamma\|_{L^\infty} &\leq \sum_{k\leq N-1} \|\Delta_k\gamma\|_{L^\infty} + \sum_{k\geq N} \|\Delta_k\gamma\|_{L^\infty}\\ &\leq CN\|\gamma\|_{B^0_{\infty,\infty}} + C\sum_{k\geq N} 2^{\frac{k}{2}}\|\Delta_k\gamma\|_{L^2}\\ &\leq CN\|\gamma\|_{B^0_{\infty,\infty}} + C2^{-N}\|\gamma\|_{H^1}.\end{aligned}$$

Setting $N = \log_2(2 + \|\gamma\|_{H^1})$, we complete the proof. □

We need another blow-up criterion for (1.2) to prove norm inflation in the critical Besov Spaces.

Lemma 4.5 *Let $\gamma_0 \in H^1$, and let T^* be the maximal existence time of the corresponding solution γ to (1.2). Then γ blows up in finite time $T^* < \infty$ if and only if*

$$\int_0^{T^*} \|\gamma(t')\|_{B^0_{\infty,\infty}} dt' = \infty.$$

Proof Applying Lemma 2.9, and since $L^\infty \hookrightarrow B^0_{\infty,\infty}$, we have

$$\|\gamma(t)\|_{H^1} \leq C(\|\gamma_0\|_{H^1} + \int_0^t \|G^1_x\|_{L^\infty}\|\gamma\|_{H^1} + \|G^1_x\|_{H^1}\|\gamma\|_{L^\infty} + \|F\|_{H^1}dt'), \tag{4.10}$$

where $G^1 = G^{-1}m,\ F = \frac{1}{2}\gamma^2 + \lambda G^{-1}m - \gamma G^{-1}m_x,\ m = G^{-1}(\gamma_x + \frac{\gamma^2}{2\lambda})$.

Note that the operator G^{-1} coincides with the convolution by the function $x \mapsto \frac{-1}{2}e^{-|x|}$, which implies that $\|G^{-1}\gamma\|_{L^\infty}$, $\|G^{-1}\gamma_x\|_{L^\infty}$ and $\|G^{-1}\gamma_{xx}\|_{L^\infty}$ can be bounded by $\|\gamma\|_{L^\infty}$. Then applying Lemma 4.1, we get

$$\|G_x^1\|_{L^\infty} \le C\left(\|\gamma\|_{L^\infty} + \|(\frac{-1}{2}e^{-|x|})_x\|_{L^\infty}\|\gamma^2\|_{L^1}\right) \le C(1+\|\gamma\|_{L^\infty}). \tag{4.11}$$

Applying Lemma 2.5, we have

$$\|G_x^1\|_{H^1} \le C(\|\gamma\|_{H^1} + \|\gamma^2\|_{L^2}) \le C(\|\gamma\|_{H^1} + \|\gamma\|_{L^\infty}), \tag{4.12}$$

and

$$\begin{aligned}\|F\|_{H^1} &\le C(\|\gamma\|_{H^1}\|\gamma\|_{L^\infty} + \|\gamma\|_{H^1} + \|G_x^1\|_{H^1}\|\gamma\|_{L^\infty} + \|G_x^1\|_{L^\infty}\|\gamma\|_{H^1})\\ &\le C\|\gamma\|_{H^1}(1+\|\gamma\|_{L^\infty}).\end{aligned} \tag{4.13}$$

Plugging (4.11), (4.12) and (4.13) into (4.2), we get

$$\|\gamma(t)\|_{H^1} \le C(\|\gamma_0\|_{H^1} + \int_0^t (1+\|\gamma\|_{L^\infty})\|\gamma\|_{H^1} dt'). \tag{4.14}$$

By Lemma 4.4, we have

$$\|\gamma(t)\|_{H^1} \le C(\|\gamma_0\|_{H^1} + \int_0^t \left(1+\|\gamma\|_{B^0_{\infty,\infty}}\log(e+\|\gamma\|_{H^1})\right)\|\gamma\|_{H^1} dt'). \tag{4.15}$$

Appling Gronwall's inequality yields that

$$\|\gamma(t)\|_{H^1} \le \|\gamma_0\|_{H^1} e^{Ct + C\int_0^t \|\gamma\|_{B^0_{\infty,\infty}}\log(e+\|\gamma\|_{H^1})dt'}.$$

Simplifying the above inequality and appling Gronwall's inequality, we get

$$\log(e+\|\gamma(t)\|_{H^1}) \le \left(\log(e+\|\gamma_0\|_{H^1}) + Ct\right)e^{C\int_0^t \|\gamma\|_{B^0_{\infty,\infty}} dt'}.$$

If T^* is finite, and $\int_0^{T^*} \|\gamma\|_{B^0_{\infty,\infty}} dt' < \infty$, then $\gamma \in L^\infty([0,T^*); H^1)$, which contradicts the assumption that T^* is the maximal existence time.

On the other hand, by Theorem 3.2 and the fact that $H^1 \hookrightarrow L^\infty \hookrightarrow B^0_{\infty,\infty}$, if $\int_0^{T^*} \|\gamma\|_{B^0_{\infty,\infty}} dt' = \infty$, then γ must blow up in finite time. □

We end up with the following theorem which proves the norm inflation and hence the ill-posedness of the modified CH equation (1.2) in $H^{\frac{1}{2}}$ and in $B^{\frac{1}{2}}_{2,r}$, $1 < r \le \infty$.

Theorem 4.6 *Let* $1 \le p \le \infty$ *and* $1 < r \le \infty$. *For any* $\epsilon > 0$, *there exists* $\gamma_0 \in H^\infty$, *such that the following holds:*

(1) $\|\gamma_0\|_{B^{\frac{1}{p}}_{p,r}} \le \epsilon$;

(2) There is a unique solution $\gamma \in C([0,T); H^\infty)$ *to the equation (1.2) with a maximal lifespan* $T < \epsilon$;

(3) $limsup_{t\to T^-}\|\gamma\|_{B^{\frac{1}{p}}_{p,r}} \ge limsup_{t\to T^-}\|\gamma\|_{B^0_{\infty,\infty}} = \infty$.

Proof Fix $1 \le p \le \infty$ and $1 < r \le \infty$, and $\epsilon > 0$. We define $g(x)$

$$g(x) = \sum_{k\ge 1} \frac{1}{2^k k^{\frac{2}{1+r}}} g_k(x) \tag{4.16}$$

with $g_k(x)$ given by the Fourier transform $\hat{g}_k(\xi) = -2^{-k}\xi\widetilde{\chi}(2^{-k}\xi)$, where $\widetilde{\chi}$ is a non-negative, non-zero C_0^∞ function such that $\widetilde{\chi}\chi_0 = \widetilde{\chi}$. Directly calculating, we have $\Delta_k g(x) = \frac{1}{2^k k^{\frac{2}{1+r}}} g_k(x)$. We also have $\|\Delta_k g(x)\|_{L^p} \sim \frac{2^{\frac{k}{p'}}}{2^k k^{\frac{2}{1+r}}}$ and

$$\|g\|_{B^{\frac{1}{p}}_{p,q}} \sim \|\frac{1}{k^{\frac{2}{1+r}}}\|_{l^q}.$$

Then we get $g \in B^{\frac{1}{p}}_{p,r} \backslash B^{\frac{1}{p}}_{p,1}$, and

$$g(0) = \int \hat{g}(\xi)d\xi = -c\sum_{k\ge 1}\frac{1}{k^{\frac{2}{1+r}}} = -\infty.$$

For any $\epsilon > 0$, let $\gamma_{0,\epsilon} = \|g\|^{-1}_{B^{\frac{1}{p}}_{p,r}} \cdot \epsilon S_K(g)$ where K is large enough such that $\gamma_{0,\epsilon}(0) < \frac{-2\sqrt{d}(e^{d\epsilon}+1)}{e^{d\epsilon}-1}$. Then $\gamma_{0,\epsilon} \in H^\infty$, $\|\gamma_{0,\epsilon}\|_{B^{\frac{1}{p}}_{p,r}} \le \epsilon$. Applying Theorem 4.3, there is a unique associated solution $\gamma \in C([0,T); H^\infty)$ with a maximal lifespan $T < \epsilon$. By Lemmas 4.2 and 4.5, we can show that $limsup_{t\to T^-}\|\gamma\|_{B^0_{\infty,\infty}} = \infty$. □

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References

1. Bahouri, H., Chemin, J.Y., Danchin, R.: Fourier analysis and nonlinear partial differential equations. Springer, Berlin (2011)
2. Bressan, A., Constantin, A.: Global dissipative solutions of the Camassa–Holm equation. Anal. Appl. **5**(01), 1–27 (2007)

3. Bressan, A., Constantin, A.: Global conservative solutions of the Camassa-Holm equation. Arch. Ration. Mech. Anal. **183**(2), 215–239 (2007)
4. Constantin, A.: The Hamiltonian structure of the Camassa–Holm equation. Expo. Math. **15**(1), 53–85 (1997)
5. Constantin, A.: Existence of permanent of solutions and breaking waves for a shallow water equation: a geometric approach. Ann. Inst. Fourier (Grenoble) **50**, 321–362 (2000)
6. Constantin, A.: On the scattering problem for the Camassa–Holm equation. Proc. R. Soc. Lond Ser A: Math. Phys. Eng. Sci. **457**, 953–970 (2001)
7. Constantin, A., Escher, J.: Global existence and blow-up for a shallow water equation. Ann. Sc. Norm. Super. Pisa Cl. Sci. **26**, 303–328 (1998)
8. Constantin, A., Escher, J.: Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation. Comm. Pure Appl. Math. **51**, 475–504 (1998)
9. Constantin, A., Escher, J.: Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. **181**, 229–243 (1998)
10. Camassa, R., Holm, D.D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**, 1661–1664 (1993)
11. Constantin, A., Lannes, D.: The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations. Arch. Ration. Mech. Anal. **192**, 165–186 (2009)
12. Constantin, A., Molinet, L.: Global weak solutions for a shallow water equation. Comm. Math. Phys. **211**, 45–61 (2000)
13. Constantin, A., Strauss, W.A.: Stability of peakons. Comm. Pure Appl. Math. **53**, 603–610 (2000)
14. Chern, S.S., Tenenblat, K.: Pseudo-spherical surfaces and evolution equations. Stud. Appl. Math. **74**, 55–83 (1986)
15. Danchin, R.: A few remarks on the Camassa–Holm equation. Differ. Integr. Equ. **14**, 953–988 (2001)
16. Danchin, R.: A note on well-posedness for Camassa–Holm equation. J. Differ. Equ. **192**, 429–444 (2003)
17. Fokas, A., Fuchssteiner, B.: Symplectic structures, their Bäcklund transformation and hereditary symmetries. Phys. D. **4**(1), 47–66 (1981)
18. Guo, Z., Liu, X., Molinet, L., et al.: Ill-posedness of the Camassa-Holm and related equations in the critical space. J. Differ. Equ. **266**, 1698–1707 (2019)
19. Gorka, P., Reyes, E.G.: The modified Camassa–Holm equation. Int. Math. Res. Not. **2011**, 2617–2649 (2010)
20. He, H., Yin, Z.: On a generalized Camassa-Holm equation with the flow generated by velocity and its gradient. Appl. Anal. **96**(4), 679–701 (2017)
21. Luo, W., Yin, Z.: Local well-posedness and blow-up criteria for a two-component Novikov system in the critical Besov space. Nonlinear Anal. Theory Methods Appl. **122**, 1–22 (2015)
22. Li, J., Yin, Z.: Well-posedness and global existence for a generalized Degasperis-Procesi equation. Nonlinear Anal. Real World Appl. **28**, 72–90 (2016)
23. Li, J., Yin, Z.: Remarks on the well-posedness of Camassa-Holm type equations in Besov spaces. J. Differ. Equ. **261**(11), 6125–6143 (2016)
24. Li, J., Yin, Z.: Well-posedness and analytic solutions of the two-component Euler–Poincare system. Monatsh. Math. **183**, 509–537 (2017)
25. Qiao, Z.J.: The Camassa–Holm hierarchy, related N-dimensional integrable systems and algebro-geometric solution on a symplectic submanifold. Commun. Math. Phys. **239**, 309–341 (2003)
26. Rodríguez-Blanco, G.: On the Cauchy problem for the Camassa-Holm equation. Nonlinear Anal. Theory Methods Appl. **46**, 309–327 (2001)
27. Reyes, E.G.: Geometric integrability of the Camassa–Holm equation. Lett. Math. Phys **59**, 117–131 (2002)
28. Zheng, R., Yin, Z.: The Cauchy problem for a generalized Novikov equation. Discrete Contin. Dyn. Syst. **37**(6), 3503–3519 (2017)
29. Xin, Z., Zhang, P.: On the weak solutions to a shallow water equation. Comm. Pure Appl. Math. **53**, 1411–1433 (2000)