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Existence and uniqueness and first order approximation of solutions to atmospheric Ekman flows

Michal Fečkan[1,2] · Yi Guan[3] · Donal O'Regan[4] · JinRong Wang[3,5]

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Abstract
In this paper, we study the classical problem of the wind in the steady atmospheric Ekman layer with classical boundary conditions and the eddy viscosity is an arbitrary height-dependent function with a finite limit value. We present existence and uniqueness and smooth results to justify computing first order approximation of solutions. Using a different argument that in previous works, we construct the Green's function to derive the solution by a perturbation approach.

Keywords Ekman layer · Variable eddy viscosity · Explicit solutions · Green function

Mathematics Subject Classification 34B05

1 Introduction

The atmospheric boundary layer has three parts [1,2], i.e., the lamina sublayer, surface (Prandtl) layer and the Ekman layer. The Ekman layer covers 90% of the atmospheric boundary layer and it is driven by a three-way balance among frictional effects, pressure gradient and the influence of the coriolis force [1,3,4]. In general, textbooks on geophysical fluid dynamics and dynamic meteorology contain the derivation for the Ekman layer with a constant eddy viscosity k [5,6]. However k usually varies with

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✉ JinRong Wang
jrwang@gzu.edu.cn

Extended author information available on the last page of the article

the height. As a result it is necessary to find the explicit solution of Ekman flows with a non-constant eddy viscosity, but unfortunately explicit solutions are scare in the literature and are restricted to linear [7,8] or quadratic and cubic poly-nominal [9]. For arbitrary $k(z)$ or $k(z,t)$, we often have to rely on approximation and numerical simulation; the authors in [10–13] apply the Wentzel, Kramers and Brillouin's method to get the approximation solution.

Constantin and Johnson [14] studied Ekman flows with variable eddy viscosity $k(z)$ and they derived the explicit solution through an unclosed form and verified the existence of the solution by transformation and the iterative technique. The authors in [15] studied the horizontal wind drift currents which spiral and decay with depth and they obtained the solution by a perturbation approach. In this paper, we adopt the linearization approach to establish existence and uniqueness results and we consider the smoothness of solutions, which justify computing first order approximation of solutions. Also, based on [15], we regard the eddy viscosity $k(z)$ as perturbations of the asymptotic reference value and we perform the variable change and get a linear, non-homogeneous second order differential equation and then we show the existence of a solution using the Green's function.

2 Model description

The Ekman layer is governed by the following equations, see [1,2]

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{dP}{dx} + fv - \frac{\partial \overline{u'w'}}{\partial z}, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{dP}{dy} - fu - \frac{\partial \overline{v'w'}}{\partial z}, \end{cases}$$

where u, v and w are the components of the wind in the x, y and z directions respectively, P is the atmospheric pressure, ρ is the reference density, $f = 2\Omega sin\theta$ is the Coriolis parameter at the fixed latitude θ, $\Omega \approx 7.29 \times 10^{-5}$ is the angular speed of the roattion of the earth in the northern Hemisphere, and $\theta \in (0, \pi/2]$ is the angle of latitude in right-handed rotating spherical cooridates, t is time and k is the eddy diffusivity for momentum.

Assuming a steady state we get $\frac{Du}{Dt} = 0$, $\frac{Dv}{Dt} = 0$. From the geostrophic balance, we have

$$\begin{cases} \frac{1}{\rho}\frac{dP}{dx} = fv_g, \\ \frac{1}{\rho} = -fu_g. \end{cases}$$

From the Flux-Gradient theory, we get

$$\begin{cases} \overline{u'w'} = -k(\frac{\partial u}{\partial z}), \\ \overline{v'w'} = -k(\frac{\partial v}{\partial z}), \end{cases}$$

where k is the eddy viscosity coefficient. Then we obtain

$$\begin{cases} f(v-v_g) = -\frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}), \\ f(u-u_g) = \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}), \end{cases} \tag{1}$$

where u_g and v_g are the corresponding constant geostrophic wind components. We use the traditional boundary conditions for (1) as

$$u = 0, \quad v = 0 \quad \text{at} \ \ z = 0, \tag{2}$$

$$u \to u_g, \quad v \to v_g \quad \text{for} \ \ z \to \infty. \tag{3}$$

Let $\Phi = (u - u_g) + i(v - v_g)$, and from (1), we will get

$$k\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial k}{\partial z}\frac{\partial \Phi}{\partial z} - i \cdot f\Phi = 0. \tag{4}$$

If $k =$constant, we have

$$\begin{cases} u(z) = u_g - e^{-\gamma z}[u_g \cos(\gamma z) + v_g \sin(\gamma z)], \\ v(z) = v_g + e^{-\gamma z}[u_g \sin(\gamma z) - v_g \cos(\gamma z)], \end{cases} \tag{5}$$

where $\gamma = \sqrt{\frac{f}{2k}}$. However, the eddy viscosity k always varies with height [10], so (4) will become

$$\begin{cases} k'(z)\frac{dv}{dz} + k(z)\frac{d^2v}{dz^2} = f(u-u_g), \\ -k'(z)\frac{du}{dz} - k(z)\frac{d^2u}{dz^2} = f(v-v_g). \end{cases} \tag{6}$$

Here, we regard the physically relevant eddy viscosity $k(z)$ as perturbations of the asymptotic reference value $k_* = \lim\limits_{z\to\infty} k(z) > 0$, so

$$k(z) = k_* + \varepsilon k_1(z), \quad \text{at} \ \ z \geq 0, \tag{7}$$

where $\varepsilon \ll 1$, and the asymptotic rate of convergence is faster than quadratic [14], that is, there exist constants $a, b, c > 0$ such that

$$|k_1(z)| \leq \frac{a}{1 + b|z|^{2+c}}, \qquad z \geq 0.$$

3 Main results

3.1 Existence and uniqueness

Let

$$x = k(z)\frac{\partial u}{\partial z},$$

$$y = k(z)\frac{\partial v}{\partial z},$$

so (1) is replaced with

$$\begin{cases} \frac{\partial u}{\partial z} = \hat{k}(z)x, \\ \frac{\partial v}{\partial z} = \hat{k}(z)y, \\ \frac{\partial x}{\partial z} = -f(v - v_g), \\ \frac{\partial y}{\partial z} = f(u - u_g) \end{cases} \tag{8}$$

for $\hat{k}(z) = \frac{1}{k(z)}$. The affine system (8) has a unique equilibrium

$$p = (u_g, v_g, 0, 0) \tag{9}$$

with the linearization

$$\begin{cases} \frac{\partial u}{\partial z} = \hat{k}(z)x, \\ \frac{\partial v}{\partial z} = \hat{k}(z)y, \\ \frac{\partial x}{\partial z} = -fv, \\ \frac{\partial y}{\partial z} = fu \end{cases} \tag{10}$$

When $\varepsilon = 0$, $k(z) = k_*$ is a constant function, and then the matrix

$$A = \begin{pmatrix} 0 & 0 & \hat{k}_* & 0 \\ 0 & 0 & 0 & \hat{k}_* \\ 0 & -f & 0 & 0 \\ f & 0 & 0 & 0 \end{pmatrix}$$

has eigenvalues

$$\lambda_1 = -\sqrt{\frac{f\hat{k}_*}{2}}(1+i), \quad \lambda_2 = -\sqrt{\frac{f\hat{k}_*}{2}}(1-i), \quad \lambda_3 = \sqrt{\frac{f\hat{k}_*}{2}}(1-i), \lambda_4 = \sqrt{\frac{f\hat{k}_*}{2}}(1+i)$$

with corresponding eigenvectors

$$\left(-\frac{(1+i)\sqrt{\hat{k}_*}}{\sqrt{2}\sqrt{f}}, -\frac{(1-i)\sqrt{\hat{k}_*}}{\sqrt{2}\sqrt{f}}, i, 1\right)$$

$$\left(-\frac{(1-i)\sqrt{\hat{k}_*}}{\sqrt{2}\sqrt{f}}, -\frac{(1+i)\sqrt{\hat{k}_*}}{\sqrt{2}\sqrt{f}}, -i, 1\right)$$

$$\left(\frac{(1-i)\sqrt{\hat{k}_*}}{\sqrt{2}\sqrt{f}}, \frac{(1+i)\sqrt{\hat{k}_*}}{\sqrt{2}\sqrt{f}}, -i, 1\right)$$

$$\left(\frac{(1+i)\sqrt{\hat{k}_*}}{\sqrt{2}\sqrt{f}}, \frac{(1-i)\sqrt{\hat{k}_*}}{\sqrt{2}\sqrt{f}}, i, 1\right).$$

Thus the linear system (10) has a stable space

$$X_s = \begin{bmatrix} \left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, 0, 1\right) \\ \left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, 1, 0\right) \end{bmatrix}.$$

Introduce a linear subspace

$$W = \{(0, 0, x, y) \mid x, y \in \mathbb{R}\}, \tag{11}$$

and condition (2) means $(u(0), v(0), x(0), y(0)) \in W$. Note

$$\mathbb{R}^n = X_s \oplus W.$$

First we study a general affine system

$$q'(z) = (A + \varepsilon B(\varepsilon, z))(q(z) - p) \tag{12}$$

for $A \in L(\mathbb{R}^n)$, $B : (-\delta, \delta) \times \mathbb{R}_+ \to L(\mathbb{R}^n)$, $\delta > 0$ and $p \in \mathbb{R}^n$, where $\mathbb{R}_+ = [0, \infty)$ and $L(\mathbb{R}^n)$ is the space of linear maps on $\mathbb{R}^n$. We suppose

(A1) A is hyperbolic, i.e., the real parts of eigenvalues of A are nonzero.
(A2) $B(\varepsilon, z)$ has continuous partial derivatives $\partial_\varepsilon^i B(\varepsilon, z)$ with

$$\sup_{(\varepsilon,z)\in(-\delta,\delta)\times\mathbb{R}_+} \|\partial_\varepsilon^i B(\varepsilon, z)\| < \infty$$

for $i = 0, 1, \cdots, r$.

Recall that (A1) is equivalent to the existence of constants $K > 0, \alpha > 0$ and a splitting $\mathbb{R}^n = X_s \oplus X_u$ such that $A(X_s) = X_s$, $A(X_u) = X_u$ along with $\|e^{A_s z}\| \le K e^{-\alpha z}$ and $\|e^{-A_u z}\| \le K e^{-\alpha z}$ for $z \in \mathbb{R}_+$, where $A_s = A/X_s$ and $A_u = A/X_u$. We consider a projection $P : \mathbb{R}^n \to \mathbb{R}^n$ with ker $P = X_u$ and im $P = X_s$.

Let $W \subset \mathbb{R}^n$ be a linear subspace such that $\mathbb{R}^n = X_s \oplus W$, i.e., W is transversal to X_s. We look for solutions of (12) satisfying

$$q(0) \in W, \quad \lim_{z\to\infty} q(z) = p. \tag{13}$$

Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be a projection with ker $Q = W$ and im $Q = X_s$.

Proposition 3.1 *Assume (A1) and (A2). Set*

$$B_\infty = \sup_{(\varepsilon,z)\in(-\delta,\delta)\times\mathbb{R}_+} \|B(\varepsilon, z)\|.$$

For any $\varepsilon \in (-\delta, \delta)$ *satisfying,*

$$|\varepsilon| < \frac{\alpha}{4K B_\infty(\|P\| + (1 + K\|Q\|)\|I - P\|)}, \tag{14}$$

there is a unique solution $q(\varepsilon, z)$ of (12) *satisfying* (13)*. Moreover, $q(\varepsilon, z)$ is C^r-smooth in ε.*

Proof Following [16], any bounded solution of (12) is given by

$$q(z) = p + e^{Az}q_s + \varepsilon \int_0^z e^{A(z-r)} PB(\varepsilon, r)(q(r) - p)dr \\ -\varepsilon \int_z^\infty e^{A(r-z)}(I - P)B(\varepsilon, r)(q(r) - p)dr \tag{15}$$

for $q_s \in X_s$. We consider a Banach space $C_b(\mathbb{R}_+, \mathbb{R}^n)$ of all bounded and continuous functions $q : \mathbb{R}_+ \to \mathbb{R}^n$ endowed with a norm $\|q\| = \sup_{z\in\mathbb{R}_+} |q(z)|$. A solution of (15) is a fixed point of the map

$$H(\varepsilon, q_s, q)(z) = p + e^{Az}q_s + \varepsilon \int_0^z e^{A(z-r)} PB(\varepsilon, r)(q(r) - p)dr \\ -\varepsilon \int_z^\infty e^{A(r-z)}(I - P)B(\varepsilon, r)(q(r) - p)dr.$$

Now H is linear in q and C^r-smooth in ε. For any $q_1, q_2 \in C_b(\mathbb{R}_+, \mathbb{R}^n)$, we have

$$\begin{aligned} &|H(\varepsilon, q_s, q_1)(z) - H(\varepsilon, q_s, q_2)(z)| \\ &\quad\le |\varepsilon| \int_0^z |e^{A(z-r)} PB(\varepsilon, r)(q_1(r) - q_2(r))|dr \\ &\qquad + |\varepsilon| \int_z^\infty |e^{A(z-r)}(I - P)B(\varepsilon, r)(q_1(r) - q_2(r))|dr \\ &\quad\le |\varepsilon|K\|P\|B_\infty \int_0^z e^{-\alpha(z-r)}dr\|q_1 - q_2\| \\ &\qquad + |\varepsilon|K\|I - P\|B_\infty \int_z^\infty e^{\alpha(z-r)}dr\|q_1 - q_2\| \\ &\quad\le |\varepsilon|KB_\infty\alpha^{-1}(\|P\| + \|I - P\|)\|q_1 - q_2\|. \end{aligned}$$

Condition (14) guarantees the contraction of $H(\varepsilon, q_s, q)$ in q, so the Banach fixed point theorem ensures a unique solution $\hat{q}(\varepsilon, q_s, z)$ of (15). Next, for any $q_{s1}, q_{s2} \in X_s$, we have

$$|H(\varepsilon, q_{s1}, q)(z) - H(\varepsilon, q_{s2}, q)(z) \le |e^{Az}(q_{s1} - q_{s2})| \le K|q_{s1} - q_{s2}|.$$

Consequently, $\hat{q}(\varepsilon, q_s, z)$ is globally Lipschitz in q_s with a constant

$$l_1 = \frac{K}{1 - |\varepsilon|KB_\infty\alpha^{-1}(\|P\| + \|I - P\|)}$$

and C^r-smooth in ε. To finish the proof, we need to solve

$$Q\hat{q}(\varepsilon, q_s, 0) = 0. \tag{16}$$

We set

$$\hat{q}(\varepsilon, q_s, 0) = p + q_s - \varepsilon\tilde{q}(\varepsilon, q_s).$$

From (15), for any $q_{s1}, q_{s2} \in X_s$, we have

$$\begin{aligned} |\tilde{q}(\varepsilon, q_{s1}) - \tilde{q}(\varepsilon, q_{s2})| &\leq \int_z^\infty |e^{A(z-r)}(I-P)B(\varepsilon, r)(\hat{q}(\varepsilon, q_{s1}, r) - \hat{q}(\varepsilon, q_{s2}, r))|dr \\ &\leq KB_\infty\alpha^{-1}\|I-P\|l_1\|q_1 - q_2\|. \end{aligned}$$

Clearly (16) is equivalent to

$$\varepsilon Q\tilde{q}(\varepsilon, q_s) - Qp = q_s. \tag{17}$$

Assumption (14) ensures that the map

$$q_s \to \varepsilon Q\tilde{q}(\varepsilon, q_s) - Qp$$

is a contraction. Thus (17) has a unique solutions $q_s(\varepsilon)$. Next, by using [16, Proposition 1, p. 34], we see that (12) is dichotomous for any ε satisfying (14). Consequently, $q(\varepsilon, z) = \hat{q}(\varepsilon, q_s(\varepsilon, z)$ is the desired unique solution of (12) satisfying (13). The proof is finished. □

Now we apply Proposition 3.1 to (8). We already verified (A1). To show (A2), we note that

$$B(\varepsilon, z) = \begin{pmatrix} 0 & 0 & \hat{k}_2(\varepsilon, z) & 0 \\ 0 & 0 & 0 & \hat{k}_2(\varepsilon, z) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for

$$\hat{k}_2(\varepsilon, z) = \frac{k_1(z)}{k_*(k_* + \varepsilon k_1(z))}.$$

Thus supposing that $k_1(z) \geq 0$ is continuous with $\sup_{z\in\mathbb{R}_+} |k_1(z)| < \infty$, condition (A2) holds. Consequently, we have the following result.

Theorem 3.2 *Assume $k_1(z) \geq 0$ is continuous with $\sup_{z\in\mathbb{R}_+} |k_1(z)| < \infty$. Then for any ε small, there is a unique solution of* (8) *satisfying* (13) *with p and W given by* (9) *and* (11)*, respectively. This solution is C^∞ smooth in ε.*

Since $k(z)$ is continuous, here we have a solution $u(z)$, $v(z)$ of (12) such that $u(z)$, $v(z)$, $\frac{\partial u(z)}{\partial z}$, $\frac{\partial v(z)}{\partial z}$, $\frac{\partial}{\partial z}(k(z)\frac{\partial u(z)}{\partial z})$ and $\frac{\partial}{\partial z}(k(z)\frac{\partial u(z)}{\partial z})$ exist and are continuous on $\mathbb{R}_+$.

3.2 First order approximation

We use Theorem 3.2 to consider first order approximations of solutions. Consider the following second order boundary-value problem

$$\begin{cases} w''(s) - \frac{i\cdot f}{k_*} w(s) = 0, & s \in (0,t) \cup (t,\infty), \quad k_* > 0, \\ w(0) = 0, \quad w'(t_-) - w'(t_+) = 1, & w(s) \to 0 \ \textit{as} \ s \to \infty. \end{cases} \tag{18}$$

Lemma 3.3 *If $w(\cdot) := G(\cdot, t)$, $t \in (0,\infty)$ (called the Green's function) solves (18), then*

$$w(s) = \begin{cases} a_1 x_1(s) + a_2 x_2(s), & s \in (0,t), \\ b x_2(s), & s \in (t,\infty), \end{cases}$$

where

$$x_1(s) = e^{(1+i)\lambda s}, \ x_2(s) = e^{-(1+i)\lambda s}, \ \lambda = \sqrt{f/2k_*},$$
$$a_1 = \frac{1}{2(1+i)\lambda} e^{-(1+i)\lambda t}, \ a_2 = -\frac{1}{2(1+i)\lambda} e^{-(1+i)\lambda t},$$

and

$$b = -\frac{1}{2(1+i)\lambda} e^{-(1+i)\lambda t} + \frac{1}{2(1+i)\lambda} e^{(1+i)\lambda t}.$$

Proof Note x_1 and x_2 are two linearly independent solutions of $w''(s) - \frac{i\cdot f}{k_*} w(s) = 0$. Suppose

$$w(s) = \begin{cases} a_1 x_1(s) + a_2 x_2(s), & s \in (0,t), \\ b_1 x_1(s) + b_2 x_2(s), & s \in (t,\infty), \end{cases}$$

is a solution of (18). Then using the boundary conditions one obtains

$$b_1 = 0, \quad a_1 + a_2 = 0,$$

and

$$a_1(1+i)\lambda e^{(1+i)\lambda t} - a_2(1+i)\lambda e^{-(1+i)\lambda t} = -b_2(1+i)\lambda e^{-(1+i)\lambda t} + 1.$$

Using the continuity of w at $s = t$, we find

$$a_1 e^{(1+i)\lambda t} + a_2 e^{-(1+i)\lambda t} = b_2 e^{-(1+i)\lambda t}.$$

From the above we get the value of a_1, a_2, b_1 and b_2, and therefore we obtain

$$w(s) = \begin{cases} \frac{1}{2(1+i\lambda)} [e^{(1+i)\lambda(s-t)} - e^{-(1+i)\lambda(s+t)}], & s \in (0,t), \\ \frac{1}{2(1+i\lambda)} [e^{(1+i)\lambda t} - e^{-(1+i)\lambda t}] e^{-(1+i)\lambda s}, & s \in (t,\infty). \end{cases}$$

The proof is complete. □

Let

$$s = s(z) = k_* \int_0^z \frac{1}{k(t)} dt. \tag{19}$$

Then

$$\frac{ds}{dz} = \frac{k_*}{k(z)}.$$

Let

$$k_2(s) = \frac{f}{k_*^2} k_1(z), \quad \Psi(s) = U(s) + iV(s), \tag{20}$$

where

$$U(s) = u(z) - u_g, \quad V(s) = v(z) - v_g. \tag{21}$$

Set

$$\Psi(s) = \Psi_0(s) + \epsilon\varphi(s) + o(\epsilon), \quad s \geq 0, \tag{22}$$

where $\Psi_0(s)$ is the classical Ekman solution

$$\Psi_0(s) = -e^{-(1+i)\lambda s}(u_g + iv_g).$$

Theorem 3.4 *The function defined by* (22) *is the solution of* (6) *with* (2) *and* (3) *if*

$$\begin{aligned}\varphi(s) =& \frac{v_g - iu_g}{2(1+i)\lambda} \int_0^s k_2(t) e^{-(1+i)\lambda(s+t)} (e^{(1+i)\lambda t} - e^{-(1+i)\lambda t}) dt \\ &+ \frac{v_g - iu_g}{2(1+i)\lambda} \int_s^\infty k_2(t) (e^{(1+i)\lambda(s-t)} - e^{-(1+i)\lambda(s+t)}) e^{-(1+i)\lambda t} dt.\end{aligned} \tag{23}$$

Proof Suppose for simplicity that $k(z)$ is C^2-smooth. From the definition of $\Psi(s)$, we have

$$\Psi'(s) = \frac{k(z)}{k_*} \frac{dU}{dz} + i \frac{k(z)}{k_*} \frac{dV}{dz} \tag{24}$$

and

$$\Psi''(s) = \frac{k'(z)k(z)}{k_*^2} \frac{dV}{dz} + \frac{k^2(z)}{k_*^2} \frac{d^2U}{dz^2} + i(\frac{k'(z)k(z)}{k_*^2} \frac{dV}{dz} + \frac{k^2(z)}{k_*^2} \frac{d^2U}{dz^2}). \tag{25}$$

Using (21), (6) becomes

$$\begin{cases} k'(z)\frac{dV}{dz} + k(z)\frac{d^2V}{dz^2} = fU, \\ k'(z)\frac{dU}{dz} + k(z)\frac{d^2U}{dz^2} = -fV. \end{cases}$$

From (24) and (25) we obtain

$$\Psi''(s) = i\frac{k(z)}{k_*^2} f\Psi(s), \tag{26}$$

and the corresponding boundary conditions are

$$\begin{aligned} \Psi(s) &= -(u_g + iv_g), \quad \text{at } s = 0, \\ \Psi(s) &\to 0, \quad \text{as } s \to \infty. \end{aligned}$$

Inserting (22) and (7) into (26) one obtains

$$\Psi_0''(s) + \epsilon\varphi''(s) = i\left(\frac{f}{k_*} + \epsilon k_2(s)\right)(\Psi_0(s) + \epsilon\varphi(s) + o(\epsilon)).$$

From the definition of $\Psi_0(s)$, we have

$$\Psi_0''(s) = \frac{if}{k_*}\Psi_0(s).$$

Dividing by ϵ and letting $\epsilon \to 0$, this yields a second order differential equation for φ, namely

$$\varphi''(s) - \frac{if}{k_*}\varphi(s) = ik_2(s)\Psi_0(s), \tag{27}$$

The boundary conditions are transformed to

$$\varphi(0) = 0, \quad \varphi \to 0 \quad \text{as } s \to \infty. \tag{28}$$

From Lemma 3.3, we know that $G(\cdot, \cdot)$ is the Green's function of (27) and (28), and the solution can be expressed as the convolution

$$\varphi(s) = i\int_0^\infty k_2(t)\Psi_0(t)G(s,t)dt \qquad s \geq 0.$$

Therefore the solution to (27) and (28) is given by (23). □

4 An example

Constantin and Johnson [14] considered the case of an eddy viscosity which is constant above a certain height, with a non-constant below, that is

$$k(z) = \begin{cases} a + bz, & 0 \le z \le z_0, \\ a + bz_0, & z > z_0, \end{cases} \tag{29}$$

where $a > 0,\ z_0 > 0, b \in R$. Motivated by [14], we change (29) to

$$k(z) = \begin{cases} a + bz, & 0 \le z \le z_0, \\ k_* + e^{-z}, & z > z_0, \end{cases}$$

here $a > 0,\ z_0 > 0$, and we assume $a + bz_0 = k_* + e^{-z_0}$ for continuity. Here, $\varepsilon = 1$. From (19), we get

$$s = \begin{cases} \frac{k_*}{b} \ln \frac{a+bz}{a}, & 0 \le s \le s_0, \\ s_0 + \ln \frac{k_0 e^{z}+1}{k_0 e^{z_0}+1}, & s > s_0, \end{cases}$$

with $s_0 = \frac{k_*}{b} \ln \frac{k_*}{a} > 0$. Note that

$$k_3(s) = \frac{f}{k_*^2} k(z) = \begin{cases} \frac{af}{k_*^2} e^{\frac{bs}{k_*}}, & 0 \le s \le s_0, \\ \frac{f}{k_*} + \frac{f}{k_*} \frac{1}{(k_* e^{z_0}+1)e^{s-s_0} - 1}, & s > s_0, \end{cases} \tag{30}$$

with $z_0 = \frac{a}{b}(e^{\frac{bs_0}{k^*}} - 1)$.

For the solution in the lower part of the layer $s \in [0, s_0]$, we set

$$\Psi(s) = \Phi(x), \quad \text{with } x = e^{\frac{bs}{2k_*}}.$$

Then (26) will be transformed into the Bessel equation

$$x^2 \frac{d^2\Phi}{dx^2} + x\frac{d\Phi}{dx} - \frac{4iaf}{b^2} x^2 \Phi = 0.$$

Let $\beta = \sqrt{-\frac{4iaf}{b^2}} = \frac{2\sqrt{af}}{|b|} e^{-\frac{\pi}{4}i}$, and the general solution of above Bessel equation can be expressed as

$$\Phi(x) = c_1 J_0(\beta x) + c_2 Y_0(\beta x),$$

where $J_0(x)$ and $Y_0(x)$ are the first and second kind of the Bessel functions with order 0 respectively. In the upper part of the layer, $s \in (s_0, \infty)$, we obtain the solution $\varphi(s)$ by (23).

Also motivated by [14] and (7) with

$$k_1(z) = \begin{cases} e^{-z_0} - bz_0 + bz, & 0 \le z \le z_0, \\ e^{-z}, & z > z_0, \end{cases}$$

we could change (29) to

$$k(z) = \begin{cases} k_* + \varepsilon(e^{-z_0} - bz_0 + bz), & 0 \le z \le z_0, \\ k_* + \varepsilon e^{-z}, & z > z_0, \end{cases}$$

with $k_\star > 0$, $b > 0$ and $z_0 > 0$. It follows from (19) that

$$s = \begin{cases} \frac{k_*}{b\varepsilon} \ln \frac{k_* + e^{-z_0}\varepsilon - \varepsilon b z_0 + \varepsilon b z}{a}, & 0 \le s \le s_0, \\ s_0 + \ln \frac{k_* e^z + \varepsilon}{k_* e^{z_0} + \varepsilon}, & s > s_0, \end{cases}$$

with $s_0 = \frac{k_*}{b\varepsilon} \ln \frac{k_* + e^{-z_0}\varepsilon}{k_* + e^{-z_0}\varepsilon - b\varepsilon z_0} > 0$. From (20), we see that

$$k_2(s) = \begin{cases} \frac{f e^{\frac{b\varepsilon s}{k_*}} (e^{-z_0} - bz_0 + \frac{k_*}{\varepsilon})}{k_*^2} - \frac{f}{\varepsilon k_*}, & 0 \le s \le s_0, \\ \frac{f}{k_*(k_* e^{z_0} + \varepsilon) e^{s - s_0} - \varepsilon}, & s > s_0, \end{cases} \tag{31}$$

then we obtain the solution $\varphi(s)$ by (23) and (31).

Next, we change (29) to

$$k(z) = \begin{cases} 5e^{-z} - 2, & 0 \le z \le z_0, \\ 5e^{-z_0} - 2, & z > z_0, \end{cases}$$

where z_0 satisfies $k_* = 5 \cdot e^{-z_0} - 2 > 0$. From (19), we have

$$s = \begin{cases} -\frac{k_*}{2} \ln \frac{a + bz}{a}, & 0 \le s \le s_0, \\ s_0 + \ln \frac{k_0 e^z + 1}{k_0 e^{z_0} + 1}, & s > s_0, \end{cases}$$

with $s_0 = -\frac{k_*}{2} \ln \frac{5 - 2e^{z_0}}{3} > 0$. Then we have

$$k_3(s) = \frac{f}{k_*^2} k(z) = \frac{f}{k_*^2} \left(5 \cdot \frac{2}{5 - 3e^{-\frac{2}{k_*}s}} - 2 \right) = \frac{2f}{k_*^2} \frac{\frac{3}{5} e^{-\frac{2}{k_*}s}}{1 - \frac{3}{5} e^{-\frac{2}{k_*}s}}, \qquad 0 < s \le s_0.$$

Therefore

$$k_3(s) = \begin{cases} \frac{2f}{k_*^2} \frac{\frac{3}{5} e^{-\frac{2}{k_*}s}}{1 - \frac{3}{5} e^{-\frac{2}{k_*}s}}, & 0 \le s \le s_0, \\ \frac{f}{5e^{-z_0} - 2}, & s > s_0. \end{cases} \tag{32}$$

For the solution in the upper part of the layer, $s > s_0$, we have

$$\Psi''(s) = i\frac{f}{5e^{-z_0}-2}\Psi(s) = \frac{(1+i)^2}{2}\frac{f}{5e^{-z_0}-2}\Psi(s),$$

so the general solution can be expressed by

$$\Psi(s) = c_1 e^{(1+i)\sqrt{\frac{f}{2(5e^{-z_0}-2)}}s} + c_2 e^{-(1+i)\sqrt{\frac{f}{2(5e^{-z_0}-2)}}s}.$$

For the solution in the lower part of the layer $s \in [0, s_0]$, we set

$$\zeta = \frac{3}{5}e^{-\frac{2}{k_*}s}, \quad \Psi(s) = \Phi(\zeta).$$

From (32), $k_3(s)$ can be expressed by

$$k_3(s) = \frac{2f}{k_*^2}\frac{\zeta}{1-\zeta}.$$

Now (26) can be written as

$$\frac{36}{25k_*^2}e^{-\frac{4s}{k_*}}\frac{d^2\Phi(\zeta)}{d\zeta^2} + \frac{12}{5k_*^2}e^{-\frac{2s}{k_*}}\frac{d\Phi(\zeta)}{d\zeta} = i\frac{2f}{k_*^2}\frac{\zeta}{1-\zeta}\Phi(\zeta),$$

and then we get the hypergeometric equation

$$\frac{d^2\Phi(\zeta)}{d\zeta^2} + \frac{1}{\zeta}\frac{d\Phi(\zeta)}{d\zeta} = if\frac{25}{18}\frac{\zeta}{1-\zeta}\Phi(\zeta). \tag{33}$$

If we let

$$\beta = \frac{5(1+i)}{6}\sqrt{f}, \quad \gamma = -\beta,$$

then the solution of (33) is

$$\Phi(\zeta) = cF(\beta, \gamma, 1, \zeta),$$

where c is an arbitrary complex constant and F is the Gauss hypergeometric function.

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Affiliations

Michal Fečkan[1,2] · Yi Guan[3] · Donal O'Regan[4] · JinRong Wang[3,5]

Michal Fečkan
Michal.Feckan@fmph.uniba.sk

Yi Guan
xyguanyi@163.com

Donal O'Regan
donal.oregan@nuigalway.ie

1 Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia

2 Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

3 Department of Mathematics, Guizhou University, Guiyang 550025, Guizhou, China

4 School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland

5 School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China