Monatshefte für Mathematik (2020) 193:573–589
https://doi.org/10.1007/s00605-020-01411-w

Local-in-space blow-up and symmetry of traveling wave solutions to a generalized two-component Dullin–Gottwald–Holm system

Wenguang Cheng[1,2] · **Tianzhou Xu**[1]

Received: 29 February 2020 / Accepted: 30 March 2020 / Published online: 15 April 2020

Abstract
In this paper, we are concerned with the Cauchy problem for a generalized two-component Dullin–Gottwald–Holm system arising from the shallow water regime with nonzero constant vorticity. We provide new sufficient conditions on the initial data which lead to the local-in-space blow-up. In addition, it is shown that horizontally symmetric weak solutions to this system must be traveling wave solutions.

Keywords Generalized two-component Dullin–Gottwald–Holm system · Local-in-space blow-up · Traveling waves

Mathematics Subject Classification 35B44 · 35G55 · 35C07

1 Introduction

In this paper, we study the Cauchy problem for the following generalized two-component Dullin–Gottwald–Holm (g2DGH) system [6,27,38]

Communicated by Adrian Constantin.

✉ Wenguang Cheng
wgchengmath@yahoo.com

1 School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, People's Republic of China

2 Department of Mathematics, Yuxi Normal University, Yuxi 653100, People's Republic of China

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_xu_{xx} + uu_{xxx}) + \gamma u_{xxx} & \\ +\rho\rho_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

which was derived in the shallow water regime using Ivanov's modeling approach in [30]. Here $u(t, x)$ stands for the horizontal velocity of the fluid, $\rho(t, x)$ is in connection with the free surface elevation from equilibrium, the parameter A is related to a linear underlying shear flow, and σ is a dimensionless parameter providing the competition (or balance) in fluid convection between nonlinear steepening and amplification due to stretching. In the derivation of system (1.1), the boundary conditions $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$ are required.

System (1.1) includes several classical shallow water wave models. For example, if $\sigma = 1$, system (1.1) becomes the two-component Dullin–Gottwald–Holm (2DGH) system [8,24,33,49]

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - 2u_xu_{xx} - uu_{xxx} + \gamma u_{xxx} + \rho\rho_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0, \quad x \in \mathbb{R}. \end{cases} \tag{1.2}$$

If we further take $\gamma = 0$ in the 2DGH system (1.2), then it recovers the integrable two-component Camassa–Holm (2CH) system

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - 2u_xu_{xx} - uu_{xxx} + \rho\rho_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0, \quad x \in \mathbb{R}, \end{cases} \tag{1.3}$$

which was originally introduced by Olver and Rosenau [36] and rigorously derived in the context of shallow water regime [11,30]. The Cauchy problem of the 2CH system has been studied widely [5,15,17–23,31,40,44]. Here we give a brief review. The local well-posedness of the 2CH system with initial data in the Sobolev spaces and Besov spaces was established in [15,21,22]. It was shown that the 2CH system admits global strong solutions [11,19,22] and also finite time blow-up solutions [11,15,17,19,22, 23,40,44]. Besides, it has global weak solutions [18,20,31]. Moreover, the Lipschitz continuous dependence of conservative solutions to the 2CH system was investigated in [5].

If $\rho = 0$, $\sigma = 1$, system (1.1) is reduced to the Dullin–Gottwald–Holm (DGH) equation [13], modeling the unidirectional propagation of shallow water waves over a flat bottom,

$$u_t - u_{txx} - Au_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \gamma u_{xxx} = 0, \quad t > 0, \quad x \in \mathbb{R}. \tag{1.4}$$

Equation (1.4) was deduced by Dullin, Gottwald and Holm [13] using the asymptotic analysis and a near-identity normal form transformation from water wave theory. This equation is integrable in the sense that it admits the Lax pair and bi-Hamiltonian structure [13]. For the DGH equation, the local well-posedness of strong solutions was proved in [37,41,43]. The precise blow-up scenario and several blow-up results for strong solutions to the DGH equation were presented in [25,28,29,34,35,39,43, 45,47,48]. For the global existence of weak and strong solutions, we refer the readers to [34,39,41,43,46] and the references therein. Furthermore, the orbital stability of single peakon and the train of peakons for the DGH equation were proved in [26] and [32], respectively. It is worth pointing out that the study of the DGH equation was based on methods that were developed for the Camassa–Holm (CH) equation [4,16], corresponding to the choice $\gamma = 0$ in (1.4). In particular, the idea to associate diffeomorphisms to derive invariance properties was pioneered by Constantin–Escher [10] and Constantin [9], while Constantin and Strauss [12] verified that the peakons of the CH equation are orbitally stable.

Han et al. [27] investigated the blow-up mechanism to the g2DGH system (1.1) on the line and provided two sufficient conditions for wave breaking of strong solutions in finite time, they also classified traveling wave solutions for this system. Chen and Yan [6] studied the wave breaking phenomena and global existence of the g2DGH system (1.1) on the circle. Recently, inspired by the ideas of [1–3], Wang and Zhu [38] established a local-in-space blow-up criterion for the g2DGH system (1.1) with $\sigma = 1$, $\gamma = 0$ (corresponding to the 2CH system (1.3)), that is, a blow-up condition involving only the values of $u_{0,x}(x_0)$ and $u_0(x_0)$ in a single point x_0 of the real line. One of our purposes in the present paper is to improve this result to $1 \le \sigma \le 4$, $\gamma \in \mathbb{R}$.

Another interesting issue investigated here is concerned with the traveling wave solutions of the g2DGH system. Following the ideas of [7,14], we show that horizontally symmetric weak solutions of the g2DGH system must be traveling wave solutions. Denote $G(x) = \frac{1}{2}e^{-|x|}$, the fundamental solution of the operator $(1-\partial_x^2)^{-1}$ on $\mathbb{R}$, which satisfies $(1-\partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{R})$. Therefore, the Cauchy problem (1.1) can be reformulated as

$$\begin{cases} u_t + (\sigma u - \gamma)u_x = -\partial_x G * \left[\frac{3-\sigma}{2}u^2 + (\gamma - A)u + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2\right], & t > 0, \quad x \in \mathbb{R}, \\ \rho_t + u\rho_x = -u_x\rho, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.5}$$

The entire paper is organized as follows. In the next section, we recall several useful results including the local well-posedness and the precise blow-up scenario for the g2DGH system, as well as the convolution estimates. Then we establish a new local-in-space blow-up criterion with suitable conditions on the initial data. Finally in Sect. 3, we prove that an x-symmetric weak solution of the g2DGH system is necessarily a traveling wave solution.

2 Local-in-space blow-up

This section is devoted to investigating the local-in-space blow-up criterion for the Cauchy problem (1.5). We first recall the local well-posedness result and the precise blow-up scenario of (1.5).

Lemma 2.1 ([27]) *If $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 2$, then there exists a maximal time $T = T(\|(u_0, \rho_0 - 1)\|_{H^s \times H^{s-1}}) > 0$ and a unique solution $(u, \rho - 1)$ to the Cauchy problem (1.5) such that $(u, \rho - 1) \in C([0, T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$. Moreover, the solution depends continuously on the initial data, and T can be chosen to be independent of s.*

Lemma 2.2 ([27]) *Suppose that $\sigma \neq 0$ and $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 2$. Then the corresponding solution (u, ρ) to the initial value problem (1.5) blows up in finite time $T > 0$ if and only if*

$$\liminf_{t \to T} \left(\inf_{x \in \mathbb{R}} \sigma u_x(t, x) \right) = -\infty.$$

We define two convolution operators G_+ and G_- as

$$\begin{aligned} G_+ * f(x) &= \frac{e^{-x}}{2} \int_{-\infty}^{x} e^y f(y)\, dy, \\ G_- * f(x) &= \frac{e^x}{2} \int_x^{\infty} e^{-y} f(y)\, dy, \end{aligned} \tag{2.1}$$

it is easy to see that

$$G = G_+ + G_-, \quad G_x = G_- - G_+. \tag{2.2}$$

The following convolution estimates are crucial to prove the local-in-space blow-up criterion.

Lemma 2.3 ([1,2,42]) *Let $1 \leq \sigma \leq 4$.*

(i) *If $\gamma \neq A$ and $\sigma \neq 3$, then*

$$G_\pm * \left[\frac{3-\sigma}{2} \left(u + \frac{\gamma - A}{3 - \sigma} \right)^2 + \frac{\sigma}{2} u_x^2 \right] \geq \frac{\sqrt{\sigma}}{8} (\sqrt{12 - 3\sigma} - \sqrt{\sigma}) \left(u + \frac{\gamma - A}{3 - \sigma} \right)^2.$$

(ii) *If $\gamma = A$, then*

$$G_\pm * \left(\frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 \right) \geq \frac{\sqrt{\sigma}}{8} (\sqrt{12 - 3\sigma} - \sqrt{\sigma}) u^2.$$

We are now in a position to state a local-in-space type of blow-up mechanism for the Cauchy problem (1.5).

Theorem 2.1 *Let* $1 \leq \sigma \leq 4$, $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s \geq 2$ *and* $T > 0$ *be the maximal time of existence of the corresponding solution* (u, ρ) *to* (1.5). *Assume that there exists a point* $x_0 \in \mathbb{R}$ *such that*

$$\rho_0(x_0) = 0,$$

and

$$u_{0,x}(x_0) < \begin{cases} -\beta_\sigma \left| u_0(x_0) + \dfrac{\gamma - A}{3 - \sigma} \right|, & \text{if } \gamma \neq A,\ \sigma \neq 3, \\ -\beta_\sigma |u_0(x_0)|, & \text{if } \gamma = A, \end{cases} \tag{2.3}$$

where

$$\beta_\sigma = \sqrt{-\frac{1}{2} + \frac{3}{\sigma} - \frac{\sqrt{12 - 3\sigma}}{2\sqrt{\sigma}}}. \tag{2.4}$$

Then the solution (u, ρ) *blows up in finite time with an estimate of the blow-up time* T^* *as*

$$T^* \leq \begin{cases} \dfrac{2}{\sigma \sqrt{u_{0,x}^2(x_0) - \beta_\sigma^2 \left(u_0(x_0) + \dfrac{\gamma - A}{3 - \sigma} \right)^2}}, & \text{if } \gamma \neq A,\ \sigma \neq 3, \\ \dfrac{2}{\sigma \sqrt{u_{0,x}^2(x_0) - \beta_\sigma^2 u_0^2(x_0)}}, & \text{if } \gamma = A. \end{cases}$$

Proof The two associated Lagrangian scales of the g2DGH system are established by

$$\begin{cases} \dfrac{\partial q_1}{\partial t} = \sigma u(t, q_1) - \gamma, & 0 < t < T, \\ q_1(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{2.5}$$

and

$$\begin{cases} \dfrac{\partial q_2}{\partial t} = u(t, q_2), & 0 < t < T, \\ q_2(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{2.6}$$

where $u \in C^1([0, T), H^{s-1}(\mathbb{R}))$ is the first component of the solution (u, ρ) to (1.5). A direct calculation yields

$$q_{1,tx} = \sigma u_x(t, q_1) q_{1,x},$$

and

$$q_{2,tx} = u_x(t, q_2)q_{2,x}.$$

Hence for $t > 0$, $x \in \mathbb{R}$, we have

$$q_{1,x}(t, x) = e^{\sigma \int_0^t u_x(\tau, q_1(\tau, x))\, d\tau} > 0,$$

and

$$q_{2,x}(t, x) = e^{\int_0^t u_x(\tau, q_2(\tau, x))\, d\tau} > 0,$$

which implies that $q_i(t, \cdot) : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are two diffeomorphisms of the line for every $t \in [0, T)$. It is inferred that there exists an $x_1(t) \in \mathbb{R}$ such that

$$q_2(t, x_1(t)) = q_1(t, x_0).$$

When $t = 0$, we obtain

$$x_1(0) = q_2(0, x_1(0)) = q_1(0, x_0) = x_0.$$

Along with the trajectory of $q_2(t, x_1(t))$, we have

$$\frac{d}{dt}\rho(t, q_2(t, x_1(t))) = -u_x(t, q_2(t, x_1(t)))\rho(t, q_2(t, x_1(t))). \tag{2.7}$$

Since $\rho_0(x_0) = 0$, integrating the above equation gives

$$\begin{aligned} \rho(t, q_2(t, x_1(t))) &= \rho(0, q_2(0, x_1(0)))e^{-\int_0^t u_x(\tau, q_2(\tau, x_1(\tau)))\, d\tau} \\ &= \rho_0(x_0)e^{-\int_0^t u_x(\tau, q_2(\tau, x_1(\tau)))\, d\tau} = 0, \end{aligned}$$

that is

$$\rho(t, q_1(t, x_0)) = \rho(t, q_2(t, x_1(t))) = 0.$$

Differentiating the first equation in system (1.5) with respect to x and using the identity $-\partial_x^2 G * f = f - G * f$, one gets

$$\begin{aligned} u_{tx} + (\sigma u - \gamma)u_{xx} &= \frac{3-\sigma}{2}u^2 + (\gamma - A)u - \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 \\ &\quad - G * \left[\frac{3-\sigma}{2}u^2 + (\gamma - A)u + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2\right]. \end{aligned} \tag{2.8}$$

From the convolution estimates in Lemma 2.3, we infer that

$$
\begin{aligned}
&\frac{d}{dt}u_x(t,q_1(t,x_0))\\
&=[u_{tx}+(\sigma u-\gamma)u_{xx}](t,q_1(t,x_0))\\
&=\left\{\frac{3-\sigma}{2}u^2+(\gamma-A)u-\frac{\sigma}{2}u_x^2-G*\left[\frac{3-\sigma}{2}u^2+(\gamma-A)u+\frac{\sigma}{2}u_x^2\right.\right.\\
&\left.\left.+\frac{1}{2}\rho^2\right]\right\}(t,q_1(t,x_0))\\
&\le\left\{\frac{3-\sigma}{2}u^2+(\gamma-A)u-\frac{\sigma}{2}u_x^2-(G_++G_-)*\left[\frac{3-\sigma}{2}u^2+(\gamma-A)u\right.\right.\\
&\left.\left.+\frac{\sigma}{2}u_x^2\right]\right\}(t,q_1(t,x_0))\\
&=\left\{\frac{3-\sigma}{2}\left(u+\frac{\gamma-A}{3-\sigma}\right)^2-\frac{\sigma}{2}u_x^2-(G_++G_-)*\left[\frac{3-\sigma}{2}\left(u+\frac{\gamma-A}{3-\sigma}\right)^2\right.\right.\\
&\left.\left.+\frac{\sigma}{2}u_x^2\right]\right\}(t,q_1(t,x_0))\\
&\le\left[\frac{3-\sigma}{2}\left(u+\frac{\gamma-A}{3-\sigma}\right)^2-\frac{\sigma}{2}u_x^2-\frac{\sqrt{\sigma}}{4}(\sqrt{12-3\sigma}\right.\\
&\left.-\sqrt{\sigma})\left(u+\frac{\gamma-A}{3-\sigma}\right)^2\right](t,q_1(t,x_0))\\
&=\frac{\sigma}{2}\left[\beta_\sigma^2\left(u+\frac{\gamma-A}{3-\sigma}\right)^2-u_x^2\right](t,q_1(t,x_0)),
\end{aligned}\tag{2.9}
$$

where β_σ is given in (2.4). Due to the right-hand side of (2.9), we set

$$
\begin{aligned}
M(t)&=\left[\beta_\sigma\left(u+\frac{\gamma-A}{3-\sigma}\right)-u_x\right](t,q_1(t,x_0)),\\
N(t)&=\left[\beta_\sigma\left(u+\frac{\gamma-A}{3-\sigma}\right)+u_x\right](t,q_1(t,x_0)).
\end{aligned}
$$

Then (2.9) becomes

$$
\frac{d}{dt}u_x(t,q_1(t,x_0))\le\frac{\sigma}{2}M(t)N(t).\tag{2.10}
$$

For $1\le\sigma\le4$, notice that $0\le\beta_\sigma\le1$. Similarly using Lemma 2.3 it follows that

$$
\begin{aligned}
&\frac{dM}{dt}=\{\beta_\sigma[u_t+(\sigma u-\gamma)u_x]-[u_{tx}+(\sigma u-\gamma)u_{xx}]\}(t,q_1(t,x_0))\\
&=\left\{-\frac{3-\sigma}{2}u^2-(\gamma-A)u+\frac{\sigma}{2}u_x^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+G*\left[\frac{3-\sigma}{2}u^2+(\gamma-A)u+\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2\right]\\
&-\beta_\sigma\partial_xG*\left[\frac{3-\sigma}{2}u^2+(\gamma-A)u+\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2\right]\Bigg\}(t,q_1(t,x_0))\\
=&\left\{-\frac{3-\sigma}{2}u^2-(\gamma-A)u+\frac{\sigma}{2}u_x^2+(1-\beta_\sigma)G_-\right.\\
&*\left[\frac{3-\sigma}{2}u^2+(\gamma-A)u+\frac{\sigma}{2}u_x^2\right]\\
&+(1+\beta_\sigma)G_+*\left[\frac{3-\sigma}{2}u^2+(\gamma-A)u+\frac{\sigma}{2}u_x^2\right]+(1-\beta_\sigma)G_-*\left(\frac{1}{2}\rho^2\right)\\
&\left.+(1+\beta_\sigma)G_+*\left(\frac{1}{2}\rho^2\right)\right\}(t,q_1(t,x_0))\\
\geq&\left\{-\frac{3-\sigma}{2}\left(u+\frac{\gamma-A}{3-\sigma}\right)^2+\frac{\sigma}{2}u_x^2+(1-\beta_\sigma)G_-\right.\\
&*\left[\frac{3-\sigma}{2}\left(u+\frac{\gamma-A}{3-\sigma}\right)^2+\frac{\sigma}{2}u_x^2\right]\\
&\left.+(1+\beta_\sigma)G_+*\left[\frac{3-\sigma}{2}\left(u+\frac{\gamma-A}{3-\sigma}\right)^2+\frac{\sigma}{2}u_x^2\right]\right\}(t,q_1(t,x_0))\\
\geq&\left[-\frac{3-\sigma}{2}\left(u+\frac{\gamma-A}{3-\sigma}\right)^2+\frac{\sigma}{2}u_x^2+\frac{\sqrt{\sigma}}{4}(\sqrt{12-3\sigma}\right.\\
&\left.-\sqrt{\sigma})\left(u+\frac{\gamma-A}{3-\sigma}\right)^2\right](t,q_1(t,x_0))\\
=&-\frac{\sigma}{2}\left[\beta_\sigma^2\left(u+\frac{\gamma-A}{3-\sigma}\right)^2-u_x^2\right](t,q_1(t,x_0))\\
=&-\frac{\sigma}{2}MN,
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
&\frac{dN}{dt}=\{\beta_\sigma[u_t+(\sigma u-\gamma)u_x]+u_{tx}+(\sigma u-\gamma)u_{xx}\}(t,q_1(t,x_0))\\
=&\left\{\frac{3-\sigma}{2}u^2+(\gamma-A)u-\frac{\sigma}{2}u_x^2-G\right.\\
&*\left[\frac{3-\sigma}{2}u^2+(\gamma-A)u+\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2\right]\\
&\left.-\beta_\sigma\partial_xG*\left[\frac{3-\sigma}{2}u^2+(\gamma-A)u+\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2\right]\right\}(t,q_1(t,x_0))
\end{aligned}
$$

$$
\begin{aligned}
&= \Bigg\{ \frac{3-\sigma}{2}u^2 + (\gamma - A)u - \frac{\sigma}{2}u_x^2 - (1+\beta_\sigma)G_- \\
&\quad * \left[\frac{3-\sigma}{2}u^2 + (\gamma - A)u + \frac{\sigma}{2}u_x^2 \right] \\
&\quad - (1-\beta_\sigma)G_+ * \left[\frac{3-\sigma}{2}u^2 + (\gamma - A)u + \frac{\sigma}{2}u_x^2 \right] - (1+\beta_\sigma)G_- * \left(\frac{1}{2}\rho^2 \right) \\
&\quad - (1-\beta_\sigma)G_+ * \left(\frac{1}{2}\rho^2 \right) \Bigg\} (t, q_1(t, x_0)) \\
&\leq \Bigg\{ \frac{3-\sigma}{2}\left(u + \frac{\gamma - A}{3-\sigma} \right)^2 - \frac{\sigma}{2}u_x^2 - (1+\beta_\sigma)G_- \\
&\quad * \left[\frac{3-\sigma}{2}\left(u + \frac{\gamma - A}{3-\sigma} \right)^2 + \frac{\sigma}{2}u_x^2 \right] \\
&\quad - (1-\beta_\sigma)G_+ * \left[\frac{3-\sigma}{2}\left(u + \frac{\gamma - A}{3-\sigma} \right)^2 + \frac{\sigma}{2}u_x^2 \right] \Bigg\} (t, q_1(t, x_0)) \\
&\leq \Bigg[\frac{3-\sigma}{2}\left(u + \frac{\gamma - A}{3-\sigma} \right)^2 - \frac{\sigma}{2}u_x^2 - \frac{\sqrt{\sigma}}{4}(\sqrt{12-3\sigma} \\
&\quad - \sqrt{\sigma}) \left(u + \frac{\gamma - A}{3-\sigma} \right)^2 \Bigg] (t, q_1(t, x_0)) \\
&= \frac{\sigma}{2}\left[\beta_\sigma^2 \left(u + \frac{\gamma - A}{3-\sigma} \right)^2 - u_x^2 \right] (t, q_1(t, x_0)) \\
&= \frac{\sigma}{2}MN.
\end{aligned} \tag{2.12}
$$

By the assumption (2.3) on $u_0(x_0)$, we know that

$$
\begin{aligned}
&M(0) > 0, \quad N(0) < 0, \\
&M'(0) \geq -\frac{\sigma}{2}M(0)N(0) > 0, \\
&N'(0) \leq \frac{\sigma}{2}M(0)N(0) < 0.
\end{aligned} \tag{2.13}
$$

We now claim for $t \in [0, T)$

$$
M'(t) > 0, \quad N'(t) < 0. \tag{2.14}
$$

If (2.14) does not hold, then there exists $t_0 \in [0, T)$ such that

$$
t_0 = \min\{t \in [0, T) | M'(t) = 0 \ \text{ or } \ N'(t) = 0\}. \tag{2.15}
$$

Then (2.13) implies that $t_0 > 0$. The definition of t_0 given in (2.15) together with (2.11) and (2.12) leads to

$$0 = M'(t_0) \geq -\frac{\sigma}{2}M(t_0)N(t_0) \quad \text{or} \quad 0 = N'(t_0) \leq \frac{\sigma}{2}M(t_0)N(t_0). \tag{2.16}$$

On the other hand, since $M(t)$ is increasing and $N(t)$ is decreasing on $[0, t_0]$, we have

$$M(t_0) \geq M(0) > 0, \quad N(t_0) \leq N(0) < 0.$$

Thus,

$$-\frac{\sigma}{2}M(t_0)N(t_0) > 0 \quad \text{and} \quad \frac{\sigma}{2}M(t_0)N(t_0) < 0,$$

which contradict with (2.16). Therefore (2.14) holds for all $t \in [0, T)$.

Furthermore from (2.13) we see that

$$M(t) \geq M(0) > 0, \quad N(t) \leq N(0) < 0. \tag{2.17}$$

Let

$$H(t) = \sqrt{-M(t)N(t)}.$$

In view of (2.11), (2.12) and (2.17), it follows that

$$\begin{aligned} H'(t) &= -\frac{M'N + MN'}{2\sqrt{-MN}} \\ &\geq -\frac{-\frac{\sigma}{2}MN^2 + \frac{\sigma}{2}M^2N}{2\sqrt{-MN}} \\ &= \frac{M-N}{2\sqrt{-MN}}\left(-\frac{\sigma}{2}MN\right) \\ &\geq \frac{\sigma}{2}H^2(t). \end{aligned} \tag{2.18}$$

Solving the above differential inequality, we get

$$H(t) \geq \frac{2H(0)}{2 - \sigma H(0)t}.$$

It is thereby inferred that

$$-u_x(t, q_1(t, x_0)) = \frac{M-N}{2} \geq H(t) \geq \frac{2H(0)}{2 - \sigma H(0)t} \to +\infty, \quad \text{as} \quad t \to \frac{2}{\sigma H(0)},$$

which, in view of Lemma 2.2, implies that the solution (u, ρ) blows up at a finite time T^* with

$$T^* \le \frac{2}{\sigma H(0)}.$$

The proof of Theorem 2.1 is completed. □

3 Symmetry of traveling wave solutions

In this section, we will consider the unique x-symmetric weak solution of the g2DGH system and prove that such a solution must be a traveling wave solution. We first rewrite system (1.1) with $\rho = 1 + \eta$ ($\eta \to 0$ as $|x| \to \infty$) as

$$\begin{cases} (1-\partial_x^2)u_t - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \gamma u_{xxx} + (1+\eta)\eta_x = 0, \\ \eta_t + ((1+\eta)u)_x = 0. \end{cases} \tag{3.1}$$

A weak solution of system (3.1) is defined as follows.

Definition 3.1 If $(u, \eta) \in C(\mathbb{R}_+, H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ satisfies

$$\begin{cases} \int\int_{\mathbb{R}_+\times\mathbb{R}} \Big[u(1-\partial_x^2)\phi_t + \left(\frac{3}{2}u^2 - Au + \frac{\sigma}{2}u_x^2 + \frac{(1+\eta)^2}{2}\right)\phi_x \\ \quad - \left(\frac{\sigma}{2}u^2 - \gamma u\right)\phi_{xxx}\Big]\,dtdx = 0, \\ \int\int_{\mathbb{R}_+\times\mathbb{R}} [\eta\phi_t + (1+\eta)u\phi_x]\,dtdx = 0, \end{cases} \tag{3.2}$$

for all $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$. Then (u, η) is a weak solution to system (3.1).

Using the notation $\langle\cdot,\cdot\rangle$ for distributions, (3.2) can be rewritten as

$$\begin{cases} \langle u, (1-\partial_x^2)\phi_t\rangle + \langle \frac{3}{2}u^2 - Au + \frac{\sigma}{2}u_x^2 + \frac{(1+\eta)^2}{2}, \phi_x\rangle - \langle \frac{\sigma}{2}u^2 - \gamma u, \phi_{xxx}\rangle = 0, \\ \langle \eta, \phi_t\rangle + \langle (1+\eta)u, \phi_x\rangle = 0. \end{cases} \tag{3.3}$$

Now we give the definition of x-symmetric solution.

Definition 3.2 A solution $(u(t,x), \eta(t,x))$ is x-symmetric if there exists a function $b(t) \in C^1(\mathbb{R}_+)$ such that for every $t > 0$,

$$(u(t,x), \eta(t,x)) = (u(t, 2b(t) - x), \eta(t, 2b(t) - x))$$

for a.e. $x \in \mathbb{R}$. We say that $b(t)$ is the symmetric axis of $(u(t,x), \eta(t,x))$.

The next lemma gives the form of a weak solution of (3.1).

Lemma 3.1 *Suppose that* $(U(x), V(x)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ *and satisfies*

$$\begin{cases} \int_{\mathbb{R}} \Big[-cU(1-\partial_x^2)\varphi_x + \left(\frac{3}{2}U^2 - AU + \frac{\sigma}{2}U_x^2 + \frac{(1+V)^2}{2}\right)\varphi_x \\ \quad - \left(\frac{\sigma}{2}U^2 - \gamma U\right)\varphi_{xxx}\Big]\,dx = 0, \\ \int_{\mathbb{R}} [-cV\varphi_x + (1+V)U\varphi_x]\,dx = 0, \end{cases} \tag{3.4}$$

for all $\varphi \in C_0^\infty(\mathbb{R})$. *Then* (u, η) *given by*

$$(u(t,x), \eta(t,x)) = (U(x - c(t - t_0)), V(x - c(t - t_0))) \tag{3.5}$$

is a weak solution of system (3.1) *for any fixed* $t_0 \in \mathbb{R}$.

Proof Without loss of generality, suppose $t_0 = 0$. Following the arguments in [14], we obtain that (u, η) belongs to $C(\mathbb{R}, H^1(\mathbb{R}) \times L^2(\mathbb{R}))$. For any $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, letting $\phi_c(t,x) = \phi(t, x + ct)$, we find that

$$\partial_x(\phi_c) = (\phi_x)_c, \quad \partial_t(\phi_c) = (\phi_t)_c + c(\phi_x)_c. \tag{3.6}$$

Assume $(u(t,x), \eta(t,x)) = (U(x - ct), V(x - ct))$, it is easy to see that

$$\begin{aligned}
&\langle u, \phi\rangle = \langle U, \phi_c\rangle, \quad \langle u^2, \phi\rangle = \langle U^2, \phi_c\rangle, \quad \langle u_x^2, \phi\rangle = \langle U_x^2, \phi_c\rangle,\\
&\langle \eta, \phi\rangle = \langle V, \phi_c\rangle,\\
&\langle (1+\eta)^2, \phi\rangle = \langle (1+V)^2, \phi_c\rangle, \quad \langle (1+\eta)u, \phi\rangle = \langle (1+V)U, \phi_c\rangle,
\end{aligned} \tag{3.7}$$

where $(U, V) = (U(x), V(x))$. It follows from (3.6) and (3.7) that

$$\begin{aligned}
&\langle u, (1-\partial_x^2)\phi_t\rangle = \left\langle U, \left((1-\partial_x^2)\phi_t\right)_c\right\rangle = \langle U, (1-\partial_x^2)(\partial_t\phi_c - c\partial_x\phi_c)\rangle,\\
&\left\langle \frac{3}{2}u^2 - Au + \frac{\sigma}{2}u_x^2 + \frac{(1+\eta)^2}{2}, \phi_x\right\rangle = \left\langle \frac{3}{2}U^2 - AU + \frac{\sigma}{2}U_x^2 + \frac{(1+V)^2}{2}, \partial_x\phi_c\right\rangle,\\
&\left\langle \frac{\sigma}{2}u^2 - \gamma u, \phi_{xxx}\right\rangle = \left\langle \frac{\sigma}{2}U^2 - \gamma U, \partial_x^3\phi_c\right\rangle,\\
&\langle \eta, \phi_t\rangle = \langle V, (\phi_t)_c\rangle = \langle V, \partial_t\phi_c - c\partial_x\phi_c\rangle,\\
&\langle (1+\eta)u, \phi_x\rangle = \langle (1+V)U, \partial_x\phi_c\rangle.
\end{aligned} \tag{3.8}$$

Notice that (U, V) is independent of time, for T sufficiently large such that it does not belong to the support of ϕ_c, it is then deduced that

$$\begin{aligned}
\langle U, (1-\partial_x^2)\partial_t\phi_c\rangle &= \int_{\mathbb{R}} U(x) \int_{\mathbb{R}_+} \partial_t(1-\partial_x^2)\phi_c \, dt dx\\
&= \int_{\mathbb{R}} U(x)[(1-\partial_x^2)\phi_c(T,x) - (1-\partial_x^2)\phi_c(0,x)]\, dx = 0,\\
\langle U, \partial_t\phi_c\rangle &= \int_{\mathbb{R}} U(x) \int_{\mathbb{R}_+} \partial_t\phi_c \, dt dx = \int_{\mathbb{R}} U(x)[\phi_c(T,x)\\
&\quad - \phi_c(0,x)]\, dx = 0,\\
\langle V, \partial_t\phi_c\rangle &= \int_{\mathbb{R}} V(x) \int_{\mathbb{R}_+} \partial_t\phi_c \, dt dx = \int_{\mathbb{R}} V(x)[\phi_c(T,x)\\
&\quad - \phi_c(0,x)]\, dx = 0.
\end{aligned} \tag{3.9}$$

Combining (3.8) with (3.9) gives rise to

$$
\begin{aligned}
&\langle u, (1-\partial_x^2)\phi_t\rangle + \left\langle \frac{3}{2}u^2 - Au + \frac{\sigma}{2}u_x^2 + \frac{(1+\eta)^2}{2}, \phi_x \right\rangle - \left\langle \frac{\sigma}{2}u^2 - \gamma u, \phi_{xxx} \right\rangle \\
&\quad = \langle U, -c(1-\partial_x^2)\partial_x\phi_c\rangle + \left\langle \frac{3}{2}U^2 - AU + \frac{\sigma}{2}U_x^2 + \frac{(1+V)^2}{2}, \partial_x\phi_c \right\rangle \\
&\qquad - \left\langle \frac{\sigma}{2}U^2 - \gamma U, \partial_x^3\phi_c \right\rangle \\
&\quad = \int_{\mathbb{R}_+}\int_{\mathbb{R}} \left[-cU(1-\partial_x^2)\partial_x\phi_c + \left(\frac{3}{2}U^2 - AU + \frac{\sigma}{2}U_x^2 + \frac{(1+V)^2}{2} \right)\partial_x\phi_c \right. \\
&\qquad \left. - \left(\frac{\sigma}{2}U^2 - \gamma U \right)\partial_x^3\phi_c \right] dxdt \\
&\quad = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\langle \eta, \phi_t\rangle + \langle (1+\eta)u, \phi_x\rangle &= \langle V, -c\partial_x\phi_c\rangle + \langle (1+V)U, \partial_x\phi_c\rangle \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}} [-cV\partial_x\phi_c + (1+V)U\partial_x\phi_c]\, dxdt = 0,
\end{aligned}
$$

where we used (3.4) with $\varphi(x) = \phi_c(t, x)$, which belongs to $C_0^\infty(\mathbb{R})$, for every given $t \geq 0$. This completes the proof of the lemma. □

Finally, we state the main result of this section.

Theorem 3.1 *If $(u(t, x), \eta(t, x))$ is a unique weak solution of system (3.1) and is x-symmetric, then $(u(t, x), \eta(t, x))$ is a traveling wave solution.*

Proof Recalling Definition 3.1 and noting that $C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ is dense in $C_0^1(\mathbb{R}_+, C_0^3(\mathbb{R}))$, thus we only consider the test function ϕ belonging to $C_0^1(\mathbb{R}_+, C_0^3(\mathbb{R}))$. Let us introduce the notation

$$
\phi_b(t, x) = \phi(t, 2b(t) - x), \quad b(t) \in C^1(\mathbb{R}).
$$

Then we derive that $(\phi_b)_b = \phi$ and

$$
\begin{aligned}
&\partial_x u_b = -(\partial_x u)_b, \quad \partial_x\phi_b = -(\partial_x\phi)_b, \\
&\partial_t\phi_b = (\partial_t\phi)_b + 2\dot{b}(\partial_x\phi)_b,
\end{aligned}
\tag{3.10}
$$

where $\dot{b}$ denotes the derivative of b with respect to t. Furthermore, we obtain that

$$
\begin{aligned}
&\langle u_b, \phi\rangle = \langle u, \phi_b\rangle, \langle u_b^2, \phi\rangle = \langle u^2, \phi_b\rangle, \quad \langle (\partial_x u_b)^2, \phi\rangle = \langle (\partial_x u)^2, \phi_b\rangle, \\
&\langle \eta_b, \phi\rangle = \langle \eta, \phi_b\rangle, \langle (1+\eta_b)^2, \phi\rangle = \langle (1+\eta)^2, \phi_b\rangle, \\
&\langle (1+\eta_b)u_b, \phi\rangle = \langle (1+\eta)u, \phi_b\rangle.
\end{aligned}
\tag{3.11}
$$

Since (u,η) is x-symmetric, using (3.10) and (3.11) we have

$$\begin{aligned} \langle u,(1-\partial_x^2)\phi_t\rangle &= \langle u,((1-\partial_x^2)\partial_t\phi)_b\rangle = \langle u,(1-\partial_x^2)(\partial_t\phi_b+2\dot b\partial_x\phi_b)\rangle,\\ \left\langle \frac{3}{2}u^2-Au+\frac{\sigma}{2}u_x^2+\frac{(1+\eta)^2}{2},\phi_x\right\rangle &= -\left\langle \frac{3}{2}u^2-Au+\frac{\sigma}{2}u_x^2+\frac{(1+\eta)^2}{2},\partial_x\phi_b\right\rangle,\\ \left\langle \frac{\sigma}{2}u^2-\gamma u,\phi_{xxx}\right\rangle &= -\left\langle \frac{\sigma}{2}u^2-\gamma u,\partial_x^3\phi_b\right\rangle,\\ \langle \eta,\phi_t\rangle &= \langle \eta,\partial_t\phi_b+2\dot b\partial_x\phi_b\rangle,\\ \langle (1+\eta)u,\phi_x\rangle &= -\langle (1+\eta)u,\partial_x\phi_b\rangle. \end{aligned} \tag{3.12}$$

In view of (3.3), one deduces

$$\begin{aligned} &\langle u,(1-\partial_x^2)\phi_t\rangle + \left\langle \tfrac{3}{2}u^2-Au+\tfrac{\sigma}{2}u_x^2+\tfrac{(1+\eta)^2}{2},\phi_x\right\rangle - \left\langle \tfrac{\sigma}{2}u^2-\gamma u,\phi_{xxx}\right\rangle\\ &= \langle u,(1-\partial_x^2)(\partial_t\phi_b+2\dot b\partial_x\phi_b)\rangle - \left\langle \tfrac{3}{2}u^2-Au+\tfrac{\sigma}{2}u_x^2+\tfrac{(1+\eta)^2}{2},\partial_x\phi_b\right\rangle\\ &\quad + \left\langle \tfrac{\sigma}{2}u^2-\gamma u,\partial_x^3\phi_b\right\rangle = 0,\\ &\langle \eta,\phi_t\rangle + \langle (1+\eta)u,\phi_x\rangle\\ &= \langle \eta,\partial_t\phi_b+2\dot b\partial_x\phi_b\rangle - \langle (1+\eta)u,\partial_x\phi_b\rangle = 0. \end{aligned} \tag{3.13}$$

Thanks to $(\phi_b)_b=\phi$, by taking $\phi=\phi_b$ in (3.13), we get

$$\begin{cases} \langle u,(1-\partial_x^2)(\partial_t\phi+2\dot b\partial_x\phi)\rangle - \left\langle \tfrac{3}{2}u^2-Au+\tfrac{\sigma}{2}u_x^2+\tfrac{(1+\eta)^2}{2},\partial_x\phi\right\rangle\\ \quad + \left\langle \tfrac{\sigma}{2}u^2-\gamma u,\partial_x^3\phi\right\rangle = 0,\\ \langle \eta,\partial_t\phi+2\dot b\partial_x\phi\rangle - \langle (1+\eta)u,\partial_x\phi\rangle = 0. \end{cases} \tag{3.14}$$

Combining (3.3) and (3.14), we find

$$\begin{cases} \langle u,\dot b(1-\partial_x^2)\partial_x\phi\rangle - \left\langle \tfrac{3}{2}u^2-Au+\tfrac{\sigma}{2}u_x^2+\tfrac{(1+\eta)^2}{2},\partial_x\phi\right\rangle + \left\langle \tfrac{\sigma}{2}u^2-\gamma u,\partial_x^3\phi\right\rangle = 0,\\ \langle \eta,\dot b\partial_x\phi\rangle - \langle (1+\eta)u,\partial_x\phi\rangle = 0. \end{cases} \tag{3.15}$$

For a fixed $t_0>0$ and any $\varphi\in C_0^\infty(\mathbb{R})$, let $\phi_\epsilon(t,x)=\varphi(x)\rho_\epsilon(t)$, where $\rho_\epsilon\in C_0^\infty(\mathbb{R}_+)$ is a mollifier with the property that $\rho_\epsilon\to\delta(t-t_0)$, the Dirac mass at t_0, as $\epsilon\to 0$. From (3.15), using the test function $\phi_\epsilon(t,x)$, we get

$$\begin{cases} \displaystyle\int_{\mathbb{R}}\left[(1-\partial_x^2)\partial_x\varphi\int_{\mathbb{R}_+}\dot b u\rho_\epsilon(t)\,dt\right]dx\\ \displaystyle\quad -\int_{\mathbb{R}}\left[\partial_x\varphi\int_{\mathbb{R}_+}\left(\frac{3}{2}u^2-Au+\frac{\sigma}{2}u_x^2+\frac{(1+\eta)^2}{2}\right)\rho_\epsilon(t)\,dt\right]dx\\ \displaystyle\quad +\int_{\mathbb{R}}\left[\partial_x^3\varphi\int_{\mathbb{R}_+}\left(\frac{\sigma}{2}u^2-\gamma u\right)\rho_\epsilon(t)\,dt\right]dx=0,\\ \displaystyle\int_{\mathbb{R}}\left(\partial_x\varphi\int_{\mathbb{R}_+}\dot b\eta\rho_\epsilon(t)\,dt\right)dx-\int_{\mathbb{R}}\left[\partial_x\varphi\int_{\mathbb{R}_+}(1+\eta)u\rho_\epsilon(t)\,dt\right]dx=0. \end{cases} \tag{3.16}$$

Notice that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}_+} \dot{b}u\rho_\epsilon(t)\,dt = \dot{b}(t_0)u(t_0,x),$$
$$\lim_{\epsilon\to 0}\int_{\mathbb{R}_+} \dot{b}\eta\rho_\epsilon(t)\,dt = \dot{b}(t_0)\eta(t_0,x),$$

in $L^2(\mathbb{R})$, and

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\mathbb{R}_+}\left(\frac{3}{2}u^2 - Au + \frac{\sigma}{2}u_x^2 + \frac{(1+\eta)^2}{2}\right)\rho_\epsilon(t)\,dt\\
&\quad = \frac{3}{2}u^2(t_0,x) - Au(t_0,x) + \frac{\sigma}{2}u_x^2(t_0,x) + \frac{(1+\eta(t_0,x))^2}{2},\\
&\lim_{\epsilon\to 0}\int_{\mathbb{R}_+}\left(\frac{\sigma}{2}u^2 - \gamma u\right)\rho_\epsilon(t)\,dt = \frac{\sigma}{2}u^2(t_0,x) - \gamma u(t_0,x),\\
&\lim_{\epsilon\to 0}\int_{\mathbb{R}_+}(1+\eta)u\rho_\epsilon(t)\,dt = (1+\eta(t_0,x))u(t_0,x),
\end{aligned}$$

in $L^1(\mathbb{R})$. Hence, by taking the limit of (3.16) as $\epsilon\to 0$, one obtains

$$\begin{cases}
\int_{\mathbb{R}} \dot{b}(t_0)u(t_0,x)(1-\partial_x^2)\partial_x\varphi\,dx - \int_{\mathbb{R}}\left(\frac{3}{2}u^2(t_0,x) - Au(t_0,x) + \frac{\sigma}{2}u_x^2(t_0,x)\right.\\
\quad\left. + \frac{(1+\eta(t_0,x))^2}{2}\right)\partial_x\varphi\,dx + \int_{\mathbb{R}}\left(\frac{\sigma}{2}u^2(t_0,x) - \gamma u(t_0,x)\right)\partial_x^3\varphi\,dx = 0,\\
\int_{\mathbb{R}} \dot{b}(t_0)\eta(t_0,x)\partial_x\varphi\,dx - \int_{\mathbb{R}}(1+\eta(t_0,x))u(t_0,x)\partial_x\varphi\,dx = 0.
\end{cases} \tag{3.17}$$

Therefore $(u(t_0,x),\eta(t_0,x))$ satisfies (3.4) for $c=\dot{b}(t_0)$. By means of Lemma 3.1, we see that $(\tilde{u}(t,x),\tilde{\eta}(t,x)) = (u(t_0,x-\dot{b}(t_0)(t-t_0)),\eta(t_0,x-\dot{b}(t_0)(t-t_0)))$ is a traveling wave solution of system (3.1). Due to $(\tilde{u}(t_0,x),\tilde{\eta}(t_0,x)) = (u(t_0,x),\eta(t_0,x))$ and the uniqueness of the solution of system (3.1), we deduce that $(\tilde{u}(t,x),\tilde{\eta}(t,x)) = (u(t,x),\eta(t,x))$ for any $t>0$. Thus, we complete the proof of Theorem 3.1. □

Acknowledgements This work is supported by the Scientific Research Foundation of Educational Committee of Yunnan Province (No. 2019J0735), and the Joint Special Funds of the Basic Research Foundation of Yunnan Province (No. 2019FH001-080).

References

1. Brandolese, L.: Local-in-space criteria for blowup in shallow water and dispersive rod equations. Commun. Math. Phys. **330**, 401–414 (2014)
2. Brandolese, L., Cortez, M.F.: Blowup issues for a class of nonlinear dispersive wave equations. J. Differ. Equ. **256**, 3981–3998 (2014)
3. Brandolese, L., Cortez, M.F.: On permanent and breaking waves in hyperelastic rods and rings. J. Funct. Anal. **266**, 6954–6987 (2014)

4. Camassa, R., Holm, D.D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**, 1661–1664 (1993)
5. Cai, H., Chen, G., Shen, Y.N.: Lipschitz metric for conservative solutions of the two-component Camassa–Holm system. Z. Angew. Math. Phys. **68**, 5 (2017)
6. Chen, C.X., Yan, Y.H.: On the wave breaking phenomena for the generalized periodic two-component Dullin–Gottwald–Holm system. J. Math. Phys. **53**, 103709 (2012)
7. Chen, R.M., Fan, L.L., Gao, H.J., Liu, Y.: Breaking waves and solitary waves to the rotation-two-component Camassa–Holm system. SIAM J. Math. Anal. **49**, 3573–3602 (2017)
8. Chen, Y., Gao, H.J., Liu, Y.: On the Cauchy problem for the two-component Dullin–Gottwald–Holm system. Discrete Contin. Dyn. Syst. **33**, 3407–3441 (2013)
9. Constantin, A.: Existence of permanent and breaking waves for a shallow water equation: a geometric approach. Ann. Inst. Fourier (Grenoble) **50**, 321–362 (2000)
10. Constantin, A., Escher, J.: On the blow-up rate and the blow-up set of breaking waves for a shallow water equation. Math. Z. **233**, 75–91 (2000)
11. Constantin, A., Ivanov, R.: On an integrable two-component Camassa-Holm shallow water system. Phys. Lett. A **372**, 7129–7132 (2008)
12. Constantin, A., Strauss, W.A.: Stability of peakons. Commun. Pure Appl. Math. **53**, 603–610 (2000)
13. Dullin, H.R., Gottwald, G.A., Holm, D.D.: An integrable shallow water equation with linear and nonlinear dispersion. Phys. Rev. Lett. **87**, 194501 (2001)
14. Ehrnström, M., Holden, H., Raynaud, X.: Symmetric waves are traveling waves. Int. Math. Res. Not. **2009**, 4578–4596 (2009)
15. Escher, J., Lechtenfeld, O., Yin, Z.Y.: Well-posedness and blow-up phenomena for the 2-component Camassa–Holm equation. Discrete Contin. Dyn. Syst. **19**, 493–513 (2007)
16. Fuchssteiner, B., Fokas, A.S.: Symplectic structures, their Bäcklund transformations and hereditary symmetries. Phys. D **4**, 47–66 (1981)
17. Grunert, K.: Blow-up for the two-component Camassa–Holm system. Discrete Contin. Dyn. Syst. **35**, 2041–2051 (2015)
18. Grunert, K., Holden, H., Raynaud, X.: Global solutions for the two-component Camassa–Holm system. Commun. Partial Differ. Equ. **37**, 2245–2271 (2012)
19. Guan, C.X., Yin, Z.Y.: Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system. J. Differ. Equ. **248**, 2003–2014 (2010)
20. Guan, C.X., Yin, Z.Y.: Global weak solutions for a two-component Camassa–Holm shallow water system. J. Funct. Anal. **260**, 1132–1154 (2011)
21. Guan, C.X., Zhu, H.: Well-posedness of a class of solutions to an integrable two-component Camassa-Holm system. J. Math. Anal. Appl. **470**, 413–433 (2019)
22. Gui, G.L., Liu, Y.: On the global existence and wave-breaking criteria for the two-component Camassa–Holm system. J. Funct. Anal. **258**, 4251–4278 (2010)
23. Gui, G.L., Liu, Y.: On the Cauchy problem for the two-component Camassa–Holm system. Math. Z. **268**, 45–66 (2011)
24. Guo, F., Gao, H.J., Liu, Y.: On the wave-breaking phenomena for the two-component Dullin–Gottwald–Holm system. J. Lond. Math. Soc. **86**, 810–834 (2012)
25. Guo, Z.G., Ni, L.D.: Wave breaking for the periodic weakly dissipative Dullin–Gottwald–Holm equation. Nonlinear Anal. **74**, 965–973 (2011)
26. Hakkaev, S.: Stability of peakons for an integrable shallow water equation. Phys. Lett. A **354**, 137–144 (2006)
27. Han, Y.W., Guo, F., Gao, H.J.: On solitary waves and wave-breaking phenomena for a generalized two-component integrable Dullin–Gottwald–Holm system. J. Nonlinear Sci. **23**, 617–656 (2013)
28. He, F.Y., Wei, Z.Q.: A remark on wave breaking for the Dullin–Gottwald–Holm equation. Appl. Anal. **96**, 1928–1934 (2017)
29. Hoang, D.T.: The local criteria for blowup of the Dullin–Gottwald–Holm equation and the two-component Dullin–Gottwald–Holm system. Ann. Fac. Sci. Toulouse Math. **25**, 995–1012 (2016)
30. Ivanov, R.: Two-component integrable systems modelling shallow water waves: the constant vorticity case. Wave Mot. **46**, 389–396 (2009)
31. Li, M.J., Zhang, Q.T.: Uniqueness of conservative solutions to the two-component Camassa–Holm system via characteristics. Commun. Math. Sci. **14**, 1581–1597 (2016)
32. Liu, X.X., Yin, Z.Y.: Orbital stability of the sum of N peakons for the Dullin–Gottwald–Holm equation. Nonlinear Anal. Real World Appl. **13**, 2414–2422 (2012)

33. Liu, X.X., Yin, Z.Y.: Local well-posedness and stability of solitary waves for the two-component Dullin–Gottwald–Holm system. Nonlinear Anal. **88**, 1–15 (2013)
34. Liu, Y.: Global existence and blow-up solutions for a nonlinear shallow water equation. Math. Ann. **335**, 717–735 (2006)
35. Novruzov, E.: Blow-up phenomena for the weakly dissipative Dullin–Gottwald–Holm equation. J. Math. Phys. **54**, 092703 (2013)
36. Olver, P.J., Rosenau, P.: Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support. Phys. Rev. E **53**, 1900–1906 (1996)
37. Tian, L.X., Gui, G.L., Liu, Y.: On the well-posedness problem and the scattering problem for the Dullin–Gottwald–Holm equation. Commun. Math. Phys. **257**, 667–701 (2005)
38. Wang, Y., Zhu, M.: On the persistence and blow up for the generalized two-component Dullin–Gottwald–Holm system. Monatsh. Math. **191**, 377–394 (2020)
39. Wei, L., Wang, Y.: Symmetry analysis, conserved quantities and applications to a dissipative DGH equation. J. Differ. Equ. **266**, 3189–3208 (2019)
40. Wei, L., Wang, Y., Zhang, H.Y.: Breaking waves and persistence property for a two-component Camassa–Holm system. J. Math. Anal. Appl. **445**, 1084–1096 (2017)
41. Yan, K., Yin, Z.Y.: On the solutions of the Dullin–Gottwald–Holm equation in Besov spaces. Nonlinear Anal. Real World Appl. **13**, 2580–2592 (2012)
42. Yang, S.J., Xu, T.Z.: Local-in-space blow-up and symmetric waves for a generalized two-component Camassa–Holm system. Appl. Math. Comput. **347**, 514–521 (2019)
43. Yin, Z.Y.: Well-poseness, blowup, and global existence for an integrable shallow water equation. Discrete Contin. Dyn. Syst. **11**, 393–411 (2004)
44. Zhang, P.Z., Liu, Y.: Stability of solitary waves and wave-breaking phenomena for the two-component Camassa–Holm system. Int. Math. Res. Not. **2010**, 1981–2021 (2010)
45. Zhang, S.H., Yin, Z.Y.: On the blow-up phenomena of the periodic Dullin–Gottwald–Holm equation. J. Math. Phys. **49**, 113504 (2008)
46. Zhang, S.H., Yin, Z.Y.: Global weak solutions for the Dullin–Gottwald–Holm equation. Nonlinear Anal. **72**, 1690–1700 (2010)
47. Zhang, Z.Y., Huang, J.H., Sun, M.B.: Blow-up phenomena for the weakly dissipative Dullin–Gottwald–Holm equation revisited. J. Math. Phys. **56**, 092701 (2015)
48. Zhou, Y.: Blow-up of solutions to the DGH equation. J. Funct. Anal. **250**, 227–248 (2007)
49. Zhu, M., Xu, J.X.: On the wave-breaking phenomena for the periodic two-component Dullin–Gottwald–Holm system. J. Math. Anal. Appl. **391**, 415–428 (2012)