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Pointwise persistence and shadowing

W. Jung[1] · K. Lee[1] · C. A. Morales[2]

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Abstract

We show that every pointwise persistent homeomorphism with the shadowing property is persistent. The proof relies on a new tracing property, the *persistent shadowing property*, which has its own interest. We shown for instance that expansive homeomorphisms with this property are s-topologically stable in the sense of Kawaguchi (Discrete Contin Dyn Syst 39(5):2743–2761, 2019). We also present orbital versions of our results.

Keywords Shadowing property · Persistent shadowing property · Homeomorphism

Mathematics Subject Classification Primary 37C50; Secondary 54H20

1 Introduction

Lewowicz defined the persistence systems in [9]. Roughly speaking, a dynamical system is *persistent* if its trajectories can be seen on every small perturbation of it.

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✉ C. A. Morales
morales@impa.br

W. Jung
wcjung@cnu.ac.kr

K. Lee
khlee@cnu.ac.kr

1 Department of Mathematics, Chungnam National University, Daejeon 305-764, Republic of Korea

2 Instituto de Matemática, Universidade Federal do Rio de Janeiro, P.O. Box 68530, Rio de Janeiro 21945-970, Brazil

He proved that every two- or three-dimensional expansive diffeomorphism with dense hyperbolic periodic points (including the pseudo-Anosov maps) are persistent [9]. Sakai and Kobayashi [16] observed that the full shift on two symbols is not persistent though expansive with shadowing (hence topologically stable). Every topologically stable homeomorphism of a compact manifold is persistent. The converse of this assertion is false as the pseudo-Anosov maps are persistent but not topologically stable. However, such concepts (namely topological stability and persistence) are equivalent in the group automorphisms of a solenoidal group. Sakai [15] proved that the C^1 interior of the persistent diffeomorphisms and the Axiom A diffeomorphisms with the strong transversality condition coincide on any compact manifold. Lewowicz [9] also defined *persistent point* of a dynamical system as a point whose orbit has a representative on any small perturbation of the systems. It is then natural to consider the *pointwise persistent systems* namely dynamical systems for which every point is persistent. Every persistent dynamical systems is pointwise persistent. The converse of this assertion is true for equicontinuous homeomorphisms [4].

In this paper we will prove such a converse for homeomorphisms with the shadowing property. More precisely, every pointwise persistent homeomorphism with the shadowing property of a compact metric space is persistent. The proof relies on a new tracing property referred to as *persistent shadowing property*. This property will be analyzed and, in particular, we prove that it together with expansivity implies topological stability in the strong sense [7]. Motivated by the orbital shadowing lemma [11] we present the notion of orbital persistence. We use it to present orbital versions of our results. Let us state them in a precise way.

Hereafter X is a compact metric space. The C^0-distance between maps $l, r : X \to X$ will be denoted by

$$d_{C^0}(l, r) = \sup_{x \in X} d(l(x), r(x)).$$

Let $f : X \to X$ be a homeomorphism. Following [9], we say that $K \subset X$ is a *persistent set* if for every $\epsilon > 0$ there is $\delta > 0$ such that for every $x \in K$ and every homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \le \delta$ there is $y \in X$ such that $d(f^n(x), g^n(y)) \le \epsilon$ for $n \in \mathbb{Z}$. We say that x is a *persistent point* if the single point set $K = \{x\}$ is persistent. Denote by $Persi(f)$ the set of persistent points of f.

Definition 1 We say that a homeomorphism $f : X \to X$ is :

1. *Persistent* if X is a persistent set.
2. *Pointwise persistent* if $Persi(f) = X$.

Here are some illustrating examples.

Example 1 Examples of persistent homeomorphisms are the pseudo Anosov maps on two or three dimensional manifolds and the Axiom A diffeomorphisms with the strong transversality conditions (including the Anosov or Morse-Smale diffeomorphisms). Examples which are not are the circle rotations.

Recall that a homeomorphism $f : X \to X$ is *periodic* if for every $x \in X$ there is $n \in \mathbb{N}$ such that $f^n(x) = x$.

Example 2 Let X_c be the convergent sequence space, namely, $X = \{0\} \cup \{\frac{1}{k} : k \in \mathbb{N}\}$ endowed with the Euclidean metric from $\mathbb{R}$. Let $f : X_c \to X_c$ be a periodic homeomorphism. One can easily see that for every $x \in X$ there is $\delta > 0$ (depending on x) such that $f = g$ at $\{f^n(x) : n \in \mathbb{Z}\}$ for every homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$. It follows that f is pointwise persistent.

Every persistent homeomorphism is clearly pointwise persistent. As mentioned before, the converse is true for equicontinuous homeomorphisms [4]. We shall prove that converse also for homeomorphisms with the shadowing property.

Recall that given $\delta > 0$ we say that a bi-infinite sequence $(x_n)_{n\in\mathbb{Z}}$ is a δ-*pseudo orbit* if $d(f(x_n), x_{n+1}) \leq \delta$ for every $n \in \mathbb{Z}$. We say that $(x_n)_{n\in\mathbb{Z}}$ *can be* δ-*shadowed* if there is $x \in X$ such that $d(f^n(x), x_n) \leq \delta$ for every $n \in \mathbb{Z}$. To emphasize f we just write that $(x_n)_{n\in\mathbb{Z}}$ can be (f, δ)-*shadowed*.

Definition 2 A homeomorphism has the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit can be ϵ-shadowed.

A related example is as follows.

Example 3 Persistence (and so pointwise persistence) do not imply the shadowing property. Indeed, the pseudo Anosov maps are persistent (hence pointwise persistent), expansive but not topologically stable and so without the shadowing property.

Now we can state our result.

Theorem 1 *Every pointwise persistent homeomorphism with the shadowing property of a compact metric space is persistent.*

An example where this result can be applied is as follows.

Example 4 The full two symbol shift is not persistent (proved in [16]) but has the shadowing property so it is not pointwise persistent (by Theorem 1).

We also have the following corollary.

Corollary 1 *Every periodic homeomorphism of* X_c *(Example 2) is persistent.*

Proof All such homeomorphisms are equicontinuous (Proposition 1.1 in [7]) hence with the shadowing property (since X_c is totally disconnected [10]) and also pointwise persistence (by Example 2). Then, Theorem 1 applies. □

This corollary can be also proved by using [4] and applies to the identity of X_c. An example where Theorem 1 (but not [4]) can be applied is as follows.

Example 5 Consider the subset of $\mathbb{R}^2$ defined by

$$X = \{(0, 0)\} \cup \left(\bigcup_{n\in\mathbb{N}} \left\{ \frac{1}{n} \right\} \times \left[0, \frac{1}{n} \right] \right),$$

endowed the induced metric. Define the homeomorphism $f : X \to X$ by

$$f(x, y) = \begin{cases} (0, 0), & \text{if } x = 0 \\ \left(\frac{1}{n}, ny^2\right), & \text{if} x = \frac{1}{n} \text{ for some } n \in \mathbb{N}. \end{cases}$$

Then, f is pointwise persistent with the shadowing property and so persistent by Theorem 1. However, f is not equicontinuous.

This paper is organized as follows. In Sect. 2 we introduce and study the notion of persistent shadowing property. In Sect. 3 we present the pointwise version of the persistent shadowing lemma. In Sect. 4 we prove Theorem 1. In Sect. 5 we present the notion of *orbital persistent homeomorphism* and use it to obtain an orbital version of Theorem 1.

2 Persistent shadowing property

It is natural to ask under which conditions for a given homeomorphism of a compact metric space there is some $\delta > 0$ such that every homeomorphism within δ of it has the shadowing property. We will briefly refer to this property as the *robust shadowing lemma*. Every homeomorphism of a finite metric space has that property but not the ones on compact manifolds [8]. It seems that homeomorphisms with the robust shadowing lemma only exist on totally disconnected spaces. Sakai introduced the notion of C^1 uniform pseudo orbit tracing property for diffeomorphisms on compact manifolds [13]. He proved that the sole ones with this property are the Axiom A diffeomorphisms with the strong transversality condition [15]. Gu [6] considered the C^0 uniformly pseudo orbit tracing property and Kulczycki [8] proved that no homeomorphism with that property exists on a compact manifold of positive dimension.

In this section we will present an alternative shadowing that takes into account the small perturbations of the system under consideration. This will be referred to as the *persistent shadowing property*. This property is a hybrid between persistence and shadowing and in fact is equivalent to them (justifying the name for this property). We prove that every expansive homeomorphism enjoying it is s-topologically stable in the sense of Kawaguchi. Let us present the details.

Hereafter X will denote a compact metric space and $f : X \to X$ is a homeomorphism.

Definition 3 We say that f has the *persistent shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of every homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$ can be (g, ϵ)-shadowed.

This property clearly implies the shadowing property (just take $g = f$ in the definition above). It also implies persistence by the following lemma.

Lemma 1 *Every homeomorphism with the persistent shadowing property of a compact metric space is persistent.*

Proof Suppose that a homeomorphism of a compact metric space $f : X \to X$ has the persistent shadowing property. Let $\epsilon > 0$ and δ be given by persistent shadowing for this ϵ. Fix $x \in X$ and a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \le \delta$. Since

$$d(g(f^n(x)), f^{n+1}(x)) = d(g(f^n(x)), f(f^n(x))) \le d_{C^0}(g, f) \le \delta, \quad \forall n \in \mathbb{Z},$$

one has that $(f^n(x))_{n\in\mathbb{Z}}$ is a δ-pseudo orbit of g. Then, by persistent shadowing, there is $y \in X$ such that $d(f^n(x), g^n(y)) \le \epsilon$ for every $n \in \mathbb{Z}$. Therefore, f is persistent. □

Actually the persistent shadowing is equivalent to persistence and shadowing (Corollary 2). Let us recall the definition of topological stability (Walters [18]) and the recent notion of s-topological stability (Kawaguchi [7]).

Definition 4 A homeomorphism of a compact metric space $f : X \to X$ is *topologically stable* if for every $\epsilon > 0$ there is $\delta > 0$ such that for every homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \le \delta$ there is a continuous map $h : X \to X$ such that $d_{C^0}(h, I) \le \epsilon$ and $f \circ h = h \circ g$. If, additionally, the continuous map h can be taken to be onto, then we say that f is *s-topologically stable*.

Although topological stability does not imply persistence (e.g. the full two symbol shift [16]), s-topological stability does as reported below.

Lemma 2 *Every s-topologically stable homeomorphism of a compact metric space is persistent.*

Proof Let $f : X \to X$ be a s-topologically stable homeomorphism of a compact metric space. Fix $\epsilon > 0$ and let δ be given by s-topological stability for this ϵ. If $g : X \to X$ is a homeomorphism with $d_{C^0}(f, g) \le \delta$, then there is a continuous onto map $h : X \to X$ with $d_{C^0}(h, I) \le \epsilon$ and $f \circ h = h \circ g$. If now $x \in X$, then by choosing $y \in h^{-1}(x)$ one has

$$d(f^n(x), g^n(y)) = d(f^n(h(y)), g^n(y)) = d(h(g^n(y)), g^n(y)) \le d_{C^0}(h, I) \le \epsilon,$$

for every $n \in \mathbb{Z}$ proving the result. □

Now recall the notion of expansive homeomorphism [17].

Definition 5 A homeomorphism $f : X \to X$ is *expansive* if there is $e > 0$ (called expansivity constant) such that for every distinct points $x, y \in X$ there is an integer n such that $d(f^n(x), f^n(y)) > e$.

Walters stability theorem [18] asserts that every expansive homeomorphism with the shadowing property of a compact metric space is topologically stable. The question arises what will happen if we replace shadowing by persistent shadowing in this statement. On the other hand, not every expansive homeomorphism with the shadowing property of a compact metric space is s-topological stability. Indeed, the full shift on two symbols is expansive, has the shadowing property (hence it is topologically stable), is not persistent (proved in Example 1 in [16]) and so it is not s-topologically stable (by Lemma 2).

These facts motivate the following result.

Theorem 2 *Every expansive homeomorphism with the persistent shadowing property of a compact metric space is s-topologically stable.*

Proof Let $f : X \to X$ be an expansive homeomorphism with the persistent shadowing property of a compact metric space. Then, f is expansive with the shadowing property and so topologically stable by Walters theorem. Fix $\epsilon > 0$ and take $0 < \epsilon' < \frac{\min(e,\epsilon)}{2}$. Let δ_0 and δ_1 be given by the topological stability and the persistent shadowing property of f for ϵ' respectively. Take $\delta = \min\{\delta_0, \delta_1\}$ and a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$. It follows that $d_{C^0}(f, g) \leq \delta_0$ and so there is a continuous map $h : X \to X$ such that $d_{C^0}(h, I) \leq \epsilon'$ and $f \circ h = h \circ g$. In particular, $d_{C^0}(h, I) \leq \epsilon$ and so it suffices to show that h is onto.

Fix $x \in X$. It follows from the definition of δ that $d_{C^0}(f, g) \leq \delta_1$ and so we can prove as before that the bi-infinite sequence $(f^n(x))_{n\in\mathbb{Z}}$ is a δ_1-pseudo orbit of g. Then, by persistent shadowing, there is $y \in X$ such that

$$d(f^n(x), g^n(y)) \leq \epsilon', \quad \forall n \in \mathbb{Z}.$$

But $f \circ h = h \circ g$ so

$$d(f^n(h(y)), g^n(y)) = d(h(g^n(y)), g^n(y)) \leq d_{C^0}(h, I) \leq \epsilon', \quad \forall n \in \mathbb{Z}.$$

Then,

$$d(f^n(x), f^n(h(y))) \leq d(f^n(x), g^n(y)) + d(f^n(h(y)), g^n(y)) \leq \epsilon' + \epsilon' = 2\epsilon' \leq e,$$

for every $n \in \mathbb{Z}$. Since e is an expansivity constant, one has $h(y) = x$ so h is onto. This completes the proof. □

3 Persistent shadowable points

The shadowing property was decomposed into individual shadowings in [10]. This was also carried out in [3] for the eventual shadowing property considered in [5]. In this section we will do the same for the persistent shadowing property. We will compare the resulting notion (persistently shadowable points) with the shadowable and the persistent points. Let us present the details.

We say that a bi-infinite sequence $(x_n)_{n\in\mathbb{Z}}$ of X is through some subset $K \subset X$ if $x_0 \in K$. In the special case when K reduces to a single point x we just say that the sequence is through x.

Definition 6 ([10]) We say that x is a *shadowable point* if for every $\epsilon > 0$ there is $\delta > 0$ such that such that every δ-pseudo orbit through x can be ϵ-shadowed.

Denote by $Sh(f)$ the set of shadowable points of f. We know that $Sh(f) = X$ if and only if f has the shadowing property [10].

Definition 7 We say that x is a *persistently shadowable point* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit through x of a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$ can be (g, ϵ)-shadowed.

We denote by $PSh(f)$ the set of persistently shadowable points. It is clear that $PSh(f) \subset Sh(f)$.

Lemma 3 *A homeomorphism of a compact metric space $f : X \to X$ has the persistent shadowing property if and only if $PSh(f) = X$.*

Proof First we claim that $x \in PSh(f)$ if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit through $B[x, \delta]$ of a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$ can be (g, ϵ)-shadowed. Indeed, we only have to prove the necessity in this claim. Suppose by contradiction that $x \in PSh(f)$ but there are $\epsilon > 0$, a sequence of homeomorphisms $g_k : X \to X$ with $d_{C^0}(f, g_k) \leq \frac{1}{k}$, a sequence of $\frac{1}{k}$-pseudo orbits $\xi^k = (\xi^k_n)_{n\in\mathbb{N}_0}$ of g_k with $d(x, \xi^k_0) \leq \frac{1}{k}$ such that ξ^k cannot be $(g_k, 2\epsilon)$-shadowed for every $k \in \mathbb{N}$. For this ϵ we let δ be given by the persistently shadowableness of x. We can assume that $\delta < \epsilon$.

On the one hand,

$$\begin{aligned} d(g_k(x), g_k(\xi^k_0)) &\leq d(g_k(x), f(x) + d(f(x), f(\xi^k_0)) + d(f(\xi^k_0), g_k)\xi^k_0)) \\ &\leq 2d_{C^0}(f, g_k) + d(f(x), f(\xi^k_0)) \end{aligned}$$

and, on the other, X is compact so f is uniformly continuous. Then, we can choose k large satisfying

$$\max\left\{d(g_k(x), g_k(\xi^k_0)), \frac{1}{k}\right\} \leq \frac{\delta}{2}.$$

Now define the sequence $\hat{\xi} = (\hat{\xi}_n)_{n\in\mathbb{Z}}$ by

$$\hat{\xi}_n = \begin{cases} \xi^k_n & \text{if } n \neq 0 \\ x & \text{if } n = 0. \end{cases}$$

Since for $n \neq 0, -1$ one has $d(g_k(\hat{\xi}_n), \hat{\xi}_{n+1}) = d(g_k(\xi^k_n), \xi^k_{n+1}) \leq \frac{1}{k} \leq \delta$, for $n = 0$,

$$d(g_k(\hat{\xi}_0), \hat{\xi}_1) = d(g_k(x), \xi^k_1) \leq d(g_k(x), g_k(\xi^k_0)) + d(g_k(\xi^k_0), \xi^k_1) \leq \frac{\delta}{2} + \frac{1}{k} \leq \delta$$

and, for $n = -1$,

$$d(g_k(\hat{\xi}_{-1}), \hat{\xi}_0) = d(g_k(\xi^k_{-1})), x) \leq d(g_k(\xi^k_{-1}), \xi^k_0) + d(\xi^k_0, x) \leq \frac{\delta}{2} + \frac{1}{k} \leq \delta$$

one gets that $\hat{\xi}$ is a δ-pseudo-orbit of g_k. Since $d_{C^0}(f, g_k) \leq \delta$, $\hat{\xi}$ can be (g_k, ϵ)-shadowed namely there is $y \in X$ such that $d(g^n_k(y), \hat{\xi}_n) \leq \epsilon$ for every $n \in \mathbb{Z}$.

Since for $n \neq 0$ one has $d(g_k^n(y), \xi_n^k) = d(g_k^n(y), \hat{\xi}_n) \leq \epsilon$ and for $n = 0$

$$d(y, \xi_0^k) \leq d(y, x) + d(x, \xi_k^0) \leq d(y, \hat{\xi}_0) + \frac{1}{k} \leq \epsilon + \frac{1}{k} \leq \epsilon + \frac{\delta}{2} \leq 2\epsilon,$$

one has that ξ^k can be $(g_k, 2\epsilon)$-shadowed. This is a contradiction and the claim is proved.

Now we complete the proof of the lemma. Indeed, we only have to prove the sufficiency. Therefore, suppose that $PSh(f) = X$ and choose $\epsilon > 0$.

It follows from the claim that for every $x \in X$ there is $\delta_x > 0$ such that every δ_x-pseudo-orbit through the ball $B[x, \delta_x]$ of a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta_x$ can be eventually (g, ϵ)-shadowed. Since X is compact, we can cover X with finitely many of such balls namely

$$X = \bigcup_{i=1}^{l} B[x_i, \delta_{x_i}].$$

Take $\delta = \min\{\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_l}\}$ and let $(\xi_n)_{n\in\mathbb{Z}}$ be a δ-pseudo orbit of a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$. Clearly, $\xi_0 \in B[x_i, \delta_{x_i}]$ for some $1 \leq i \leq l$. So, $\{\xi_n\}_{n\in\mathbb{Z}}$ is a δ-pseudo-orbit through $B[x_i, \delta_{x_i}]$ of g. This implies that $(\xi_n)_{n\in\mathbb{Z}}$ is a δ_{x_i}-pseudo-orbit of g through $B[x_i, \delta_{x_i}]$. Then, $(\xi_n)_{n\in\mathbb{Z}}$ can be eventually (g, ϵ)-shadowed proving the result. □

Now we prove the following identity.

Lemma 4 $PSh(f) = Persi(f) \cap Sh(f)$.

Proof Take $x \in PSh(f)$, $\epsilon > 0$ and let δ be given by persistently shadowableness of x for this ϵ. Fix a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$. Since

$$d(g(f^n(x)), f^{n+1}(x)) = d(g(f^n(x)), f(f^n(x))) \leq d_{C^0}(g, f) \leq \delta, \quad \forall n \in \mathbb{Z},$$

one has that $(f^n(x))_{n\in\mathbb{Z}}$ is a δ-pseudo orbit of g which clearly is through x. Then, by persistent shadowing, there is $y \in X$ such that $d(f^n(x), g^n(y)) \leq \epsilon$ for every $n \in \mathbb{Z}$. It follows that $x \in Persi(f)$. Since $PSh(f) \subset Sh(f)$ we get $x \in Persi(f) \cap Sh(f)$. Therefore, $PSh(f) \subset Persi(f) \cap Sh(f)$.

Now suppose that $x \in Persi(f) \cap Sh(f)$. Fix $\epsilon > 0$ and let $\bar{\delta}$ be given by both the persistence and the shadowableness of f for $\frac{\epsilon}{2}$. Take $\delta = \frac{\bar{\delta}}{2}$. Let $g : X \to X$ be a homeomorphism with $d_{C^0}(f, g) \leq \delta$ and $(x_n)_{n\in\mathbb{Z}}$ be a δ-pseudo orbit through x of g. Since

$$d(f(x_n), x_{n+1}) \leq d(f(x_n), g(x_n)) + d(g(x_n), x_{n+1}) \leq d_{C^0}(f, g) + \delta < 2\delta = \bar{\delta},$$

$(x_n)_{n\in\mathbb{Z}}$ is a $\bar{\delta}$-pseudo orbit of f. Then, there is $\bar{x} \in X$ such that

$$d(f^n(\bar{x}), x_n) \leq \frac{\epsilon}{2}, \quad \forall n \in \mathbb{Z}.$$

Again since $d_{C^0}(f, g) \leq \delta$, by persistence, there is $x \in X$ such that

$$d(f^n(\bar{x}), g^n(x)) \leq \frac{\epsilon}{2}, \quad \forall n \in \mathbb{Z}.$$

It follows that

$$d(g^n(x), x_n) \leq d(f^n(\bar{x}), g^n(x)) + d(f^n(\bar{x}), x_n) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall n \in \mathbb{Z}.$$

Hence $(x_n)_{n\in\mathbb{Z}}$ can be (g, ϵ)-shadowed. Therefore, $x \in PSh(f)$ and the proof follows. □

Corollary 2 *A homeomorphism of a compact metric space has the persistent shadowing property if and only if it has the shadowing property and is persistent.*

Proof If a homeomorphism of a compact metric space $f : X \to X$ has the shadowing property and is persistent, then $PSh(f) = X$ by Lemma 4 and so has the persistent shadowing property by Lemma 3. Conversely, if it has the persistent shadowing property, then it obviously has the shadowing property and is persistent by Lemma 1. □

Example 6 In a totally disconnected metric group X, a group automorphism with zero topological entropy is persistent (Example 3 in [16]). Since all of them have the shadowing property (Application 2 in [1]), we obtain examples of persistent homeomorphisms with the shadowing property on totally disconnected metric groups. By Corollary 2 all such homeomorphisms have the persistent shadowing property. This applies to the identity of X.

Example 7 Every s-topologically stable homeomorphism of the Cantor set has the shadowing property (Theorem 1.3 in [7]). Since such homeomorphisms are persistent (Lemma 2), they have the persistent shadowing property (Corollary 2).

We also obtain the following corollary.

Corollary 3 *Every topologically stable homeomorphism of a compact manifold has the persistent shadowing property.*

Proof Every topologically stable homeomorphism of a compact manifold has the shadowing property. This is proved in dimension two by Walters [18] and in dimension one by Morimoto (see also [19]). We can also see that every topologically stable homeomorphism of a compact manifold is s-topologically stable. Indeed, every continuous map C^0 close to the identity in such manifold is onto. Then, it is persistent (by Lemma 2) and so has the persistent shadowing property by Corollary 2. □

Example 8 Corollary 3 implies that there are homeomorphisms with the persistent shadowing property on compact manifolds (e.g. the Axiom A diffeomorphisms with the strong transversality condition). This is not the case for the uniform pseudo orbit tracing property as defined in [6] (see [8]).

4 Proof of Theorem 1

Let $f : X \to X$ be a pointwise persistent homeomorphism with the shadowing property of a compact metric space. It follows that $Sh(f) = Persi(f) = X$ and so $PSh(f) = X$ by Lemma 4. Then, f has the persistent shadowing property by Lemma 3 and so f is persistent by Lemma 1. This completes the proof. □

Combining Theorem 1 and Corollary 2 we obtain the following equivalence.

Corollary 4 *The following properties are equivalent for every homeomorphism of a compact metric space $f : X \to X$:*

1. *f has the shadowing property and is pointwise persistent.*
2. *f has the shadowing property and is persistent.*
3. *f has the persistent shadowing property.*

5 Orbital persistence

In this section we introduce the notion of *orbit persistent dynamical systems* as an orbital version of Lewowicz's persistence [9]. The motivation is the *orbital shadowing property* introduced by Pilyugin, Rodionova and Sakai [11], used in [5] to characterize those continuous maps for which the internally chain transitive and omega-limit sets coincide. Indeed, we will obtain an orbital version of Theorem 1. Let us present the details.

Let X be a compact metric space. Given $\delta > 0$ and a subset $A \subset X$ we define

$$A^\delta = \{x \in X : d(x, a) < \delta \text{ for some } a \in A\}.$$

Denote by $\overline{B}$ the closure of a subset $B \subset X$. Define the *Hausdorff metric* between $A, B \subset X$ by

$$d_H(A, B) = \inf\{\delta > 0 : A \subset B^\delta \text{ and } B \subset A^\delta\}.$$

Let $f : X \to X$ be a homeomorphism. Given $x \in X$ we denote by

$$O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$$

the orbit of x under f. We say that a bi-infinite sequence $(x_n)_{n\in\mathbb{Z}}$ can be *orbitally ϵ-shadowed* if there is $x \in X$ such that

$$d_H\left(\overline{O_f(x)}, \overline{(x_n)_{x\in\mathbb{Z}}}\right) \le \epsilon.$$

(We write *orbitally (f, ϵ)-shadowed* to emphasize f.)

Definition 8 ([11]) A homeomorphism has the *orbital shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit can be orbitally ϵ-shadowed.

This definition motivates the notion of orbital persistence below.

We say that $K \subset X$ is *orbital persistent* if for every $\epsilon > 0$ there is $\delta > 0$ such that for every $x \in K$ and every homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$ there is $y \in X$ such that $d_H\left(\overline{O_f(x)}, \overline{O_g(y)}\right) \leq \epsilon$.

We say that $x \in X$ is *orbital persistent* if the single point set $K = \{x\}$ is orbitally persistent. Denote by $OPersi(f)$ the set of orbitally persistent points of f.

Definition 9 A homeomorphism $f : X \to X$ is :

1. *Orbital persistent* if X is an orbitally persistent set.
2. *Pointwise orbital persistent* if $OPersi(f) = X$.

Persistent homeomorphisms are orbital persistent and the orbital persistent homeomorphisms are in turn pointwise orbital persistent. We obtain examples of orbital persistent homeomorphisms which are not persistent through the following lemma. Recall that a homeomorphism $f : X \to X$ is *minimal* if $\overline{O_f(x)} = X$ for every $x \in X$.

Lemma 5 *Every minimal homeomorphism of a compact metric space is orbital persistent.*

Proof Let $f : X \to X$ be a minimal homeomorphism of a compact metric space. Take $\epsilon > 0$ and a finite covering by ϵ-balls $B(x_1, \epsilon), \cdots, B(x_n, \epsilon)$. Since f is minimal, there is $\delta > 0$ such that if $d_{C^0}(f, g) \leq \delta$, then $O_g(x_1) \cap B(x_j, \epsilon)$ for every $i = 1, \cdots, n$. Then, for such g's one has $d_H\left(X, \overline{O_g(x_1)}\right) \leq \epsilon$. But $\overline{O_f(x)} = X$ for all $x \in X$ since f is minimal. Hence for every $x \in X$ there is $y = x_1$ such that $d_H\left(\overline{O_f(x)}), \overline{O_g(y)}\right) \leq \epsilon$. Therefore, f is orbital persistent. □

The converse of this lemma is false as the pseudo-Anosov map are persistent (hence orbit persistent) but not minimal. We use this lemma in the following example.

Example 9 The irrational circle rotations are minimal so orbital persistent (by Lemma 5) but not persistent. Hence, the orbital persistence separates the circle rotations in two parts: the rational ones (which are not orbital persistent) and the irrational ones (which are orbital persistent).

The result below is an orbital version of Theorem 1.

Theorem 3 *Every pointwise orbital persistent homeomorphism with the orbital shadowing property of a compact metric space is orbital persistent.*

Outline of the proof The proof follows the same lines of the proof of Theorem 1.

Let $f : X \to X$ be a homeomorphism of a compact metric space. We say that $x \in X$ is an *orbit* (resp. *orbit persistent) shadowable point* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of f (resp. of a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$) can be orbitally (f, ϵ)-shadowed (resp. orbitally (g, ϵ)-shadowed). Denote by $OSh(f)$ and $OPSh(f)$ the set of orbit shadowable and orbit persistent shadowable points of f.

On the other hand, we say that f has the *orbit persistent shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$ can be orbitally (g, ϵ)-shadowed.

It follows that f has the orbital shadowing property (resp. orbit persistent shadowable property) if and only if $OSh(f) = X$ (resp. $OPSh(f) = X$). Moreover, if f has the orbit persistent shadowing property, then f is orbital persistent. We also have that $OPSh(f) \cap OPersi(f) \cap OSh(f)$.

Then, if f has the orbital shadowing property and is pointwise orbital persistent, $OPersi(f) = OSh(f) = X$ so $OPSh(f) = X$ thus f has the orbit persistent shadowing property hence f is orbit persistent. This completes the outline of the proof. □

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References

1. Aoki, N.: The splitting of zero-dimensional automorphisms and its application. Colloq. Math. **49**(2), 161–173 (1985)
2. Bowen, R.: ω-limit sets for axiom a diffeomorphisms. J. Differ. Equ. **18**(2), 333–339 (1975)
3. Dong, M., Jung, W., Morales, C.: Eventually shadowable points. Qual. Theory Dyn. Syst. **19**, 16 (2020). https://doi.org/10.1007/s12346-020-00367-4
4. Dong, M., Lee, K., Morales, C.A.: Pointwise topological stability and persistence. J. Math. Anal. Appl. **480**(2), 123334 (2019). 12 pp
5. Good, C., Meddaugh, J.: Orbital shadowing, internal chain transitivity and ω-limit sets. Ergod. Theory Dyn. Syst. **38**(1), 143–154 (2018)
6. Gu, R.: Uniform pseudo-orbit tracing property and orbit stability. Ann. Differ. Equ. **13**(4), 350–359 (1997)
7. Kawaguchi, N.: On the topological stability and shadowing in zero-dimensional spaces. Discrete Contin. Dyn. Syst. **39**(5), 2743–2761 (2019)
8. Kulczycki, M.: Uniform pseudo-orbit tracing property for homeomorphisms and continuous mappings. Ann. Polon. Math. **75**(1), 1–6 (2000)
9. Lewowicz, J.: Persistence in expansive systems. Ergod. Theory Dyn. Syst. **3**(4), 567–578 (1983)
10. Morales, C.A.: Shadowable points. Dyn. Syst. **31**(3), 347–356 (2016)
11. Pilyugin, S.Y., Rodionova, A.A., Sakai, K.: Orbital and weak shadowing properties. Discrete Contin. Dyn. Syst. **9**(2), 287–308 (2003)
12. Robinson, C.: Stability theorems and hyperbolicity in dynamical systems, Proceedings of the Regional Conference on the Application of Topological Methods in Differential Equations (Boulder, Colo., 1976). Rocky Mt. J. Math. **7**(3), 425–437 (1977)
13. Sakai, K.: The C^1 uniform pseudo-orbit tracing property. Tokyo J. Math. **15**(1), 99–109 (1992)
14. Sakai, K.: Pseudo-orbit tracing property and strong transversality of diffeomorphisms on closed manifolds. Osaka J. Math. **31**(2), 373–386 (1994)
15. Sakai, K.: Diffeomorphisms with persistency. Proc. Am. Math. Soc. **124**(7), 2249–2254 (1996)
16. Sakai, K., Kobayashi, H.: On persistent homeomorphisms, Dynamical systems and nonlinear oscillations (Kyoto, 1985), 1–12, World Scientific Advanced Series in Dynamical Systems, 1. World Sci. Publishing, Singapore (1986)
17. Utz, W.R.: Unstable homeomorphisms. Proc. Am. Math. Soc. **1**, 769–774 (1950)

18. Walters, P.: On the pseudo-orbit tracing property and its relationship to stability. In: The Structure of Attractors in Dynamical Systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977). Lecture Notes in Mathematics, pp. 231–244, vol. 668. Springer, Berlin (1978)
19. Yano, K.: Topologically stable homeomorphisms of the circle. Nagoya Math. J. **79**, 145–149 (1980)