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On residually finite groups satisfying an Engel type identity

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Abstract
Let n, q be positive integers. We show that if G is a finitely generated residually finite group satisfying the identity $[x,_n y^q] \equiv 1$, then there exists a function $f(n)$ such that G has a nilpotent subgroup of finite index of class at most $f(n)$. We also extend this result to locally graded groups.

Keywords Engel element · Engel groups · Residually finite groups · Locally graded groups · Lie algebras

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1 Introduction

Let n be a positive integer. We say that a group G is (left) n-Engel if it satisfies the identity $[y,{}_n x] \equiv 1$, where the word $[x,_n y]$ is defined inductively by the rules

$$[x,_1 y] = x^{-1}y^{-1}xy, \quad [x,_n y] = [[x,_{n-1} y], y] \quad \text{for all } n \geq 2.$$

A important theorem of Wilson [13, Theorem 2] says that finitely generated residually finite n-Engel groups are nilpotent. More specific properties of residually finite n-Engel groups can be found for example in a theorem of Burns and Medvedev (quoted below as Theorem 5) stating that there exist functions $c(n)$ and $e(n)$ such that any residually finite n-Engel group G has a nilpotent normal subgroup N of class at most $c(n)$ such that the quotient group G/N has exponent dividing $e(n)$. The interested reader is referred to the survey [12] and references therein for further results on finite and

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Dedicated to Pavel Shumyatsky on the occasion of his 60th birthday.

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residually finite Engel groups. The purpose of the present article is to provide the proof for the following theorem.

Theorem 1 *Let G be a finitely generated residually finite group satisfying the identity $[x,_n y^q] \equiv 1$. Then there exists a function $f(n)$ such that G has a nilpotent subgroup of finite index of class at most $f(n)$.*

A group is called locally graded if every non-trivial finitely generated subgroup has a proper subgroup of finite index. The class of locally graded groups contains locally (soluble-by-finite) groups as well as residually finite groups. We can extend the Theorem 1 to the class of locally graded groups.

Corollary 1 *Let G be a finitely generated locally graded group satisfying the identity $[x,_n y^q] \equiv 1$. Then there exists a function $f(n)$ such that G has a nilpotent subgroup of finite index of class at most $f(n)$.*

In the next section we describe the Lie-theoretic machinery that will be used in the proof of Theorem 1. The proof of the theorem and of the corollary is given in Sect. 3.

2 About Lie algebras

Let L be a Lie algebra over a field K and X a subset of L. By a commutator in elements of X we mean any element of L that can be obtained as a Lie product of elements of X with some system of brackets. If $x_1, \ldots, x_k, x, y$ are elements of L, we define inductively

$$[x_1] = x_1; [x_1, \ldots, x_k] = [[x_1, \ldots, x_{k-1}], x_k]$$

and $[x,_0 y] = x$; $[x,_m y] = [[x,_{m-1} y], y]$, for all positive integers k, m. As usual, we say that an element $a \in L$ is ad-nilpotent if there exists a positive integer n such that $[x,_n a] = 0$ for all $x \in L$. Denote by F the free Lie algebra over K on countably many free generators $x_1, x_2, \ldots$. Let $f = f(x_1, x_2, \ldots, x_n)$ be a non-zero element of F. The algebra L is said to satisfy the identity $f \equiv 0$ if $f(l_1, l_2, \ldots, l_n) = 0$ for any $l_1, l_2, \ldots, l_n \in L$.

The next theorem represents the most general form of the Lie-theoretical part of the solution of the Restricted Burnside Problem [15,17,18]. It was announced by Zelmanov [15]. A detailed proof can be found in [18].

Theorem 2 *Let L be a Lie algebra over a field and suppose that L satisfies a polynomial identity. If L can be generated by a finite set X such that every commutator in elements of X is ad-nilpotent, then L is nilpotent.*

2.1 Associating a Lie ring to a group

Let G be a group. A series of subgroups

$$G = G_1 \geq G_2 \geq \ldots \qquad (*)$$

is called an N-series if it satisfies $[G_i, G_j] \leq G_{i+j}$ for all $i, j \geq 1$. Obviously any N-series is central, i.e. $G_i/G_{i+1} \leq Z(G/G_{i+1})$ for any i. Let p be a prime. An N-series is called N_p-series if $G_i^p \leq G_{pi}$ for all i. Given an N-series $(*)$, let $L^*(G)$ be the direct sum of the abelian groups $L_i^* = G_i/G_{i+1}$, written additively. Commutation in G induces a binary operation $[\cdot,\cdot]$ in $L^*(G)$. For homogeneous elements $xG_{i+1} \in L_i^*$, $yG_{j+1} \in L_j^*$ the operation is defined by

$$[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1} \in L_{i+j}^*$$

and extended to arbitrary elements of $L^*(G)$ by linearity. It is easy to check that the operation is well-defined and that $L^*(G)$ with the operations $+$ and $[\cdot,\cdot]$ is a Lie ring. If all quotients G_i/G_{i+1} of an N-series $(*)$ have prime exponent p then $L^*(G)$ can be viewed as a Lie algebra over the field with p elements. In the important case where the series $(*)$ is the p-dimension central series (also known under the name of Zassenhaus-Jennings-Lazard series) of G we write $D_i = D_i(G) = \prod_{jp^k \geq i} \gamma_j(G)^{p^k}$ for the i-th term of the series of G, $L(G)$ for the corresponding associated Lie algebra over the field with p elements and $L_p(G)$ for the subalgebra generated by the first homogeneous component D_1/D_2 in $L(G)$. Observe that the p-dimension central series is an N_p-series (see [5, p. 250] for details).

The nilpotency of $L_p(G)$ has strong influence in the structure of a finitely generated pro-p group G. The proof of the following theorem can be found in [4, 1.(k) and 1.(o) in Interlude A].

Theorem 3 *Let G be a finitely generated pro-p group. If $L_p(G)$ is nilpotent, then G is p-adic analytic.*

Let $x \in G$ and let $i = i(x)$ be the largest positive integer such that $x \in D_i$ (here, D_i is a term of the p-dimensional central series to G). We denote by $\tilde{x}$ the element $xD_{i+1} \in L(G)$. We now quote two results providing sufficient conditions for $\tilde{x}$ to be ad-nilpotent. The following lemma was established in [6, p. 131].

Lemma 1 *For any $x \in G$ we have $(ad\ \tilde{x})^p = ad\ (\widetilde{x^p})$.*

Corollary 2 *Let x be an element of a group G for which there exists a positive integer m such that x^m is n-Engel. Then $\tilde{x}$ is ad-nilpotent.*

The following theorem is a particular case of a result that was established by Wilson and Zelmanov in [14].

Theorem 4 *Let G be a group satisfying an identity. Then for each prime number p the Lie algebra $L_p(G)$ satisfies a polynomial identity.*

3 Proof of the main theorem

The following useful result is a consequence of [13, Lemma 2.1] (see also [11, Lemma 3.5] for details).

Lemma 2 *Let G be a finitely generated residually finite-nilpotent group. For each prime p let J_p be the intersection of all normal subgroups of G of finite p-power index. If G/J_p has a nilpotent subgroup of finite index of class at most c for each p, then G also has a nilpotent subgroup of finite index of class at most c.*

Proof It follows from proof of [11, Lemma 3.5] that there exists a finite set of primes π such that G embeds in the direct product $\prod_{p\in\pi} G/J_p$. We will identify G with its images in direct product. By hypothesis, for any $p \in \pi$, G/J_p contains a nilpotent subgroup of finite index H_p with class at most c. Set $H = \cap_{p\in\pi} H_p$. Thus, $G \cap H$ has finite index in G and has nilpotency class at mos c, which completes the proof.

Recall that a group is locally graded if every non-trivial finitely generated subgroup has a proper subgroup of finite index. Note that the quotient of a locally graded group need not be locally graded, since free groups are locally graded (see [10, 6.1.9]), but no finitely generated infinite simple group is locally graded. However, the following results give a sufficient conditions for a quotient to be locally graded (see [7] for details).

Lemma 3 *Let G be a locally graded group and N a normal locally nilpotent subgroup of G. Then G/N is locally graded.*

Let p be a prime and q be a positive integer. A finite p-group G is said to be powerful if and only if $[G, G] \leq G^p$ for $p \neq 2$ (or $[G, G] \leq G^4$ for $p = 2$), where G^q denotes the subgroup of G generated by all qth powers. While considering a pro-p group G we shall be interested only in closed subgroups. So by the commutator subgroup $G' = [G, G]$ we mean the closed commutator subgroup, G^q means the closed subgroup generated by the qth powers. Similarly to powerful finite p-groups, we may define the powerful pro-p groups. For more details we refer the reader to [7, Chapters 2 and 3]. In [1] the following useful result for powerful finite n-Engel p-group was established.

Lemma 4 *There exists a function $s(n)$ such that any powerful finite n-Engel p-group is nilpotent of class at most $s(n)$.*

The proof of Theorem 1 will requires the following lemma.

Lemma 5 *Let $s(n)$ be as in Lemma 4. If G is a finitely generated powerful pro-p group satisfying the identity $[x,_n y^q] \equiv 1$, then G^q has nilpotency class at most $s(n)$.*

Proof Since G satisfies the identity $[x,_n y^q] \equiv 1$, we can deduce from [4, Corollary 3.5] that $H = G^q = \{x^q \mid x \in G\}$ is a powerful n-Engel pro-p group. According to [4, Corollary 3.3], H is the inverse limit of an inverse system of powerful finite p-groups H_λ. Lemma 4 implies that any group H_λ has class at most $s(n)$, and so, H has class at most $s(n)$ as well. Finally, by a result due to Zelmanov [16, Theorem 1] saying that any torsion profinite group is locally finite we get that the quotient group G/H is finite. This completes the proof. □

The proof of Theorem 1 will also require the following result, due to Burns and Medvedev [3].

Theorem 5 *There exist functions $c(n)$ and $e(n)$ such that any residually finite n-Engel group G has a nilpotent normal subgroup N of class at most $c(n)$ such that G/N has exponent dividing $e(n)$.* □

We are now ready to embark on the proof of our main result.

Proof of Theorem 1 For any positive integer n let $s(n)$ and $c(n)$ be as in Lemma 4 and Theorem 5, respectively. Set $f(n) = \max\{s(n), c(n)\}$. Since G satisfies the identity $[x,_n y^q] \equiv 1$ we can deduce from [2, Theorem A] that $H = G^q$ is locally nilpotent. According to Lemma 3, G/H is locally graded. By Zelmanov's solution of the Restricted Burnside Problem [15,17,18], locally graded groups of finite exponent are locally finite (see for example [8, Theorem 1]), and so G/H is finite. Thus H is finitely generated and so it is nilpotent.

By Lemma 2, we can assume that H is residually (finite p-group) for some prime p. If p does not divides q, then H is finitely generated residually finite n-Engel group. By Theorem 5, H contains a nilpotent normal subgroup N of class at most $f(n)$ such that the quotient group G/N has exponent dividing $e(n)$. Thus, we can see that G/N is finite. Thereby, in what follows we can assume that H is residually (finite p-group), where p divides q.

Set $H = \langle h_1, \dots, h_t \rangle$. Let $L = L_p(H)$ be the Lie algebra associated with the p-dimensional central series of H. Then L is generated by $\tilde{h}_i = h_i D_2$, $i = 1, 2, \dots, t$. Let $\tilde{h}$ be any Lie-commutator in $\tilde{h}_i$ and h be the group-commutator in h_i having the same system of brackets as $\tilde{h}$. Since for any group commutator h in $h_1 \dots, h_t$ we have that h^q is n-Engel, Corollary 2 shows that any Lie commutator in $\tilde{h}_1 \dots, \tilde{h}_t$ is ad-nilpotent. Since H satisfies the identity $[x,_n y^q] \equiv 1$, by Theorem 4, L satisfies some non-trivial polynomial identity. According to Theorem 2 L is nilpotent.

Let $\hat{H}$ be the pro-p completion of H, that is, the inverse limit of all quotients of H which are finite p-groups. Notice that $\hat{H}$ is finitely generated, being H finitely generated.

Since the finite p-quotients of H are the same as the finite p-quotients of $\hat{H}$ by (a) and (d) of [9, Proposition 3.2.2], we get that $L_p(\hat{H}) = L$. Hence, $L_p(\hat{H})$ is nilpotent and so, $\hat{H}$ is a p-adic analytic group by Theorem 3.

By [4, 1.(a) and 1.(o) in Interlude A], $\hat{H}$ is virtually powerful, that is, $\hat{H}$ has a powerful subgroup K of finite index. By Lemma 5, K^q has class at most $f(n)$. Furthermore, it follows from [16, Theorem 1] that group K/K^q is finite. Finally, since H is residually-p, it embeds in $\hat{H}$. Thus, $H \cap K^q$ is a nilpotent subgroup of finite index in G of class at most $f(n)$. This completes the proof. □

Proof of Corollary 1 Let $f(n)$ be as in Theorem 1. It follows from [2, Theorem C] that $H = G^q$ is locally nilpotent. By Lemma 3, G/H is a locally graded group. By Zelmanov's solution of the Restricted Burnside Problem, locally graded groups of finite exponent are locally finite. Thus, G/H is finite and so, H is a finitely generated nilpotent group. Since polycyclic groups are residually finite [10, 5.4.17], we can deduce from Theorem 1 that H contains a subgroup of finite index and of class at most $f(n)$. The proof is complete.

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