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On distinguished local coordinates for locally homogeneous affine surfaces

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Abstract
We give a new short self-contained proof of the result of Opozda (Differ Geom Appl 21:173–198, 2004) classifying the locally homogeneous torsion free affine surfaces and the extension to the case of surfaces with torsion due to Arias-Marco and Kowalski (Monatsh Math 153:1–18, 2008). Our approach rests on a direct analysis of the affine Killing equations and is quite different than the approaches taken previously in the literature.

Keywords Affine surface · Locally homogeneous · Local forms · Affine Killing equations

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1 Introduction

We say that $\mathcal{M} = (M, \nabla)$ is an *affine surface* if M is a smooth connected 2-dimensional manifold and if ∇ is a connection on the tangent bundle of M. We emphasize that ∇ is permitted to have torsion. We say that $\mathcal{M}$ is *locally homogenous* if given any two points of M, there is a local diffeomorphism from a neighborhood of one point to a neighborhood of the other point which preserves ∇, i.e. is an affine map. In a system of local coordinates, sum over repeated indices to expand $\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma_{ij}{}^k \partial_{x^k}$ to define the Christoffel symbols.

During the past few years, there has been a concerted effort to classify homogeneous affine surfaces. Kowalski, Opozda and Vlášek [9] provided the first major step by classifying the homogeneous torsion free connections with skew-symmetric Ricci tensor. Derdzinski [3] then extended their result using in an essential fashion the fact that the curvature operator satisfies the identity $R(x, y) = \rho(x, y)\,\mathrm{Id}$ in this setting. Subsequently, Opozda [15] established a complete classification for locally homogeneous surfaces without torsion. Finally Arias-Marco and Kowalski [1] completed the program by extending the Theorem of Opozda to connections with torsion. The resulting full classification can be stated as follows.

Theorem 1.1 *If $\mathcal{M}$ is a locally homogeneous affine surface, then at least one of the following three possibilities holds describing the local geometry:*

(1) *There exists a coordinate atlas so that $\Gamma_{ij}{}^k \in \mathbb{R}$.*
(2) *There exists a coordinate atlas so that $\Gamma_{ij}{}^k = (x^1)^{-1} A_{ij}{}^k$ for $A_{ij}{}^k \in \mathbb{R}$.*
(3) *There exists a coordinate atlas where ∇ is the Levi–Civita connection defined by the metric of the round sphere.*

The compact case was considered in [6,16]. If M is compact, then either ∇ is torsion-free and flat, ∇ is the Levi–Civita connection of a surface of constant curvature, or ∇ is a connection with $\Gamma_{ij}{}^k \in \mathbb{R}$ and M is a torus. The special case of locally symmetric affine surfaces was addressed in [14], where it is shown that any locally symmetric affine surface is either modeled on a surface of constant curvature with the Levi–Civita connection or, up to linear equivalence, on one of two affine surfaces which have the form given in Theorem 1.1-(1). Theorem 1.1 has been useful in many works on affine surfaces, including but not limited to [4,5,10,12]. We also refer to Kowalski et al. [11] for another proof of Theorem 1.1 in the torsion free setting.

We shall give a short and self-contained proof of Theorem 1.1 by examining the affine Killing equations rather than by studying the curvature tensor or by using classification results of Lie algebras of vector fields. The structure of the Lie algebra of affine Killing vector fields $\mathfrak{K}(\mathcal{M})$ will play a crucial role in our analysis. We choose coordinate systems so that the vector field ∂_{x^2} is an affine Killing vector field. We complexify and consider the generalized eigenspaces of $\mathfrak{K}_{\mathbb{C}}(\mathcal{M})$ as an $\mathrm{ad}(\partial_{x^2})$ module. What is new in this approach is the mixture of Lie theory together with the affine Killing equations that affords, we believe, a more direct and conceptual approach to the proof of Theorem 1.1.

2 Affine Killing vector fields

We recall the following result of Kobayashi and Nomizu [8, Chapter VI].

Lemma 2.1 *Let* $\mathcal{M} = (M, \nabla)$.

(1) *The following 3 conditions are equivalent and if any is satisfied, then X is said to be an* affine Killing vector field.

  (a) *Let Φ_t^X be the local flow of X. Then* $(\Phi_t^X)_* \circ \nabla = \nabla \circ (\Phi_t^X)_*$.
  (b) *The Lie derivative of ∇ with respect to X vanishes.*
  (c) $[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0$ *for all* $Y, Z \in C^\infty(TM)$.

(2) *Let $\mathfrak{K}(\mathcal{M})$ be the vector space of affine Killing vector fields. The Lie bracket gives $\mathfrak{K}(\mathcal{M})$ a Lie algebra structure. We have that* $\dim\{\mathfrak{K}(\mathcal{M})\} \leq 6$.

Let $X = a^k \partial_{x^k}$. By Lemma 2.1 (1c), X is an affine Killing vector field if and only if X satisfies the 8 affine Killing equations for $1 \leq i, j, k \leq 2$

$$K_{ij}{}^k : \quad 0 = \frac{\partial^2 a^k}{\partial_{x^i} \partial_{x^j}} + \sum_\ell \left\{ a^\ell \frac{\partial \Gamma_{ij}{}^k}{\partial x^\ell} - \Gamma_{ij}{}^l \frac{\partial a^k}{\partial x^\ell} + \Gamma_{i\ell}{}^k \frac{\partial a^\ell}{\partial x^j} + \Gamma_{\ell j}{}^k \frac{\partial a^\ell}{\partial_{x^i}} \right\}.$$

Choose a point P of M; which point is irrelevant as we shall assume that $\mathcal{M}$ is locally homogeneous henceforth. We work at the level of germs and assume M is an arbitrarily small neighborhood of P. If, for example, we are given a vector field Ξ which does not vanish identically near P, we can choose a slightly different base point $\tilde{P}$ where $\Xi(\tilde{P}) \neq 0$. To pass to global results, we shall assume the underlying manifold M is simply connected to avoid difficulties with holonomy; in this setting, every affine Killing vector field which is locally defined extends to a globally defined affine Killing vector field. We shall not belabor these points in what follows. We say that a subset S of $\mathfrak{K}(\mathcal{M})$ is *effective* if there exist $X_i \in S$ so that $\{X_1(P), X_2(P)\}$ are linearly independent. Since $\mathcal{M}$ is locally homogeneous, $\mathfrak{K}(\mathcal{M})$ is effective [7,13]. We define the following Lie algebras by their relations

$$\begin{aligned} \mathfrak{K}_{\mathcal{A}} &:= \text{ Span } \{X, Y\} \text{ for } [X, Y] = 0, \ \mathfrak{K}_{\mathcal{B}} := \text{ Span } \{X, Y\} \text{ for } [X, Y] = Y, \\ \mathfrak{so}(3) &:= \text{ Span } \{X, Y, Z\} \text{ for } [X, Y] = Z, \ [Y, Z] = X, \ [Z, X] = Y. \end{aligned} \tag{2a}$$

Theorem 1.1 will be a consequence of the following result.

Lemma 2.2 *Let $\mathcal{M} = (M, \nabla)$ be locally homogeneous and simply connected.*

(1) *There is an effective Lie subalgebra $\tilde{\mathfrak{K}}$ of $\mathfrak{K}(\mathcal{M})$ which is isomorphic to $\mathfrak{K}_{\mathcal{A}}$, $\mathfrak{K}_{\mathcal{B}}$, or $\mathfrak{so}(3)$.*
(2) *If $\tilde{\mathfrak{K}} \approx \mathfrak{K}_{\mathcal{A}}$, then there is a coordinate atlas so that $\Gamma_{ij}{}^k \in \mathbb{R}$.*
(3) *If $\tilde{\mathfrak{K}} \approx \mathfrak{K}_{\mathcal{B}}$, then there is a coordinate atlas so that $\Gamma_{ij}{}^k = (x^1)^{-1} A_{ij}{}^k$ for $A_{ij}{}^k \in \mathbb{R}$.*
(4) *If $\tilde{\mathfrak{K}} \approx \mathfrak{so}(3)$, then there is a coordinate atlas where ∇ is the Levi–Civita connection defined by the metric of the round sphere.*

The possibilities of Assertion (2) and Assertion (3) are not exclusive; the non-flat examples such that both Assertion (2) and Assertion (3) hold along with a complete description of the Lie algebras $\mathfrak{K}(\mathcal{M})$ are given in [2]. By contrast, any $\mathcal{M}$ which admits an effective $\mathfrak{so}(3)$ Lie subalgebra of $\mathfrak{K}(\mathcal{M})$ satisfies $\dim\{\mathfrak{K}(\mathcal{M})\} = 3$ and does not admit any 2-dimensional Lie subalgebras of affine Killing vector fields. In Theorem 1.1, we do not impose the hypothesis that $\mathcal{M}$ is simply connected as the question of suitable coordinate systems is a local one. By contrast, in Lemma 2.2, we must impose the hypothesis that $\mathcal{M}$ is simply connected since the question of affine Killing vector fields is a global one.

By Lemma 2.1, $\dim\{\mathfrak{K}(\mathcal{M})\} \leq 6$. Complexify and set $\mathfrak{K}_{\mathbb{C}}(\mathcal{M}) := \mathfrak{K}(\mathcal{M}) \otimes_{\mathbb{R}} \mathbb{C}$.

Lemma 2.3 *Choose $\Xi \in \mathfrak{K}(\mathcal{M})$ with $\Xi(P) \neq 0$. For $\alpha \in \mathbb{C}$, let*

$$E(\alpha) := \{X_\alpha \in \mathfrak{K}_{\mathbb{C}}(\mathcal{M}) : (\mathrm{ad}(\Xi) - \alpha)^6 X_\alpha = 0\}$$

be the associated generalized eigenspace of $\mathrm{ad}(\Xi)$. *Then* $[E(\alpha), E(\beta)] \subset E(\alpha + \beta)$.

Proof Choose local coordinates so $\Xi = \partial_{x^2}$. Then $X_\alpha \in E(\alpha)$ if and only if

$$X_\alpha = e^{\alpha x^2} \left\{ \sum_{i=0}^{i_0} u_i(x^1)(x^2)^i \partial_{x^1} + \sum_{j=0}^{j_0} v_j(x^1)(x^2)^j \partial_{x^2} \right\}, \tag{2b}$$

for some suitably chosen $i_0, j_0 \leq 5$. This leads to an expansion for $[X_\alpha, X_\beta]$ where the relevant exponential is $e^{(\alpha+\beta)x^2}$ that shows $[X_\alpha, X_\beta] \in E(\alpha + \beta)$. □

3 The proof of Lemma 2.2

The following coordinate normalization will be used for much of our analysis; a different normalization will be used in examining the structure $\mathfrak{so}(3)$.

Lemma 3.1 *Let $\Xi \in \mathfrak{K}(\mathcal{M})$ satisfy $\Xi(P) \neq 0$. We can choose local coordinates centered at P so that $\Xi = \partial_{x^2}$ and so that*

$$\Gamma_{ij}{}^k(x^1, x^2) = \Gamma_{ij}{}^k(x^1), \quad \Gamma_{11}{}^1(x^1) = 0, \quad \Gamma_{11}{}^2(x^1) = 0.$$

Proof Choose initial coordinates (y^1, y^2) centered at P so that $\Xi = \partial_{y^2}$. Since Ξ is an affine Killing vector field, $\Gamma = \Gamma(y^1)$ and the map $(y^1, y^2) \to (y^1, y^2 + t)$ is an affine map. Let $\sigma(s)$ be a geodesic with $\sigma(0) = 0$ and with $\{\dot{\sigma}(0), \Xi(0)\}$ linearly independent. Let $T(x^1, x^2) := \sigma(x^1) + (0, x^2)$ define new coordinates with $\partial_{x^2} = \partial_{y^2}$. Since the curves $x^1 \to T(x^1, x^2)$ are geodesics for x^2 fixed and since ∂_{x^2} is an affine Killing vector field, the normalizations of the Lemma hold. □

With the coordinate normalization of Lemma 3.1, ∂_{x^2} is an affine Killing vector field. We now examine other affine Killing vector fields.

Lemma 3.2 *Use Lemma 3.1 to normalize the system of local coordinates. Set* $\mathfrak{K}_\alpha(\mathcal{M}) := \{X = e^{\alpha x^2} v(x^1)\partial_{x^2} : X \in \mathfrak{K}_{\mathbb{C}}(\mathcal{M})\}$.

(1) *If there exists* $X \in \mathfrak{K}_\alpha(\mathcal{M})$*, which is not a constant multiple of* ∂_{x^2}*, then*

$$\Gamma_{11}{}^1 = 0,\ \Gamma_{11}{}^2 = 0,\ \Gamma_{12}{}^1 = 0,\ \Gamma_{21}{}^1 = 0,\ \Gamma_{22}{}^1 = 0,\ \Gamma_{22}{}^2 = -\alpha. \tag{3a}$$

(2) *Suppose that the Christoffel symbols satisfy Eq.* (3a).

- (a) *If* $u(x^1, x^2)\partial_{x^1} + w(x^1, x^2)\partial_{x^2} \in \mathfrak{K}_{\mathbb{C}}(\mathcal{M})$*, then*
  - (i) $\alpha\, u^{(0,1)}(x^1, x^2) + u^{(0,2)}(x^1, x^2) = 0$,
  - (ii) $(\Gamma_{12}{}^2(x^1) + \Gamma_{21}{}^2(x^1))w^{(1,0)}(x^1, x^2) + w^{(2,0)} = 0$.
- (b) $\mathfrak{K}_\alpha(\mathcal{M}) = \{e^{\alpha x^2} v(x^1)\partial_{x^2} : (\Gamma_{12}{}^2(x^1) + \Gamma_{21}{}^2(x^1))v'(x^1) + v''(x^1) = 0\}$.
- (c) *Assume* $\alpha = 0$. *If* $u(x^1, x^2)\partial_{x^1} + \{\sum_n w_n(x^1)(x^2)^n\}\partial_{x^2} \in \mathfrak{K}_{\mathbb{C}}(\mathcal{M})$*, then* $w_n(x^1)\partial_{x^2} \in \mathfrak{K}_0(\mathcal{M})$ *for all n. Furthermore,* $x^2\partial_{x^2} \in \mathfrak{K}(\mathcal{M})$.

Proof Our choice of coordinate system yields $\Gamma_{11}{}^1 = \Gamma_{11}{}^2 = 0$. It is convenient to decompose the proof of Assertion (1) into two cases.

Case 1.1 Suppose $\alpha \neq 0$. Assume $0 \neq X = e^{\alpha x^2} v(x^1)\partial_{x^2} \in \mathfrak{K}_{\mathbb{C}}(\mathcal{M})$. Equation (3a) follows from the equations

$$\begin{aligned}
&K_{22}{}^1 : 0 = e^{\alpha x^2}\{2\alpha\Gamma_{22}{}^1(x^1)v(x^1)\},\ \text{so } \Gamma_{22}{}^1(x^1) = 0.\\
&K_{12}{}^1 : 0 = e^{\alpha x^2}\{\alpha\Gamma_{12}{}^1(x^1)v(x^1) + \Gamma_{22}{}^1(x^1)v'(x^1)\},\ \text{so } \Gamma_{12}{}^1 = 0.\\
&K_{21}{}^1 : 0 = e^{\alpha x^2}\{\alpha\Gamma_{21}{}^1(x^1)v(x^1) + \Gamma_{22}{}^1(x^1)v'(x^1)\},\ \text{so } \Gamma_{21}{}^1 = 0.\\
&K_{22}{}^2 : 0 = e^{\alpha x^2}\{\alpha v(x^1)(\alpha + \Gamma_{22}{}^2(x^1)) - \Gamma_{22}{}^1(x^1)v'(x^1)\},\ \text{so } \Gamma_{22}{}^2 = -\alpha.
\end{aligned}$$

Case 1.2 Suppose $\alpha = 0$. The assumption that $X = v(x^1)\partial_{x^2}$ is not a constant multiple of ∂_{x^2} implies v is non-constant so $v' \neq 0$. Equation (3a) follows from the equations

$$\begin{aligned}
&K_{11}{}^1\!: 0 = (\Gamma_{12}{}^1(x^1) + \Gamma_{21}{}^1(x^1))v'(x^1),\ K_{12}{}^1\!: 0 = \Gamma_{22}{}^1(x^1)v'(x^1),\\
&K_{12}{}^2\!: 0 = (\Gamma_{22}{}^2(x^1) - \Gamma_{12}{}^1(x^1))v'(x^1),\ K_{21}{}^2\!: 0 = (\Gamma_{22}{}^2(x^1) - \Gamma_{21}{}^1(x^1))v'(x^1).
\end{aligned}$$

Assume Eq. (3a) holds. Assertion (2-a) follows from the affine Killing equations $K_{11}{}^2$: $0 = (\Gamma_{12}{}^2 + \Gamma_{21}{}^2)w^{(1,0)} + w^{(2,0)}$ and $K_{22}{}^1$: $0 = \alpha\, u^{(0,1)} + u^{(0,2)}$. Assertion (2-b) follows as $K_{11}{}^2$: $0 = e^{\alpha x^2}((\Gamma_{12}{}^2 + \Gamma_{21}{}^2)v' + v'')$ is the only non-trivial affine Killing equation for $e^{\alpha x^2} v(x^1)\partial_{x^2}$. To prove Assertion (2-c), assume that $u(x^1, x^2)\partial_{x^1} + \{\sum_n w_n(x^1)(x^2)^n\}\partial_{x^2} \in \mathfrak{K}_{\mathbb{C}}(\mathcal{M})$. By Assertion (2-a-ii), $\sum_n\{(\Gamma_{12}{}^2(x^1) + \Gamma_{21}{}^2(x^1))w_n'(x^1) + w_n''(x^1)\}(x^2)^n = 0$. Thus each $w_n(x^1)$ satisfies the ODE individually so by Assertion (2-b), $w_n(x^1)\partial_{x^n} \in \mathfrak{K}_0(\mathcal{M})$. One verifies directly that $x^2\partial_{x^2}$ is an affine Killing vector field in this setting. □

We use Lemma 3.2 to study a Lie algebra structure which will arise subsequently.

Lemma 3.3 *Suppose there is a 3-dimensional effective Lie subalgebra of* $\mathfrak{K}(\mathcal{M})$ *which is spanned by vector fields* X, Y, Z *satisfying* $[X, Y] = 0$, $[X, Z] = aX + Y$, *and* $[Y, Z] = aY - X$ *for* $a \in \mathbb{R}$. *Then there exists an effective Lie subalgebra of* $\mathfrak{K}(\mathcal{M})$ *isomorphic to* $\mathfrak{K}_{\mathcal{A}}$.

Proof The Lemma is immediate if $\{X, Y\}$ is effective. Consequently, we assume that Y is a multiple of X and $\{X, Z\}$ is effective. Normalize the coordinate system as in Lemma 3.1 so that $X = \partial_{x^2}$ and thus $Y = v(x^1, x^2)\partial_{x^2}$. Since $[X, Y] = 0$, $\partial_{x^2} v = 0$ and thus $v = v(x^1)$. As Y is not a constant multiple of X, $v'(x^1) \neq 0$. By Lemma 3.2 (1), the relations of Eq. (3a) hold with $\alpha = 0$. By Lemma 3.2 (2-c), $x^2\partial_{x^2} \in \mathfrak{K}(\mathcal{M})$. Expand $Z = u(x^1, x^2)\partial_{x^1} + w(x^1, x^2)\partial_{x^2}$. We have

$$\begin{aligned}[X, Z] &= \partial_{x^2} u(x^1, x^2)\partial_{x^1} + \partial_{x^2} w(x^1, x^2)\partial_{x^2}\\ &= aX + Y = (a + v(x^1))\partial_{x^2}.\end{aligned}$$

Thus $u = u(x^1)$ and $w = (a + v(x^1))x^2 + v_0(x^1)$. As $\{X, Z\}$ is effective, $u \neq 0$. By Lemma 3.2 (2-c), $v_0(x^1)\partial_{x^2} \in \mathfrak{K}_0(\mathcal{M})$. Thus $\tilde{Z} := u(x^1)\partial_{x^1} + (a + v(x^1)x^2)\partial_{x^2}$ belongs to $\mathfrak{K}(\mathcal{M})$. Since $[\tilde{Z}, x^2\partial_{x^2}] = 0$, $\text{Span}\{\tilde{Z}, x^2\partial_{x^2}\}$ is an effective Lie subalgebra of $\mathfrak{K}(\mathcal{M})$ which is isomorphic to $\mathfrak{K}_{\mathcal{A}}$. □

3.1 The proof of Lemma 2.2 (1)

Use Lemma 3.1 to normalize the coordinate system. Choose $X \in E(\alpha)$ for some α so $\{X, \partial_{x^2}\}$ is effective. Expand X in the form given in Eq. (2b). Since $\{X, \partial_{x^2}\}$ is effective, $u_i \neq 0$ for some i. Choose i_0 maximal so $u_{i_0} \neq 0$. Apply $(\text{ad}(\partial_{x^2}) - \alpha)^{i_0}$ to X to assume that $i_0 = 0$ so

$$X = e^{\alpha x^2}\left\{u(x^1)\partial_{x^1} + \sum_{j=0}^{j_0} v_j(x^1)(x^2)^j\partial_{x^2}\right\} \text{ for } u \neq 0. \tag{3b}$$

We first examine $E(\alpha)$ for $\alpha \neq 0$.

Lemma 3.4 *If $\alpha \neq 0$, then there exists an effective Lie subalgebra of $\mathfrak{K}(\mathcal{M})$ isomorphic to $\mathfrak{K}_{\mathcal{A}}$, $\mathfrak{K}_{\mathcal{B}}$, or $\mathfrak{so}(3)$.*

Proof Adopt the notation established above. We wish to show $j_0 = 0$. Suppose to the contrary that $v_j \neq 0$ for some $j > 0$. Choose j_0 maximal so $v_{j_0} \neq 0$ and hence

$$0 \neq (\text{ad}(\partial_{x^2}) - \alpha)^{j_0} X = j_0!\, e^{\alpha x^2} v_{j_0}(x^1)\partial_{x^2} \in \mathfrak{K}_\alpha(\mathcal{M}).$$

By Lemma 3.2 (1), Eq. (3a) holds. We apply Lemma 3.2 (2-a-i) to see $2\alpha^2\, e^{\alpha x^2} u(x^1) = 0$ so $u = 0$ contrary to our assumption. Thus we conclude $j_0 = 0$ and $X = e^{\alpha x^2}\{u(x^1)\partial_{x^1} + v(x^1)\partial_{x^2}\}$.

Case 1 Suppose $\alpha \in \mathbb{R}$. $[\partial_{x^2}, X] = \alpha X$ so $[\alpha^{-1}\partial_{x^2}, X] = X$. Since $\{X, \partial_{x^2}\}$ is effective, we have an effective Lie subalgebra isomorphic to $\mathfrak{K}_{\mathcal{B}}$.

We therefore assume $\alpha \in \mathbb{C} - \mathbb{R}$. By rescaling x^2, we may suppose $\alpha = a + \sqrt{-1}$ for $a \geq 0$.

Case 2 Assume $a \neq 0$. Choose a maximal so there exists $X \in E(a+\sqrt{-1})$ so $\{X, \partial_{x^2}\}$ is effective. Expand $X = e^{ax^2} e^{\sqrt{-1}x^2}\{u(x^1)\partial_{x^1} + v(x^1)\partial_{x^2}\}$. We have $\bar{X} \in E(\bar{\alpha})$. Let $Y_1 := \sqrt{-1}[X, \bar{X}]$. By Lemma 2.3, $Y_1 \in E(2a)$. Since $\bar{Y}_1 = Y_1$, Y_1 is real. Decompose $Y_1 = e^{2ax^2}\{u_1(x^1)\partial_{x^1} + v_1(x^1)\partial_{x^2}\}$.

Case 2.1 If $u_1 \neq 0$, then we may apply Case 1 to Y_1.

Case 2.2 If $u_1 = 0$ and if $v_1 \neq 0$, then we apply Lemma 3.2 (1) to see that Eq. (3a) holds with ${\Gamma_{22}}^2 = -2a$. We apply Lemma 3.2 (2-a-i) to X to see $(3a^2 - 1 + 4a\sqrt{-1})e^{\alpha x^2}u(x^1) = 0$ so $u = 0$ contrary to our assumption.

Case 2.3 If $u_1 = 0$ and $v_1 = 0$, then $[X, \bar{X}] = 0$ and Lemma 3.3 pertains with respect to the Lie algebra $\{\Re(X), \Im(X), \partial_{x^2}\}$, since

$$[\Im(X), \Re(X)] = 0, \quad [\Im(X), -\partial_{x^2}] = a\Im(X) + \Re(X),$$
$$[\Re(X), -\partial_{x^2}] = a\Re(X) - \Im(X).$$

Case 3 Suppose $\alpha = \sqrt{-1}$. If $X = e^{\sqrt{-1}x^2}(u(x^1)\partial_{x^1} + v(x^1)\partial_{x^2}\}$ is a complex affine Killing vector field, then $\Re(X)$ is a real affine Killing vector field where the coefficients of ∂_{x^i} can be written in terms of $\sin(x^2)$ and $\cos(x^2)$ multiplied by suitably chosen functions of x^1 and x^2. We have X_i in $\mathfrak{K}(\mathcal{M})$ with $\{X_i, \partial_{x^2}\}$ effective where

$$X_1 = u(x^1, x^2)\partial_{x^1} + v(x^1, x^2)\partial_{x^2}, \quad X_2 = \mathrm{ad}(\partial_{x^2})X_1,$$
$$u(x^1, x^2) = u_1(x^1)\cos(x^2) + u_2(x^1)\sin(x^2),$$
$$v(x^1, x^2) = v_1(x^1)\cos(x^2) + v_2(x^1)\sin(x^2).$$

Since $\{X_1, \partial_{x^2}\}$ is effective, $u \neq 0$. Let $X_3 := [X_1, X_2] \in E(0)$. Because there are no powers of x^2 in X_1 or X_2, we have that $X_3 = u_3(x^1)\partial_{x^1} + v_3(x^1)\partial_{x^2}$.

Case 3.1 If $u_3 \neq 0$, then $\{X_3, \partial_{x^2}\}$ is an effective Lie algebra isomorphic to $\mathfrak{K}_{\mathcal{A}}$.

Case 3.2 If $u_3 = 0$ but $v_3 \neq 0$, then $X_3 = v_3(x^1)\partial_{x^2}$. If $v_3' \neq 0$, we may apply Lemma 3.2 (1) to obtain the relations of Eq. (3a) with $\alpha = 0$. We may then apply Lemma 3.2 (2-a-i) to see $u^{(0,2)} = 0$. Since $u = -u^{(0,2)}$, $u = 0$ contrary to our assumption. Thus $v_3' = 0$ and $[X_1, X_2]$ is a constant non-zero multiple of ∂_{x^2}. This gives the Lie algebra $\mathfrak{so}(3)$.

Case 3.3 If $X_3 = 0$, we have $[X_1, X_2] = 0$ and we can apply Lemma 3.3.

□

We now examine $E(0)$.

Lemma 3.5 *Assume there exists $X \in E(0)$ such that $\{X, \partial_{x^2}\}$ is effective. Then there exists an effective Lie subalgebra $\mathfrak{K}_0 \subset \mathfrak{K}(\mathcal{M})$ isomorphic to $\mathfrak{K}_{\mathcal{A}}$ or $\mathfrak{K}_{\mathcal{B}}$.*

Proof Choose $X \in E(0)$ of the form given in Eq. (3b) with $\alpha = 0$. If $j_0 = 0$, then $\{X, \partial_{x^2}\}$ is an effective algebra isomorphic to $\mathfrak{K}_{\mathcal{A}}$. We may therefore assume that

$j_0 \geq 1$. We suppose $j_0 \geq 2$ and argue for a contradiction. Since $j_0 - 1 \leq 2j_0 - 3$, $u(x^1)\partial_{x^1}$ contributes lower order terms and plays no role. Set:

$$\begin{aligned} Y_1 &:= \operatorname{ad}(\partial_{x^2})X = \{c_1 v_{j_0}(x^1)(x^2)^{j_0-1} + O((x^2)^{j_0-2})\}\partial_{x^2}, \\ Y_2 &:= [X, Y_1] = \{c_2 v_{j_0}^2(x^1)(x_2)^{2(j_0-1)} + O((x^2)^{2(j_0-1)-1})\}\partial_{x^2}, \\ &\dots \\ Y_n &:= [X, Y_{n-1}] = \{c_n v_{j_0}^n(x^1)(x^2)^{n(j_0-1)} + O((x^2)^{n(j_0-1)-1})\}\partial_{x^2}. \end{aligned}$$

One verifies all the normalizing constants c_n are non-zero so creates an infinite string of linearly independent elements of $\mathfrak{K}(\mathcal{M})$ which is not possible. We therefore suppose $j_0 = 1$ henceforth so $X = u(x^1)\partial_{x^1} + (v_1(x^1)x^2 + v_0(x^1))\partial_{x^2}$ for $v_1 \neq 0$. If $v_1' = 0$, then $[\partial_{x^2}, X] = v_1\partial_{x^2}$ is a constant multiple of ∂_{x^2} and we obtain a subalgebra isomorphic to $\mathfrak{K}_{\mathcal{B}}$. We therefore suppose that $v_1' \neq 0$ and apply Lemma 3.2 (1) to obtain the relations of Eq. (3a) with $\alpha = 0$. By Lemma 3.2 (2-c), $x^2\partial_{x^2} \in \mathfrak{K}(\mathcal{M})$ and $v_0(x^1)\partial_{x^2} \in \mathfrak{K}_0(\mathcal{M})$. Consequently we have $\tilde{X} := u(x^1)\partial_{x^1} + v_1(x^1)x^2\partial_{x^2} \in \mathfrak{K}(\mathcal{M})$. We have $[\tilde{X}, x^2\partial_{x^2}] = 0$ and thus $\operatorname{Span}\{\tilde{X}, x^2\partial_{x^2}\}$ is an effective Lie subalgebra of $\mathfrak{K}(\mathcal{M})$ isomorphic to $\mathfrak{K}_{\mathcal{A}}$. This completes the proof of Lemma 3.5 and thereby of Lemma 2.2 (1). □

3.2 The proof of Lemma 2.2 (2)

Let $\{X, Y\}$ be affine Killing vector fields which are effective and which satisfy $[X, Y] = 0$. The Frobenius Theorem lets us choose local coordinates so $X = \partial_{x^1}$ and $Y = \partial_{x^2}$; we then have $\Gamma_{ij}^k \in \mathbb{R}$. □

3.3 The proof of Lemma 2.2 (3)

The following is a useful observation.

Ansatz 3.6 Let $X = u(x^1)\partial_{x^1} + (x^2 + v(x^1))\partial_{x^2}$ where $u \neq 0$. Set $\tilde{x}^1 = x^1$ and $\tilde{x}^2 = x^2 + \varepsilon(x^1)$. Then $\partial_{\tilde{x}^1} = \partial_{x^1} - \varepsilon'(x^1)\partial_{x^2}$ and $\partial_{\tilde{x}^2} = \partial_{x^2}$. We then have $X = u(\tilde{x}^1)\partial_{\tilde{x}^1} + \{\tilde{x}^2 - \varepsilon(x^1) + v(x^1) + u(x^1)\varepsilon'(x^1)\}\partial_{\tilde{x}^2}$. We may then solve the ODE $-\varepsilon(x^1) + v(x^1) + u(x^1)\varepsilon'(x^1) = 0$ to express $X = u(\tilde{x}^1)\partial_{\tilde{x}^1} + \tilde{x}^2\partial_{\tilde{x}^2}$ where $w(\tilde{x}^1)\partial_{\tilde{x}^2} = w(x^1)\partial_{x^2}$ for any w.

Let $\{X, Y\}$ be affine Killing vector fields which are effective with $[X, Y] = Y$. Choose local coordinates so $Y = \partial_{x^2}$. Expand $X = u(x^1, x^2)\partial_{x^1} + v(x^1, x^2)\partial_{x^2}$. Since $[X, Y] = Y$, $\partial_{x^2}u = 0$ and $\partial_{x^2}v = -1$ so $X = u(x^1)\partial_{x^1} + (-x^2 + v_0(x^1))\partial_{x^2}$. Use Ansatz 3.6 to change coordinates so $X = u(x^1)\partial_{x^1} - x^2\partial_{x^2}$ without changing $Y = \partial_{x^2}$. Replace x^1 by $\hat{x}^1$ to ensure $u(x^1)\partial_{x^1} = -\hat{x}^1\partial_{\hat{x}^1}$ so $X = -\hat{x}^1\partial_{\hat{x}^1} - \hat{x}^2\partial_{\hat{x}^2}$. The affine Killing equations for X yield $K_{ij}{}^k\colon 0 = \Gamma_{ij}{}^k(\hat{x}^1) + \hat{x}^1\Gamma'_{ij}{}^k(\hat{x}^1)$ for $i, j, k = 1, 2$. This shows that the Christoffel symbols have the desired form. □

3.4 The proof of Lemma 2.2 (4)

Throughout this section, we will not use the normalizations of Lemma 3.1. We shall, however, assume always that $\partial_{x^2} \in \mathfrak{K}(\mathcal{M})$ so $\Gamma_{ij}{}^k = \Gamma_{ij}{}^k(x^1)$. We begin by showing:

Lemma 3.7 *Suppose $\mathcal{M}$ has an effective Lie subalgebra isomorphic to $\mathfrak{so}(3)$. Then the connection is torsion free, the Ricci tensor ρ is positive definite and symmetric, and $\nabla\rho = 0$.*

Proof Let $\text{Span}\{X, Y, Z\}$ be an effective Lie subalgebra of $\mathfrak{K}(\mathcal{M})$ satisfying the relations of Eq. (2a) defining $\mathfrak{so}(3)$. Choose coordinates so $Z = \partial_{x^2}$. Decompose $X = u(x^1, x^2)\partial_{x^1} + v(x^1, x^2)\partial_{x^2}$. We then have $u^{(0,2)} = -u$. We may then express $u(x^1, x^2) = r(x^1)\cos(x^2 + \theta(x^1))$ where by hypothesis $r(x^1) \neq 0$. Use Ansatz 3.6 to replace x^2 by $x^2 + \theta(x^1)$ and rewrite $X = r_1(y^1)\cos(y^2)\partial_{y^1} + v_1(x^1, x^2)\partial_{y^2}$ without changing ∂_{x^2}. Choose coordinates $(z^1, z^2) = (f(y^1), y^2)$ so that $\partial_{z^1} = r_1(y^1)\partial_{y^1}$ and $\partial_{z^2} = \partial_{y^2}$. Since $v_1^{(0,2)} = -v_1$, the bracket relation $[Z, X] = Y$ gives

$$\begin{aligned} X &= \cos(z^2)\partial_{z^1} + \{v_c(z^1)\cos(z^2) + v_s(z^1)\sin(z^2)\}\partial_{z^2}, \\ Y &= -\sin(z^2)\partial_{z^1} + \{-v_c(z^1)\sin(z^2) + v_s(z^1)\cos(z^2)\}\partial_{z^2}. \end{aligned}$$

The bracket relation $[X, Y] = Z$ now yields $-v_c(x^1)^2 - v_s(x^1)^2 + v_s'(x^1) = 1$ and $v_c(x^1) = 0$. We solve this to obtain $v_c(x^1) = 0$ and $v_s(x^1) = \tan(x^1 + c)$; we can further normalize the coordinates so $c = 0$. Thus, after a suitable change of notation, we have $Z = \partial_{x^2}$,

$$\begin{aligned} X &= \cos(x^2)\partial_{x^1} + \tan(x^1)\sin(x^2)\partial_{x^2}, \\ Y &= -\sin(x^2)\partial_{x^1} + \tan(x^1)\cos(x^2)\partial_{x^2}. \end{aligned}$$

We have $\Gamma_{ij}{}^k = \Gamma_{ij}{}^k(x^1)$. We evaluate the affine Killing equations corresponding to X at $x^2 = 0$ to obtain

$$\begin{aligned} &K_{11}{}^1 : 0 = \Gamma'_{11}{}^1(x^1), \qquad K_{11}{}^2 : 0 = -\Gamma_{11}{}^2(x^1)\tan(x^1) + \Gamma'_{11}{}^2(x^1), \\ &K_{12}{}^1 : 0 = \Gamma_{12}{}^1(x^1)\tan(x^1) + \Gamma'_{12}{}^1(x^1), \quad K_{12}{}^2 : 0 = \sec^2(x^1) + \Gamma'_{12}{}^2(x^1), \\ &K_{21}{}^1 : 0 = \Gamma_{21}{}^1(x^1)\tan(x^1) + \Gamma'_{21}{}^1(x^1), \quad K_{212} : 0 = \sec^2(x^1) + \Gamma'_{21}{}^2(x^1), \\ &K_{22}{}^1 : 0 = -1 + 2\Gamma_{22}{}^1(x^1)\tan(x^1) + \Gamma'_{22}{}^1(x^1), \\ &K_{22}{}^2 : 0 = \Gamma_{22}{}^2(x^1)\tan(x^1) + \Gamma'_{22}{}^2(x^1). \end{aligned}$$

We solve these ODEs to obtain real constants $a_{ij}{}^k$ so that

$$\begin{aligned} &\Gamma_{11}{}^1(x^1) = a_{11}{}^1, && \Gamma_{11}{}^2(x^1) = a_{11}{}^2\sec(x^1), \\ &\Gamma_{12}{}^1(x^1) = a_{12}{}^1\cos(x^1), && \Gamma_{12}{}^2(x^1) = a_{12}{}^2 - \tan(x^1), \\ &\Gamma_{21}{}^1(x^1) = a_{21}{}^1\cos(x^1), && \Gamma_{21}{}^2(x^1) = a_{21}{}^2 - \tan(x^1), \\ &\Gamma_{22}{}^1(x^1) = a_{22}{}^1\cos(x^1)^2 + \cos(x^1)\sin(x^1), && \Gamma_{22}{}^2(x^1) = a_{22}{}^2\cos(x^1). \end{aligned}$$

Let $i \neq j$ and $1 \leq i, j \leq 2$. We evaluate the affine Killing equations for X at $(x^1, x^2) = (0, \frac{\pi}{2})$ to obtain

$$\begin{array}{ll} K_{ii}{}^i : 0 = a_{ii}{}^j + a_{ij}{}^i + a_{ji}{}^i, & K_{ii}{}^j : 0 = -a_{ii}{}^i + a_{ij}{}^j + a_{ji}{}^j, \\ K_{ij}{}^i : 0 = -a_{ii}{}^i + a_{ij}{}^j + a_{jj}{}^i, & K_{ij}{}^j : 0 = -a_{ii}{}^j - a_{ij}{}^i + a_{jj}{}^j. \end{array}$$

These equations imply that all the $a_{ij}{}^k$ vanish and thus the non-zero Christoffel symbols are $\Gamma_{12}{}^2(x^1) = \Gamma_{21}{}^2(x^1) = -\tan(x^1)$ and $\Gamma_{22}{}^1(x^1) = \cos(x^1)\sin(x^1)$. We complete the proof of the Lemma by computing that $\rho = (dx^1)^2 + \cos(x^1)^2(dx^2)^2$ and $\nabla\rho = 0$. □

We apply Lemma 3.7. We have shown ∇ is torsion free. Let $ds^2 = \rho$. We have $\nabla\rho = 0$. This shows ∇ is the Levi–Civita connection of ds^2; this is a positive definite metric with Einstein constant 1. Thus this geometry is modeled on the round sphere. This completes the proof of Lemma 2.2 and thereby of Theorem 1.1. □

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