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Blow-up phenomena and local well-posedness for a generalized Camassa–Holm equation in the critical Besov space

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Abstract

In this paper we mainly study the Cauchy problem for a generalized Camassa–Holm equation in a critical Besov space. First, by using the Littlewood–Paley decomposition, transport equations theory, logarithmic interpolation inequalities and Osgood's lemma, we establish the local well-posedness for the Cauchy problem of the equation in the critical Besov space $B^{\frac{1}{2}}_{2,1}$. Next we derive a new blow-up criterion for strong solutions to the equation. Then we give a global existence result for strong solutions to the equation. Finally, we present two new blow-up results and the exact blow-up rate for strong solutions to the equation by making use of the conservation law and the obtained blow-up criterion.

Keywords A generalized Camassa–Holm equation · Local well-posedness · The critical Besov space · Blow-up · Global existence

Mathematics Subject Classification 35Q53 · 35A01 · 35B44 · 35B65

Contents

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1 Introduction

In this paper we consider the Cauchy problem for the following generalized Camassa–Holm equation,

$$\begin{cases} u_t - u_{txx} = \frac{1}{2}(3u_x^2 - 2u_x u_{xxx} - u_{xx}^2), \quad t > 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

which can be rewritten as

$$\begin{cases} m = u - u_{xx}, \\ m_t - u_x m_x = -\frac{1}{2}m^2 + um + \frac{1}{2}u_x^2 - \frac{1}{2}u^2, \quad t > 0, \\ m(0,x) = u(0,x) - u_{xx}(0,x) = m_0(x). \end{cases} \tag{1.2}$$

The Eq. (1.1) was proposed recently by Novikov in [38]. He showed that the equation (1.1) is integrable by using as definition of integrability the existence of an infinite hierarchy of quasi-local higher symmetries [38] and it belongs to the following class [38]:

$$(1 - \partial_x^2)u_t = F(u, u_x, u_{xx}, u_{xxx}), \tag{1.3}$$

which has attracted much interest, particularly in the possible integrable members of (1.3).

The most celebrated integrable members of (1.3) which have quadratic nonlinearity are the well-known Camassa–Holm (CH) equation [4] and the famous Degasperis-Procesi (DP) equation [23]:

$$(1 - \partial_x^2)u_t = 3uu_x - 2u_x u_{xx} - uu_{xxx}, \tag{1.4}$$

$$(1 - \partial_x^2)u_t = 4uu_x - 3u_x u_{xx} - uu_{xxx}. \tag{1.5}$$

Both the CH equation and the DP equation can be regarded as a shallow water wave equation [4,16,24]. They are completely integrable. That means that the system can be transformed into a linear flow at constant speed in suitable action-angle variables (in the sense of infinite-dimensional Hamiltonian systems), for a large class of initial data [4, 8,17,22]. They also have a bi-Hamiltonian structure [7,22,27], and admit exact peaked solitons, which are orbitally stable [19]. It is worth mentioning that the peaked solitons present the characteristic for the traveling water waves of greatest height and largest amplitude and arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, cf. [5,10,14,15,40]. The main difference between DP equation and CH equation is that DP equation has short waves [36] and the periodic shock waves [26].

The local well-posedness and ill-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces were discussed in [11,12,20,29,35,39]. It was shown that there exist global strong solutions to the CH equation [9,11,12] and finite time blow-up strong solutions to the CH equation [9,11–13]. The existence and uniqueness of global weak solutions to the CH equation were proved in [18,44]. The global conservative and dissipative solutions of CH equation were investigated in [2,3].

The local well-posedness of the Cauchy problem of the DP equation in Sobolev spaces and Besov spaces was established in [28,30,47]. Similar to the CH equation, the DP equation has also global strong solutions [33,48,50] and finite time blow-up solutions [25,26,33,34,47–50]. It also has global weak solutions [6,25,49,50].

The third celebrated integrable member of (1.3) which has cubic nonlinearity is the known Novikov equation [38]:

$$(1-\partial_x^2)u_t = 3uu_x u_{xx} + u^2 u_{xxx} - 4u^2 u_x. \tag{1.6}$$

It was showed that the Novikov equation is integrable, possesses a bi-Hamiltonian structure, and admits exact peakon solutions $u(t,x) = \pm\sqrt{c}e^{|x-ct|}$ with $c > 0$ [31]. The local well-posedness for the Novikov equation in Sobolev spaces and Besov spaces was studied in [42,43,45,46]. The global existence of strong solutions under some sign conditions were established in [42] and the blow-up phenomena of the strong solutions were shown in [46]. The global weak solutions for the Novikov equation were studied in [41].

Recently, the Cauchy problem of (1.1) in the Besov spaces $B^s_{p,r},\ s > max\{\frac{1}{p}, \frac{1}{2}\}$ has been studied in [32]. To our best knowledge, the Cauchy problem of (1.1) in the critical Besov space $B^{\frac{1}{2}}_{2,1}$ has not been studied yet. In this paper we first investigate the local well-posedness of (1.2) with initial data in the critical Besov space $B^{\frac{1}{2}}_{2,1}$. The main idea is based on the Littlewood–Paley theory, transport equations theory, logarithmic interpolation inequalities and Osgood's lemma. Then, we prove a new blow-up criteria by the energy method, which is more precise than the blow-up criteria derived in [32]. By virtue of a conservation law, we obtain two new blow-up results. Finally, we conclude the exact blow-up rate of blowing-up solutions $m(t,x)$ to (1.1).

The paper is organized as follows. In Sect. 2 we introduce some preliminaries which will be used in sequel. In Sect. 3 we prove the local well-posedness of (1.1) by using Littlewood–Paley and transport equations theory, logarithmic interpolation inequalities and Osgood's lemma. In Sect. 4, we derive a conservation law and a blow-up criterion. In Sect. 5, we show the global existence of strong solution to (1.1). Section 6 is devoted to the study of blow-up phenomena of (1.1). We present two blow-up results and the exact blow-up rate of blowing-up solutions to (1.1).

2 Preliminaries

In this section, we first recall the Littlewood–Paley decomposition and Besov spaces (for more details to see [1]). Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^d \big| \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. There exist radial functions χ and φ, valued in the interval [0, 1], belonging respectively to $\mathcal{D}(B(0, \frac{4}{3}))$ and $\mathcal{D}(\mathcal{C})$, and such that

$$\begin{aligned}
&\forall \xi \in \mathbb{R}^d,\ \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1,\\
&|j - j'| \geq 2 \Rightarrow Supp\ \varphi(2^{-j}\xi) \cap Supp\ \varphi(2^{-j'}\xi) = \emptyset,\\
&j \geq 1 \Rightarrow Supp\ \chi(\xi) \cap Supp\ \varphi(2^{-j'}\xi) = \emptyset.
\end{aligned}$$

Define the set $\widetilde{\mathcal{C}} = B(0, \frac{2}{3}) + \mathcal{C}$. Then we have

$$|j - j'| \geq 5 \Rightarrow 2^{j'}\widetilde{\mathcal{C}} \cap 2^j \mathcal{C} = \emptyset.$$

Further, we have

$$\forall \xi \in \mathbb{R}^d,\ \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1.$$

Denote $\mathcal{F}$ by the Fourier transform and $\mathcal{F}^{-1}$ by its inverse. From now on, we write $h = \mathcal{F}^{-1}\varphi$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$. The nonhomogeneous dyadic blocks Δ_j are defined by

$$\begin{aligned}
&\Delta_j u = 0\ \ if\ \ j \leq -2, \quad \Delta_{-1} u = \chi(D) u = \int_{\mathbb{R}^d} \widetilde{h}(y) u(x - y) dy,\\
&and, \quad \Delta_j u = \varphi(2^{-j} D) u = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) u(x - y) dy \quad if \ \ j \geq 0,\\
&S_j u = \sum_{j' \leq j-1} \Delta_{j'} u.
\end{aligned}$$

The nonhomogeneous Besov spaces are denoted by $B^s_{p,r}(\mathbb{R}^d)$

$$B^s_{p,r} = \left\{ u \in S' \Big| \|u\|_{B^s_{p,r}(\mathbb{R}^d)} = \left(\sum_{j \geq -1} 2^{rjs} \|\Delta_j u\|^r_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{r}} < \infty \right\}.$$

Next we introduce some useful lemmas and propositions about Besov spaces which will be used in the sequel.

Proposition 2.1 [1] *Let* $1 \leq p_1 \leq p_2 \leq \infty$ *and* $1 \leq r_1 \leq r_2 \leq \infty$, *and let* s *be a real number. Then we have*

$$B^s_{p_1,r_1}(\mathbb{R}^d) \hookrightarrow B^{s - d(\frac{1}{p_1} - \frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}^d).$$

If $s > \frac{d}{p}$ *or* $s = \frac{d}{p}$, $r = 1$, *we then have*

$$B^s_{p,r}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d).$$

Lemma 2.2 [1] *A constant C exists which satisfies the following properties. If s_1 and s_2 are real numbers such that $s_1 < s_2$ and $\theta \in (0, 1)$, then we have, for any $(p, r) \in [1, \infty]^2$ and $u \in \mathcal{S}'_h$,*

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|^{\theta}_{B^{s_1}_{p,r}} \|u\|^{1-\theta}_{B^{s_2}_{p,r}} \quad \text{and} \tag{2.1}$$

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_2 - s_1}\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right) \|u\|^{\theta}_{B^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{B^{s_2}_{p,\infty}}. \tag{2.2}$$

Lemma 2.3 [1] *For any positive real number s and any (p, r) in $[1, \infty]^2$, the space $L^\infty(\mathbb{R}^d) \cap B^s_{p,r}(\mathbb{R}^d)$ is an algebra, and a constant C exists such that*

$$\|uv\|_{B^s_{p,r}(\mathbb{R}^d)} \leq C\left(\|u\|_{L^\infty(\mathbb{R}^d)}\|v\|_{B^s_{p,r}(\mathbb{R}^d)} + \|u\|_{B^s_{p,r}(\mathbb{R}^d)}\|v\|_{L^\infty(\mathbb{R}^d)}\right).$$

If $s > \frac{d}{p}$ *or* $s = \frac{d}{p}$, $r = 1$, *then we have*

$$\|uv\|_{B^s_{p,r}(\mathbb{R}^d)} \leq C\|u\|_{B^s_{p,r}(\mathbb{R}^d)}\|v\|_{B^s_{p,r}(\mathbb{R}^d)}.$$

The following two lemmas is crucial to study well-posedness in the critical space $B^{\frac{1}{2}}_{2,1}(\mathbb{R})$.

Lemma 2.4 [37] *For any $a \in B^{-\frac{1}{2}}_{2,\infty}(\mathbb{R})$ and $b \in B^{\frac{1}{2}}_{2,1}(\mathbb{R})$, there exists a constant C such that*

$$\|ab\|_{B^{-\frac{1}{2}}_{2,\infty}(\mathbb{R})} \leq C\|a\|_{B^{-\frac{1}{2}}_{2,\infty}(\mathbb{R})}\|b\|_{B^{\frac{1}{2}}_{2,1}(\mathbb{R})}. \tag{2.3}$$

Lemma 2.5 (Morse-type estimate, [1,20]) *Let $s > \max\{\frac{d}{p}, \frac{d}{2}\}$ and (p, r) in $[1, \infty]^2$ or $s = \frac{d}{2}$, $p = 2$, $r = 1$. For any $a \in B^{s-1}_{p,r}(\mathbb{R}^d)$ and $b \in B^s_{p,r}(\mathbb{R}^d)$, there exists a constant C such that*

$$\|ab\|_{B^{s-1}_{p,r}(\mathbb{R}^d)} \leq C\|a\|_{B^{s-1}_{p,r}(\mathbb{R}^d)}\|b\|_{B^s_{p,r}(\mathbb{R}^d)}.$$

Lemma 2.6 [21] *For any $s \in \mathbb{R}$, $\varepsilon \in (0, 1]$ and $f \in B^{s+\varepsilon}_{2,1}(\mathbb{R})$, there exists a constant C such that*

$$\|f\|_{B^s_{2,1}(\mathbb{R})} \leq \frac{C}{\varepsilon}\|f\|_{B^s_{2,\infty}(\mathbb{R})} \ln\left(e + \frac{\|f\|_{B^{s+\varepsilon}_{2,1}(\mathbb{R})}}{\|f\|_{B^s_{2,\infty}(\mathbb{R})}}\right).$$

Remark 2.7 [1] Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. Then the following properties hold true:

(i) $B^s_{p,r}(\mathbb{R}^d)$ is a Banach space and continuously embedding into $\mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{S}'(\mathbb{R}^d)$ is the dual space of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.
(ii) If $p, r < \infty$, then $\mathcal{S}(\mathbb{R}^d)$ is dense in $B^s_{p,r}(\mathbb{R}^d)$.
(iii) If u_n is a bounded sequence of $B^s_{p,r}(\mathbb{R}^d)$, then an element $u \in B^s_{p,r}(\mathbb{R}^d)$ and a subsequence u_{n_k} exist such that

$$\lim_{k\to\infty} u_{n_k} = u \ \ in \ \ \mathcal{S}'(\mathbb{R}^d) \ \ and \ \ \|u\|_{B^s_{p,r}(\mathbb{R}^d)} \le C \liminf_{k\to\infty} \|u_{n_k}\|_{B^s_{p,r}(\mathbb{R}^d)}.$$

(iv) $B^s_{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$.

The following Osgood's lemma appears as a substitute for Gronwall's lemma.

Lemma 2.8 (Osgood's lemma, [1]) *Let $\rho \ge 0$ be a measurable function, $\gamma > 0$ be a locally integrable function and μ be a continuous and increasing function. Assume that, for some nonnegative real number c, the function ρ satisfies*

$$\rho(t) \le c + \int_{t_0}^{t} \gamma(t')\mu(\rho(t'))dt'.$$

If $c > 0$, then $-\mathcal{M}(\rho(t)) + \mathcal{M}(c) \le \int_{t_0}^{t} \gamma(t')dt'$ with $\mathcal{M}(x) = \int_x^1 \frac{dr}{\mu(r)}$.
If $c = 0$ and μ verifies the condition $\int_0^1 \frac{dr}{\mu(r)} = +\infty$, then the function $\rho = 0$.

Remark 2.9 In this paper, we set $\mu(r) = r(1 - \ln r)$ which satisfies the condition $\int_0^1 \frac{dr}{\mu(r)} = +\infty$. A simple calculation shows that $\mathcal{M}(x) = \ln(1 - \ln x)$, we deduce that

$$\rho(t) \le ec^{\exp \int_{t_0}^{t} -\gamma(t')dt'}, \quad if \ \ c > 0.$$

Now we introduce a priori estimates for the following transport equation.

$$\begin{cases} f_t + v\nabla f = g, \\ f|_{t=0} = f_0. \end{cases} \tag{2.4}$$

Lemma 2.10 *(A priori estimates in Besov spaces,* [1,20,21]*) Let $1 \le p \le p_1 \le \infty$, $1 \le r \le \infty$, $s \ge -d\min(\frac{1}{p_1}, \frac{1}{p'})$. For the solution $f \in L^\infty([0,T]; B^s_{p,r}(\mathbb{R}^d))$ of* (2.4) *with velocity $\nabla v \in L^1([0,T]; B^s_{p,r}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$, initial data $f_0 \in B^s_{p,r}(\mathbb{R}^d)$ and $g \in L^1([0,T]; B^s_{p,r}(\mathbb{R}^d))$, we have the following statements. If $s \ne 1 + \frac{1}{p}$ or $r = 1$,*

$$\begin{aligned}&\|f(t)\|_{B^s_{p,r}(\mathbb{R}^d)}\\&\quad\le\|f_0\|_{B^s_{p,r}(\mathbb{R}^d)}+\int_0^t\Big(\|g(t')\|_{B^s_{p,r}(\mathbb{R}^d)}+CV'_{p_1}(t')\|f(t')\|_{B^s_{p,r}(\mathbb{R}^d)}\Big)dt', \qquad (2.5)\\&\|f\|_{(B^s_{p,r}(\mathbb{R}^d))}\\&\quad\le\Big(\|f_0\|_{B^s_{p,r}(\mathbb{R}^d)}+\int_0^t\exp(-CV_{p_1}(t'))\|g(t')\|_{B^s_{p,r}(\mathbb{R}^d)}dt'\Big)\exp(CV_{p_1}(t)), \quad (2.6)\end{aligned}$$

where $V_{p_1}(t)=\int_0^t\|\nabla v\|_{B^{\frac{d}{p_1}}_{p_1,\infty}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)}dt'$, *if* $s<1+\frac{d}{p_1}$; $V_{p_1}(t)=\int_0^t\|\nabla v\|_{B^{s-1}_{p_1,r}(\mathbb{R}^d)}dt'$, *if* $s>1+\frac{d}{p_1}$ *or* $s=1+\frac{d}{p_1}, r=1$, *and* C *is a constant depending only on* $s,\ p,\ p_1$ *and* r.

Lemma 2.11 [1] *Let* s *be as in the statement of Lemma 2.10. Let* $f_0\in B^s_{p,r}(\mathbb{R}^d)$, $g\in L^1([0,T];B^s_{p,r}(\mathbb{R}^d))$, *and* v *be a time-dependent vector field such that* $v\in L^\rho([0,T];B^{-M}_{\infty,\infty}(\mathbb{R}^d))$ *for some* $\rho>1$ *and* $M>0$, *and*

$$\begin{aligned}&\nabla v\in L^1([0,T];B^{\frac{d}{p_1}}_{p_1,\infty}(\mathbb{R}^d)),\quad if\ s<1+\frac{d}{p_1},\\&\nabla v\in L^1([0,T];B^{s-1}_{p_1,\infty}(\mathbb{R}^d)),\quad if\ s>1+\frac{d}{p_1}\ \ or\ \ s=1+\frac{d}{p_1}\ and\ r=1.\end{aligned}$$

Then, (2.4) *has a unique solution* f *in*

- *the space* $\mathcal{C}([0,T];B^s_{p,r}(\mathbb{R}^d))$, *if* $r<\infty$,
- *the space* $(\bigcap_{s'<s}\mathcal{C}([0,T];B^{s'}_{p,\infty}(\mathbb{R}^d)))\bigcap\mathcal{C}_w([0,T];B^s_{p,\infty}(\mathbb{R}^d)))$, *if* $r=\infty$. *Moreover, the inequalities of Lemma 2.10 hold true.*

Lemma 2.12 [35] *Let* $s>\frac{1}{p}, r<\infty$ (*or* $s=\frac{1}{p}, 1\le p<\infty, r=1$), $k\in\mathbb{N}$ *and a constant* C *depending only on* $s,p,r,v,$ *and* g. *If* (2.4) *satisfies the following conditions for all* $f,\widetilde{f}\in B^s_{p,r}$,

$$\begin{aligned}&(1)\|v\|_{B^{s+1}_{p,r}}\le C(1+\|f\|^k_{B^s_{p,r}}),\\&(2)\|g\|_{B^s_{p,r}}\le C(1+\|f\|^k_{B^s_{p,r}}),\\&(3)\|v(f)-v(\widetilde{f})\|_{B^s_{p,r}}\le C\|f-\widetilde{f}\|_{B^s_{p,r}}(1+\|f\|^{k-1}_{B^s_{p,r}}),\\&(4)\|g(f)-g(\widetilde{f})\|_{B^s_{p,r}}\le C\|f-\widetilde{f}\|_{B^s_{p,r}}(1+\|f\|^k_{B^s_{p,r}}).\end{aligned}$$

Denote $n\in\mathbb{N}$. *If* u_0^n *tends to* $u_0\in B^s_{p,r}$ *and* u^n *tends to* $u\in C([0,T];B^{s-1}_{p,r})$, *then* u^n *tends to* $u\in C([0,T];B^s_{p,r})$.

Notations Since all space of functions in the following sections are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity.

3 Local well-posedness

In this section, we establish local well-posedness of (1.2) in the critical Besov space $B^{\frac{1}{2}}_{2,1}$. Our main result can be stated as follows.

Theorem 3.1 *Let* $m_0 \in B^{\frac{1}{2}}_{2,1}$. *Then there exists some* $T > 0$, *such that* (1.2) *has a unique solution* m *in* $C([0,T); B^{\frac{1}{2}}_{2,1}) \cap C^1([0,T); B^{-\frac{1}{2}}_{2,1})$.

Proof In order to prove Theorem 3.1, we proceed as the following five steps.

Step 1 First, we construct approximate solutions which are smooth solutions of some linear equations. Starting for $m_0(t,x) \triangleq m(0,x) = m_0$, we define by induction sequences $(m_n)_{n\in\mathbb{N}}$ by solving the following linear transport equations:

$$\begin{cases} \partial_t m_{n+1} - \partial_x u_n \partial_x m_{n+1} & = \frac{1}{2}(\partial_x u_n)^2 - \frac{1}{2}(u_n - m_n)^2 \\ & = u_n m_n + \frac{1}{2}(\partial_x u_n)^2 - \frac{1}{2}u_n^2 - \frac{1}{2}m_n^2 \\ & = F(m_n, u_n), \\ m_{n+1}(t,x)|_{t=0} = S_{n+1} m_0. \end{cases} \tag{3.1}$$

We assume that $m_n \in L^\infty(0,T; B^{\frac{1}{2}}_{2,1})$. Note that $B^{\frac{1}{2}}_{2,1}$ is an algebra and $B^{\frac{1}{2}}_{2,1} \hookrightarrow L^\infty$, which leads to $F(m_n,u_n) \in L^\infty(0,T; B^{\frac{1}{2}}_{2,1})$. Hence, from Lemma 2.11, the Eq. (3.1) has a global solution m_{n+1} which belongs to $C([0,T); B^{\frac{1}{2}}_{2,1}) \cap C^1([0,T); B^{-\frac{1}{2}}_{2,1})$ for all positive T.

Step 2 Next, we are going to find some positive T such that for this fixed T the approximate solutions are uniformly bounded on $[0,T]$. We define that $U_n(t) \triangleq \int_0^t \|m_n(t')\|_{B^{\frac{1}{2}}_{2,1}} dt'$. By Lemma 2.10, we infer that

$$\begin{aligned} \|m_{n+1}\|_{B^{\frac{1}{2}}_{2,1}} &\le e^{C\int_0^t \|\partial_x^2 u_n\|_{B^{\frac{1}{2}}_{2,1}} dt'} \Big(\|S_{n+1} m_0\|_{B^{\frac{1}{2}}_{2,1}} \\ &\quad + \int_0^t e^{-C\int_0^{t'} \|\partial_x^2 u_n\|_{B^{\frac{1}{2}}_{2,1}} d\tau} \|F(m_n,u_n)\|_{B^{\frac{1}{2}}_{2,1}} dt' \Big) \\ &\le e^{CU_n(t)} \Big(\|S_{n+1} m_0\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t e^{-CU_n(t')} \|F(m_n,u_n)\|_{B^{\frac{1}{2}}_{2,1}} dt' \Big). \end{aligned} \tag{3.2}$$

Since $B^{\frac{1}{2}}_{2,1}$ is an algebra and $B^{\frac{1}{2}}_{2,1} \hookrightarrow L^\infty$, we deduce that

$$\begin{aligned} &\left\| \frac{1}{2}(\partial_x u_n)^2 - \frac{1}{2}(u_n - m_n)^2 \right\|_{B^{\frac{1}{2}}_{2,1}} \\ &\le \frac{1}{2}\|(\partial_x u_n)^2\|_{B^{\frac{1}{2}}_{2,1}} + \frac{1}{2}\|(u_n - m_n)^2\|_{B^{\frac{1}{2}}_{2,1}} \\ &\le \|\partial_x u_n\|_{B^{\frac{1}{2}}_{2,1}} \|\partial_x u_n\|_{L^\infty} + \|u_n - m_n\|_{B^{\frac{1}{2}}_{2,1}} \|u_n - m_n\|_{L^\infty} \end{aligned}$$

$$\leq C\|m_n\|^2_{B^{\frac{1}{2}}_{2,1}}. \tag{3.3}$$

Plugging (3.3) into (3.2), we obtain

$$\begin{aligned}\|m_{n+1}\|_{B^{\frac{1}{2}}_{2,1}} &\leq e^{CU_n(t)}\left(\|S_{n+1}m_0\|_{B^{\frac{1}{2}}_{2,1}} + C\int_0^t e^{-CU_n(t')}\|m_n\|^2_{B^{\frac{1}{2}}_{2,1}}\,dt'\right)\\ &\leq e^{CU_n(t)}\left(C\|m_0\|_{B^{\frac{1}{2}}_{2,1}} + C\int_0^t e^{-CU_n(t')}\|m_n\|^2_{B^{\frac{1}{2}}_{2,1}}\,dt'\right),\end{aligned} \tag{3.4}$$

where we take $C \geq 1$.

We fix a $T > 0$ such that $2C^2T\|m_0\|_{B^{\frac{1}{2}}_{2,1}} < 1$. Suppose that

$$\|m_n(t)\|_{B^{\frac{1}{2}}_{2,1}} \leq \frac{C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}{1-2C^2\|m_0\|_{B^{\frac{1}{2}}_{2,1}}\,t} \leq \frac{C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}{1-2C^2\|m_0\|_{B^{\frac{1}{2}}_{2,1}}\,T} \triangleq \mathbf{M}, \qquad \forall t \in [0, T]. \tag{3.5}$$

Since $U_n(t) = \int_0^t \|m_n(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau$, it follows that

$$\begin{aligned}e^{CU_n(t)-CU_n(t')} &\leq \exp\left\{\int_{t'}^t \frac{C^2\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}{1-2C^2\|m_0\|_{B^{\frac{1}{2}}_{2,1}}\,t}d\tau\right\}\\ &\leq \exp\left\{-\frac{1}{2}\int_{t'}^t \frac{d\left(1-2C^2\tau\|m_0\|_{B^{\frac{1}{2}}_{2,1}}\right)}{1-2C^2\tau\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}\right\}\\ &= \left(\frac{1-2C^2t'\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}{1-2C^2t\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}\right)^{\frac{1}{2}}.\end{aligned} \tag{3.6}$$

Set $U_n(t') = 0$ when $t' = 0$. We obtain

$$e^{CU_n(t)} = \left(\frac{1}{1-2C^2t\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}\right)^{\frac{1}{2}}. \tag{3.7}$$

By using (3.5), (3.6) and (3.7), we have

$$
\begin{aligned}
&\|m_{n+1}(t)\|_{B^{\frac{1}{2}}_{2,1}}\\
&\le Ce^{CU_n(t)}\|m_0\|_{B^{\frac{1}{2}}_{2,1}}+C\int_0^t e^{CU_n(t)-CU_n(t')}\|m_n(t')\|^2_{B^{\frac{1}{2}}_{2,1}}\,dt'\\
&\le\left(\frac{1}{1-2C^2t\|m_0\|^2_{B^{\frac{1}{2}}_{2,1}}}\right)^{\frac{1}{2}}\left\{C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}+\int_0^t\left(\frac{C^3\|m_0\|^2_{B^{\frac{1}{2}}_{2,1}}}{\left(1-2C^2t'\|m_0\|_{B^{\frac{1}{2}}_{2,1}}\right)^{1+\frac{1}{2}}}\right)dt'\right\}\\
&\le\left(\frac{1}{1-2C^2t\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}\right)^{\frac{1}{2}}\left\{C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}-\frac{C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}{2}\int_0^t\frac{d\left(1-2C^2t'\|m_0\|_{B^{\frac{1}{2}}_{2,1}}\right)}{\left(1-2C^2t'\|m_0\|_{B^{\frac{1}{2}}_{2,1}}\right)^{1+\frac{1}{2}}}\right\}\\
&\le\left(\frac{1}{1-2C^2t\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}\right)^{\frac{1}{2}}\left\{C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}+C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}\left(\frac{1}{1-2C^2t'\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}\right)^{\frac{1}{2}}\Big|_0^t\right\}\\
&=\frac{C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}{1-2C^2t\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}\\
&\le\frac{C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}{1-2C^2T\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}=\mathbf{M}.
\end{aligned}
\tag{3.8}
$$

Thus, $(m_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;B^{\frac{1}{2}}_{2,1})$.

Step 3 From now on, we are going to prove that m_n is a Cauchy sequence in $L^\infty(0,T;B^{-\frac{1}{2}}_{2,\infty})$. For this purpose, we deduce from (3.1) that

$$
\begin{cases}
\partial_t(m_{n+l+1}-m_{n+1})-\partial_x u_{n+l}\partial_x(m_{n+l+1}-m_{n+1})\\
\quad=\partial_x(u_{n+l}-u_n)\partial_x m_{n+l}+\frac{1}{2}\partial_x(u_{n+l}-u_n)\partial_x(u_{n+l}+u_n)\\
\qquad-\frac{1}{2}(u_{n+l}-u_n-m_{n+l}+m_n)(u_{n+l}+u_n-m_{n+l}-m_n)\\
\quad=\partial_x(u_{n+l}-u_n)\partial_x[m_{n+l}+\frac{1}{2}(u_{n+l}+u_n)]-\frac{1}{2}(u_{n+l}-u_n)(u_{n+l}+u_n-m_{n+l}-m_n)\\
\qquad+\frac{1}{2}(m_{n+l}-m_n)(u_{n+l}+u_n-m_{n+l}-m_n)\\
\quad=\partial_x(u_{n+l}-u_n)\partial_x R^1_{n,l}-\frac{1}{2}(u_{n+l}-u_n)R^2_{n,l}+\frac{1}{2}(m_{n+l}-m_n)R^2_{n,l},\\
m_{n+l+1}-m_{n+1}|_{t=0}=(S_{n+l+1}-S_{n+1})m_0,
\end{cases}
\tag{3.9}
$$

where

$$R^1_{n,l} = m_{n+l} + \frac{1}{2}(u_{n+l} + u_n),$$
$$R^2_{n,l} = u_{n+l} + u_n - m_{n+l} - m_n.$$

By Lemma 2.10 and using the fact that m_n is bounded in $L^\infty(0, T; B^{\frac{1}{2}}_{2,1})$, we infer that

$$\begin{aligned}
&\|m_{n+l+1}(t) - m_{n+1}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \\
&\leq C_T\Bigg(\|(S_{n+l+1} - S_{n+1})m_0\|_{B^{-\frac{1}{2}}_{2,\infty}} + \int_0^t \|\partial_x(u_{n+l} - u_n)\partial_x R^1_{n,l}\|_{B^{-\frac{1}{2}}_{2,\infty}} \\
&\quad + \|\frac{1}{2}(u_{n+l} - u_n)R^2_{n,l}\|_{B^{-\frac{1}{2}}_{2,\infty}} + \|\frac{1}{2}(m_{n+l} - m_n)R^2_{n,l}\|_{B^{-\frac{1}{2}}_{2,\infty}} dt'\Bigg).
\end{aligned} \tag{3.10}$$

Taking advantage of Lemma 2.4, we have

$$\begin{aligned}
&\|\partial_x(u_{n+l} - u_n)\partial_x R^1_{n,l}\|_{B^{-\frac{1}{2}}_{2,\infty}} \\
&\leq \|\partial_x(u_{n+l} - u_n)\|_{B^{\frac{1}{2}}_{2,1}} \|\partial_x\left[m_{n+l} + \frac{1}{2}(u_{n+l} + u_n)\right]\|_{B^{-\frac{1}{2}}_{2,\infty}} \\
&\leq \|m_{n+l} - m_n\|_{B^{-\frac{1}{2}}_{2,1}} \|m_{n+l} + \frac{1}{2}(u_{n+l} + u_n)\|_{B^{\frac{1}{2}}_{2,\infty}} \\
&\leq C\|m_{n+l} - m_n\|_{B^{-\frac{1}{2}}_{2,1}} \left(\|m_{n+1}\|_{B^{\frac{1}{2}}_{2,1}} + \|u_{n+l}\|_{B^{\frac{1}{2}}_{2,1}} + \|u_n\|_{B^{\frac{1}{2}}_{2,1}}\right) \\
&\leq 3C\mathbf{M}\|m_{n+l} - m_n\|_{B^{-\frac{1}{2}}_{2,1}},
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\left\|\frac{1}{2}(u_{n+l} - u_n)R^2_{n,l}\right\|_{B^{-\frac{1}{2}}_{2,\infty}} \\
&\leq \|u_{n+l} - u_n\|_{B^{\frac{1}{2}}_{2,1}} \|u_{n+l} + u_n - m_{n+l} - m_n\|_{B^{-\frac{1}{2}}_{2,\infty}} \\
&\leq C\|m_{n+l} - m_n\|_{B^{-\frac{1}{2}}_{2,1}} \left(\|m_{n+l}\|_{B^{\frac{1}{2}}_{2,1}} + \|m_n\|_{B^{\frac{1}{2}}_{2,1}} + \|u_{n+l}\|_{B^{\frac{1}{2}}_{2,1}} + \|u_n\|_{B^{\frac{1}{2}}_{2,1}}\right) \\
&\leq 4C\mathbf{M}\|m_{n+l} - m_n\|_{B^{-\frac{1}{2}}_{2,1}},
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\left\|\frac{1}{2}(m_{n+l} - m_n)R^2_{n,l}\right\|_{B^{-\frac{1}{2}}_{2,\infty}} \\
&\leq \|m_{n+l} - m_n\|_{B^{-\frac{1}{2}}_{2,1}} \|u_{n+l} + u_n - m_{n+l} - m_n\|_{B^{-\frac{1}{2}}_{2,\infty}}
\end{aligned}$$

$$\le C\|m_{n+l}-m_n\|_{B^{-\frac{1}{2}}_{2,1}}\left(\|m_{n+l}\|_{B^{\frac{1}{2}}_{2,1}}+\|m_n\|_{B^{\frac{1}{2}}_{2,1}}+\|u_{n+l}\|_{B^{\frac{1}{2}}_{2,1}}+\|u_n\|_{B^{\frac{1}{2}}_{2,1}}\right)$$
$$\le 4C\mathbf{M}\|m_{n+l}-m_n\|_{B^{-\frac{1}{2}}_{2,1}}, \tag{3.13}$$

where $C>1$. Plugging (3.11)–(3.13) into (3.10) yields that

$$\|m_{n+l+1}(t)-m_{n+1}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}\le C_T\Bigg(\|(S_{n+l+1}-S_{n+1})m_0\|_{B^{-\frac{1}{2}}_{2,\infty}} + \int_0^t 11C\mathbf{M}\|m_{n+l}-m_n\|_{B^{-\frac{1}{2}}_{2,1}}dt'\Bigg). \tag{3.14}$$

Applying Lemma 2.6 to the above inequality, we have

$$\|m_{n+l+1}(t)-m_{n+1}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}\le C_T\Bigg(\|(S_{n+l+1}-S_{n+1})m_0\|_{B^{-\frac{1}{2}}_{2,\infty}} + \int_0^t 11C\mathbf{M}\|m_{n+l}(t')-m_n(t')\|_{B^{-\frac{1}{2}}_{2,\infty}}\ln\left(e+\frac{\|m_{n+l}(t')-m_n(t')\|_{B^{\frac{1}{2}}_{2,1}}}{\|m_{n+l}(t')-m_n(t')\|_{B^{-\frac{1}{2}}_{2,\infty}}}\right)dt'\Bigg). \tag{3.15}$$

Since

$$\left\|\sum_{q=n+1}^{n+l}\Delta_q m_0\right\|_{B^{-\frac{1}{2}}_{2,\infty}}\le C2^{-n}\|m_0\|_{B^{-\frac{1}{2}}_{2,\infty}},$$

and that $(m_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,T];B^s_{p,r})$, then it follows that

$$\begin{aligned}
&\|m_{n+l+1}(t)-m_{n+1}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
&\le \widetilde{C_T}\left(2^{-n}+\int_0^t\|m_{n+l}-m_n\|_{B^{-\frac{1}{2}}_{2,\infty}}\ln\left(e+\frac{C}{\|m_{n+l}-m_n\|_{B^{-\frac{1}{2}}_{2,\infty}}}\right)d\tau\right)\\
&\le \widetilde{C_T}\left(2^{-n}+\int_0^t\|m_{n+l}-m_n\|_{B^{-\frac{1}{2}}_{2,\infty}}\left(1-\ln\frac{\|m_{n+l}-m_n\|_{B^{-\frac{1}{2}}_{2,\infty}}}{C}\right)d\tau\right).
\end{aligned}$$

Noticing that the function $x(1-\ln\frac{x}{C})$ is nondecreasing in $x\in[0,C)$, we get

$$\|m_{n+l+1}(t)-m_{n+1}(t)\|_{L_t^\infty(B_{2,\infty}^{-\frac{1}{2}})} \leq \widetilde{C_T}\left(2^{-n}+\int_0^t \|m_{n+l}-m_n\|_{L_t^\infty(B_{2,\infty}^{-\frac{1}{2}})}\left(1-\ln\frac{\|m_{n+l}-m_n\|_{L_t^\infty(B_{2,\infty}^{-\frac{1}{2}})}}{C}\right)d\tau\right). \tag{3.16}$$

Let $g_n(t)=\sup_{l\in\mathbb{N}}\|m_{n+l}(t)-m_n(t)\|_{L_t^\infty(B_{2,\infty}^{-\frac{1}{2}})}$. Noticing that the function $x(1-\ln\frac{x}{C})$ is nondecreasing in $x\in[0,C)$, we get

$$g_{n+1}(t)\leq \widetilde{C_T}\left(2^{-n}+\int_0^T g_n(t)\left(1-\ln\frac{g_n(t)}{C}\right)dt\right), \tag{3.17}$$

which along with the Fatou–Lebesgue theorem leads to

$$\widetilde{g}(t)\triangleq \limsup_{n\to\infty} g_{n+1}(t)\leq C\int_0^T \widetilde{g}(t)\left(1-\ln\frac{\widetilde{g}(t)}{C}\right)dt. \tag{3.18}$$

By Lemma 2.8, we infer that $m_{n+l+1}(t)-m_{n+1}(t)=0$ as $n\to\infty$. In other words, $(m_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(0,T;B_{2,\infty}^{-\frac{1}{2}})$ and converges to some limit function $m\in L^\infty(0,T;B_{2,\infty}^{-\frac{1}{2}})$.

Step 4 We now prove the existence of solutions. We prove that m belongs to $C([0,T);B_{2,1}^{\frac{1}{2}})\cap C^1([0,T);B_{2,1}^{-\frac{1}{2}})$ and satisfies the Eq. (1.2) in the sense of distributions. Since $(m_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;B_{2,1}^{\frac{1}{2}})$, the Fatou property for the Besov spaces ensures that $m\in L^\infty(0,T;B_{2,1}^{\frac{1}{2}})$.

Taking limit in the Eq. (3.1), we conclude that m is indeed a solution of (1.2). Note that $m\in L^\infty(0,T;B_{2,1}^{\frac{1}{2}})$. Then

$$\begin{aligned}
&\left\|\frac{1}{2}(\partial_x u)^2-\frac{1}{2}(u-m)^2\right\|_{B_{2,1}^{\frac{1}{2}}}\\
&\quad\leq \frac{1}{2}\|(\partial_x u)^2\|_{B_{2,1}^{\frac{1}{2}}}+\frac{1}{2}\|(u-m)^2\|_{B_{2,1}^{\frac{1}{2}}}\\
&\quad\leq \|\partial_x u\|_{B_{2,1}^{\frac{1}{2}}}\|\partial_x u\|_{L^\infty}+\|u-m\|_{B_{2,1}^{\frac{1}{2}}}\|u-m\|_{L^\infty}\\
&\quad\leq C\|m\|^2_{B_{2,1}^{\frac{1}{2}}}.
\end{aligned} \tag{3.19}$$

This means that the right-hand side of (1.2) also belongs to $L^\infty(0,T;B_{2,1}^{\frac{1}{2}})$. Hence, according to Lemma 2.11, the function m belongs to $C([0,T);B_{2,1}^{\frac{1}{2}})$. Lemma 2.5 implies that $(4u-2u_x)m_x$ is bounded in $L^\infty(0,T;B_{2,1}^{-\frac{1}{2}})$. Again using the equation

(1.2) and high regularity of u, we see that $\partial_t u$ is in $C([0,T); B^{-\frac{1}{2}}_{2,1})$. Then, we know that $u \in C([0,T); B^{\frac{1}{2}}_{2,1}) \cap C^1([0,T); B^{-\frac{1}{2}}_{2,1})$.

Step 5 Finally, we prove the uniqueness of strong solutions to (1.2). Suppose that $M = (1-\partial_x^2)u,\ N = (1-\partial_x^2)v \in E^s_{p,r}$ are two solutions of (1.2). Set $W = M - N$. Hence, we obtain that

$$\begin{cases} \partial_t W - \partial_x u \partial_x W \\ = \partial_x (u-v)\partial_x G^1 - \frac{1}{2}(u-v)G^2 + \frac{1}{2}WG^2, \\ W|_{t=0} = M(0,x) - N(0,x) = W(0), \end{cases} \tag{3.20}$$

where

$$G^1 = N + \frac{1}{2}(u+v),$$
$$G^2 = u + v - M - N.$$

We define that $U(t) \triangleq \int_0^t \|m(t')\|_{B^{\frac{1}{2}}_{2,1}} dt'$. By Lemma 2.10 and using the fact that m is bounded in $L^\infty(0,T; B^{\frac{1}{2}}_{2,1})$, we infer that

$$\|W\|_{B^{-\frac{1}{2}}_{2,\infty}} \le Ce^{CU(t)}\Bigg(\|W(0)\|_{B^{-\frac{1}{2}}_{2,\infty}} + \int_0^t e^{-CU(t')}\Bigg(\left\|\frac{1}{2}\partial_x(u-v)\partial_x G^1\right\|_{B^{-\frac{1}{2}}_{2,\infty}} + \frac{1}{2}\|(u-v)G^2\|_{B^{-\frac{1}{2}}_{2,\infty}} + \frac{1}{2}\|WG^2\|_{B^{-\frac{1}{2}}_{2,\infty}}\Bigg)dt'\Bigg). \tag{3.21}$$

Taking advantage of Lemma 2.4, we have

$$\begin{aligned} \left\|\frac{1}{2}\partial_x(u-v)\partial_x G^1\right\|_{B^{-\frac{1}{2}}_{2,\infty}} &\le \|\partial_x(u-v)\|_{B^{\frac{1}{2}}_{2,1}} \left\|\partial_x\left(N + \frac{1}{2}(u+v)\right)\right\|_{B^{-\frac{1}{2}}_{2,\infty}} \\ &\le \|W\|_{B^{-\frac{1}{2}}_{2,1}} \left\|N + \frac{1}{2}(u+v)\right\|_{B^{\frac{1}{2}}_{2,\infty}} \\ &\le C\|W\|_{B^{-\frac{1}{2}}_{2,1}} \left(\|N\|_{B^{\frac{1}{2}}_{2,1}} + \|u\|_{B^{\frac{1}{2}}_{2,1}} + \|v\|_{B^{\frac{1}{2}}_{2,1}}\right) \\ &\le 3C\mathbf{M}\|W\|_{B^{-\frac{1}{2}}_{2,1}}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \|(u-v)G^2\|_{B^{-\frac{1}{2}}_{2,\infty}} &\le \|u-v\|_{B^{\frac{1}{2}}_{2,1}} \|u+v-M-N\|_{B^{-\frac{1}{2}}_{2,\infty}} \\ &\le C\|W\|_{B^{-\frac{1}{2}}_{2,1}} \left(\|M\|_{B^{\frac{1}{2}}_{2,1}} + \|N\|_{B^{\frac{1}{2}}_{2,1}} + \|u\|_{B^{\frac{1}{2}}_{2,1}} + \|v\|_{B^{\frac{1}{2}}_{2,1}}\right) \\ &\le 4C\mathbf{M}\|W\|_{B^{-\frac{1}{2}}_{2,1}}, \end{aligned} \tag{3.23}$$

$$\begin{aligned}\|WG^2\|_{B^{-\frac{1}{2}}_{2,\infty}} &\le \|W\|_{B^{-\frac{1}{2}}_{2,\infty}}\|u+v-M-N\|_{B^{\frac{1}{2}}_{2,1}}\\ &\le C\|W\|_{B^{-\frac{1}{2}}_{2,\infty}}\left(\|M\|_{B^{\frac{1}{2}}_{2,1}}+\|N\|_{B^{\frac{1}{2}}_{2,1}}+\|u\|_{B^{\frac{1}{2}}_{2,1}}+\|v\|_{B^{\frac{1}{2}}_{2,1}}\right)\\ &\le 4C\mathbf{M}\|W\|_{B^{-\frac{1}{2}}_{2,1}}. \end{aligned}\tag{3.24}$$

Plugging (3.22)–(3.24) into (3.21) yields that

$$\begin{aligned}&e^{-CU(t)}\|W(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}\\ &\le C\|W(0)\|_{B^{-\frac{1}{2}}_{2,\infty}}+\int_0^t 11C\mathbf{M}e^{-CU(t')}\|W(t')\|_{B^{-\frac{1}{2}}_{2,1}}dt'\\ &\le C\|W(0)\|_{B^{-\frac{1}{2}}_{2,\infty}}+\int_0^t 11C\mathbf{M}e^{-CU(t')}\|W(t')\|_{B^{-\frac{1}{2}}_{2,\infty}}\ln\left(e+\frac{\|W(t)\|_{B^{\frac{1}{2}}_{2,1}}}{\||W(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}}\right)dt'.\end{aligned}\tag{3.25}$$

Now define $\widetilde{W}(t)\triangleq e^{-CU(t)}\|W(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}$. Since the function $xln(e+\frac{C}{x})$ is nondecreasing and m is bounded in $L^\infty(0,T;B^{\frac{1}{2}}_{2,1})$, it follows that

$$\begin{aligned}\widetilde{W}(t) &\le C_1\left(\widetilde{W}(0)+\int_0^t \widetilde{W}(t')\ln\left(e+\frac{C_1}{\widetilde{W}(t')}\right)dt'\right)\\ &\le C_1\left(\widetilde{W}(0)+\int_0^t \widetilde{W}(t')\left(1-\ln\frac{\widetilde{W}(t')}{C_1}\right)dt'\right).\end{aligned}\tag{3.26}$$

By virtue of Lemma 2.8 and Remark 2.9 with $\rho=\frac{\widetilde{W}(t)}{C_1}$, we verifies that

$$\widetilde{W}(t)\le C_1\widetilde{W}(0)^{\exp\{-C_1t\}}$$

which leads to

$$\|W(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}\le C_2\|W(0)\|_{B^{-\frac{1}{2}}_{2,\infty}}\exp\{C_2t\}\le C_2\|W(0)\|_{B^{-\frac{1}{2}}_{2,1}}\exp\{C_2T\}.\tag{3.27}$$

Taking advantage of the interpolation argument ensures that

$$\|W(t)\|_{L^\infty([0,T],B^{s'}_{2,1})}\le C_3\|W(0)\|^{\theta}_{B^{\frac{1}{2}}_{2,1}}\exp\{C_2T\},\tag{3.28}$$

where $\theta=\frac{1}{2}-s'\in(0,1]$. The above inequality implies the uniqueness. This completes the proof of Theorem 3.1. □

Next, we prove the solution of (1.2) guaranteed by Theorem 3.1 depends continuously on the initial data.

Theorem 3.2 *Denote $\overline{\mathbb{N}} = \mathbb{N} \cup \infty$. Let $(m_n)_{n\in\overline{\mathbb{N}}}$ be the corresponding solution of (1.2) guaranteed by Theorem 3.1 with the initial data $m_0^n(x) \in B^{\frac{1}{2}}_{2,1}$. If m_0^n tends to m_0^∞ in $B^{\frac{1}{2}}_{2,1}$, then $m^n(t,x)$ tends to $m^\infty(t,x) \in C([0,T]; B^{\frac{1}{2}}_{2,1})$ with $2C^2T\|m_0\|_{B^{\frac{1}{2}}_{2,1}} < 1$.*

Proof By Theorem 3.1, we can find $M > 0$ such that for all $n \in \overline{\mathbb{N}}$,

$$\sup_{n\in\overline{\mathbb{N}}} \|u^n\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} \le M.$$

From (3.27), we have $\|m^n - m^\infty\|_{L^\infty(0,T;B^{-\frac{1}{2}}_{2,\infty})}$ tends to zero as $n \to \infty$.

For fixed $\varphi \in B^{-\frac{1}{2}}_{2,1}$, we write

$$\begin{aligned}\langle m^n - m^\infty, \varphi\rangle &= \langle S_j[m^n - m^\infty], \varphi\rangle + \langle (Id - S_j)[m^n - m^\infty], \varphi\rangle \\ &= \langle m^n - m^\infty, S_j\varphi\rangle + \langle m^n - m^\infty, (Id - S_j)\varphi\rangle. \end{aligned} \tag{3.29}$$

Direct computations show that

$$|\langle m^n - m^\infty, S_j\varphi\rangle| \le CM\|m^n - m^\infty\|_{L^\infty([0,T);B^{-\frac{1}{2}}_{2,\infty})} \|S_j\varphi\|_{B^{\frac{1}{2}}_{2,1}}, \tag{3.30}$$

and

$$|\langle m^n - m^\infty, (Id - S_j)\varphi\rangle| \le CM\|\varphi - S_j\varphi\|_{B^{-\frac{1}{2}}_{2,1}}. \tag{3.31}$$

Note that $\|\varphi - S_j\varphi\|_{B^{-\frac{1}{2}}_{2,1}}$ tends to zero as $j \to \infty$ and $\|m^n - m^\infty\|_{L^\infty([0,T);B^{-\frac{1}{2}}_{2,1})}$ tends to zero as $n \to \infty$. Then (3.31) may be made arbitrarily small for j large enough. For fixed j, we then let n tend to infinity so that (3.30) tends to zero. Thus, we conclude that $\langle m^n - m^\infty, \varphi\rangle$ tends to zero. Then we obtain m^n tends to m^∞ in $\mathbb{C}_w([0,T); B^{\frac{1}{2}}_{2,\infty})$. Because $B^{\frac{1}{2}}_{2,\infty} \hookrightarrow B^{-\frac{1}{2}}_{2,1}$, we obtain m^n tends to m^∞ in $\mathbb{C}([0,T); B^{-\frac{1}{2}}_{2,1})$.

Note that for all $m = u - u_{xx}, \widetilde{m} = \widetilde{u} - \widetilde{u}_{xx}, \in B^{\frac{1}{2}}_{2,1}$,

$$\|u_x - \widetilde{u}_x\|_{B^{\frac{1}{2}}_{2,1}} \le C\|m - \widetilde{m}\|_{B^{-\frac{1}{2}}_{2,1}}, \tag{3.32}$$

$$\begin{aligned}&\left\|\frac{1}{2}u_x^2 - \frac{1}{2}(u-m)^2 - \frac{1}{2}\widetilde{u}_x^2 + \frac{1}{2}(\widetilde{u} - \widetilde{m})^2\right\|_{B^{\frac{1}{2}}_{2,1}} \\ &\quad \le \frac{1}{2}\|u_x^2 - \widetilde{u}_x^2\|_{B^{\frac{1}{2}}_{2,1}} + \frac{1}{2}\|(u-m)^2 - (\widetilde{u} - \widetilde{m})^2\|_{B^{\frac{1}{2}}_{2,1}}\end{aligned}$$

$$\leq \frac{1}{2}\|(u_x-\widetilde{u}_x)(u_x+\widetilde{u}_x)\|_{B^{\frac{1}{2}}_{2,1}}+\frac{1}{2}\|(u-\widetilde{u}-m+\widetilde{m})(u+\widetilde{u}-m-\widetilde{m})\|_{B^{\frac{1}{2}}_{2,1}}$$
$$\leq CM\|m-\widetilde{m}\|_{B^{-\frac{1}{2}}_{2,1}}. \tag{3.33}$$

Then by Lemma 2.12 and (3.19), we have that u^n tends to $u \in C([0,T]; B^{\frac{1}{2}}_{2,1})$. □

4 A blow-up criterion

After establishing local well-posedness theory, a natural question is whether the corresponding solution exists globally in time or not. This section is devoted to investigating a blow-up criterion for (1.2). At first, we show a conservation law and an a priori estimate for strong solutions to (1.2).

Lemma 4.1 *Let $u_0 \in H^s$, $s > \frac{5}{2}$. Then the corresponding solution u to (1.2) has constant energy integral*

$$\int_{\mathbb{R}}(u_x^2+u_{xx}^2)dx=\int_{\mathbb{R}}\left((u_0')^2+(u_0'')^2\right)dx=\|u_0'\|^2_{H^1}.$$

Proof Arguing by density, it suffices to consider the case where $u \in C_0^\infty$. Applying integration by parts, we obtain

$$\int_{\mathbb{R}}u_xm_xdx=\int_{\mathbb{R}}u_x(u_x-u_{xxx})dx=\int_{\mathbb{R}}u_x^2dx+\int_{\mathbb{R}}u_{xx}^2dx.$$

Taking advantage of Lemmas 4.1 and 4.2, we infer that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}u_xm_xdx\\&\quad=\int_{\mathbb{R}}(\partial_tu_xm_x+\partial_tm_xu_x)dx\\&\quad=2\int_{\mathbb{R}}\partial_tm_xu_xdx=2\int_{\mathbb{R}}u_x[u_xm_{xx}+2u_{xx}m_x]dx\\&\quad=-2\int_{\mathbb{R}}u_x^2m_{xx}+2u_xu_{xx}m_xdx\\&\quad=0.\end{aligned}$$

□

Lemma 4.2 *Let $u_0 \in H^s$, $s > \frac{5}{2}$, and let T be the maximal existence time of the corresponding solution u to (1.2). Then we have*

$$\|u\|_{L^2}\leq\|u_0\|_{L^2}+2\|u_0'\|^2_{H^1}T. \tag{4.1}$$

Proof Arguing by density, it suffices to consider the case where $u \in C_0^\infty$.

Note that $G(x) = \frac{1}{2}e^{-|x|}$ and $G(x)\star f = (1-\partial_x^2)^{-1} f$ for all $f \in L^2(\mathbb{R})$ and $G\star m = u$. Then we can rewrite (1.1) as follows:

$$u_t - \frac{1}{2}u_x^2 = G * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]. \tag{4.2}$$

By (4.2), we infer that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} u^2 dx \\
&\quad = 2\int_{\mathbb{R}} \partial_t u u dx \\
&\quad = 2\int_{\mathbb{R}} u\left[\frac{1}{2}u_x^2 + G * \left(u_x^2 + \frac{1}{2}u_{xx}^2\right)\right] dx \\
&\quad \le \|u\|_{L^2}\left[\|u_x\|_{L^2}\|u_x\|_{L^\infty} + \|G\|_{L^2}(2\|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2)\right] \\
&\quad \le 4\|u\|_{L^2}\|u_0'\|_{H^1}^2.
\end{aligned}$$

Thus we have

$$\|u\|_{L^2} \le \|u_0\|_{L^2} + 2\|u_0'\|_{H^1}^2 t \le \|u_0\|_{H^2} + 2\|u_0'\|_{H^1}^2 T. \tag{4.3}$$

□

Remark 4.3 From Lemmas 4.1 and 4.2, we obtain

$$\|u\|_{H^2} \le \|u\|_{L^2} + \|u_x\|_{L^2} + \|u_{xx}\|_{L^2} \le \|u_0\|_{H^2} + 2\|u_0'\|_{H^1}^2 T. \tag{4.4}$$

Then we present a blow-up criterion for (1.2).

Lemma 4.4 *Let $u_0(x) \in H^s,\ s > \frac{5}{2}$, and let T be the maximal existence time of the solution $u(x,t)$ to (1.2) with the initial data $u_0(x)$. Then the corresponding solution blows up in finite time if and only if*

$$\liminf_{t\to T} \inf_{x\in\mathbb{R}} m = -\infty.$$

Proof Arguing by density, it suffices to consider the case where $u \in C_0^\infty$.

A direct computation yields

$$\begin{aligned}
\|m_x\|_{L^2}^2 &= \int_{\mathbb{R}} (u_x - u_{xxx})^2 dx \\
&= \int_{\mathbb{R}} (u_x^2 + 2u_{xx}^2 + u_{xxx}^2) dx.
\end{aligned}$$

Hence,

$$\|u_x\|_{H^2} \leq \|m_x\|_{L^2} \leq 2\|u_x\|_{H^2}. \tag{4.5}$$

Differentiating both sides of (1.2) with respect to x, taking L^2 inner product with m_x, and then integrating by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} m_x^2 dx \\
&\quad = \int_{\mathbb{R}} 2m_x \partial_t m_x dx \\
&\quad = 2\int_{\mathbb{R}} m_x[2(u-m)m_x + u_x m_{xx}]dx \\
&\quad = \int_{\mathbb{R}} [4(u-m)m_x^2 + 2u_x m_x m_{xx}]dx \\
&\quad = \int_{\mathbb{R}} [4(u-m)m_x^2 - u_{xx} m_x^2]dx \\
&\quad = \int_{\mathbb{R}} 3(u-m)m_x^2 dx.
\end{aligned}$$

Suppose that m is bounded from above on $[0, T)$ and $T < \infty$. By (4.5) and Lemmas 4.1–4.2, we get

$$\begin{aligned}
&\frac{d(\|m_x\|_{L^2}^2)}{dt} \\
&\quad = \int_{\mathbb{R}} [3(u-m)m_x^2]dx \\
&\quad \leq C\|m_x\|_{L^2}^2 + 3\|u\|_{L^\infty}\|m_x\|_{L^2}^2 \\
&\quad \leq C_1\|m_x\|_{L^2}^2,
\end{aligned} \tag{4.6}$$

here $C_1 > 0$. An application of Gronwall's inequality yields

$$\|m\|_{H^1} \leq e^{C_1 t}\|m_0\|_{H^1}, \quad \forall\, t \in [0, T). \tag{4.7}$$

So in view of (4.5) and (4.7), we obtain that if m is bounded from above on $[0, T)$, then so does the H^2-norm of u_x, which contradicts the assumption that $T < \infty$ is the maximal existence time. This completes the proof. □

5 Global existence

In this section, we present a global existence result for the Cauchy problem (1.2).

Theorem 5.1 *Assume that $u_0 \in H^4$ is such that the associated potential $m_0' = u_0' - u_0'''$ satisfies $m_0'(x_0) > 0$, $m_0'(x) \geq 0$ on $(-\infty, x_0)$ and $m_0'(x) \leq 0$ on $(x_0, +\infty)$ for some point $x_0 \in \mathbb{R}$. Then the corresponding solution to (1.2) exists globally in time.*

Proof Applying Theorem 3.1 and a simple density argument, it suffices to consider $u_0 \in H^4$ to prove the above theorem. Given $u_0 \in H^4$, T^* is the maximal existence time of the corresponding solution to (1.2) with the initial data u_0.

We consider the following initial value problem

$$\begin{cases} \frac{dq(t,x)}{dt} = -u_x(t, q(t,x)), & t \in [0, T^*), \quad x \in \mathbb{R}, \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{5.1}$$

By applying classical results in the theory of ordinary differential equations, we infer that (5.1) has a unique solution $q \in C^1([0, T) \times \mathbb{R}; \mathbb{R})$. Moreover, the map $q(\cdot, t)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(t,x) = \exp\left(\int_0^t -u_{xx}(\tau, q(\tau, x))d\tau\right) > 0, \quad \forall\, (t,x) \in [0, T) \times \mathbb{R}.$$

Hence, from (1.2), the following identity can be proved:

$$m_x(t, q(t,x))q_x(t,x) = m_0'(x)e^{\int_0^t u_{xx}dt'}, \quad \forall\, (t,x) \in [0, T) \times \mathbb{R}. \tag{5.2}$$

In fact, a direct computation yields

$$\begin{aligned} &\frac{d}{dt}\{m_x(t, q(t,x))q_x(t,x)\} \\ &\quad = m_{tx}q_x + m_{xx}q_t q_x - m_x q_{xt} \\ &\quad = q_x(m_{tx} - u_x m_{xx} - u_{xx}m_x) \\ &\quad = q_x u_{xx} m_x. \end{aligned} \tag{5.3}$$

Applying Gronwall's inequality, we obtain (5.2).

Since $q(t,x)$ is an increasing diffeomorphism of $\mathbb{R}$ as long as $t \in [0, T)$, we deduce

$$\begin{cases} m'(t,x) \geq 0, \ if \quad x \leq q(t, x_0), \\ m'(t,x) \leq 0, \ if \quad x \geq q(t, x_0). \end{cases} \tag{5.4}$$

Using the fact that the flow $q(t,x)$ is a differmorphism and $m_0(x_0) > 0$, and by (5.4) we see that x_0 is the maximum value point. (5.4) tells us that $m(t)$ will increase monotonously at the interval $(-\infty, q(t, x_0))$ along the flow and decrease monotonously at the interval $(q(t, x_0), +\infty)$ along the flow. Moreover, since $m(t, q(t, x_0))$ belongs to H^s, $m(t, q(t, x_0))$ will not be less than zero. Otherwise, it will contradict with the decay of infinity in H^s. As a result, Theorem 4.4 ensure that the solution $u(t,x)$ exists globally in time. This completes the proof of the theorem. □

Applying D_x to both sides of the Eq. (4.2) yield

$$(u_x)_t - u_x u_{xx} = \partial_x G * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right], \tag{5.5}$$

which indicates that u_x satisfies the CH Eq. (1.4). Thus solution to (4.2) or (5.5) are actually the velocity potentials of the solution to the CH equation. Then by comparing the global existence of these two equations, we obtain the following Remark.

Remark 5.2 The Eq. (1.2) and the CH equation are truly different for the global existence. As for the CH equation, the global strong solutions exist globally in time under some conditions that m_0 don't change the sign and that m_0 satisfies $m_0 \leq 0$ on $(-\infty, x_0)$ and $m_0 \geq 0$ on $(x_0, +\infty,)$ for some point $x_0 \in \mathbb{R}$ [9]. But as for the equation (4.2), there only exist zero solutions under some conditions that m_0' don't change the sign. The corresponding solutions to (1.2) exist globally with some certain conditions that m_0 satisfies $m_0(x_0) > 0$, $m_0' \geq 0$ on $(-\infty, x_0)$ and $m_0' \leq 0$ on $(x_0, +\infty,)$ for some point $x_0 \in \mathbb{R}$.

6 Blow-up phenomena

In this section we prove that there are some initial data for which the corresponding solutions to (1.2) with some certain conditions will blow up in finite time.

Theorem 6.1 *Assume that $u_0 \in H^s$, $s > \frac{5}{2}$. And m_0' satisfy $m_0' \leq 0$ on $(-\infty, x_0)$ and $m_0' \geq 0$ on $(x_0, +\infty,)$ for some point $x_0 \in \mathbb{R}$, then the corresponding solution of (1.2) blows up in finite time.*

Proof Applying Theorem 3.1 and a simple density argument, it suffices to consider $u_0 \in H^4$ to prove the above theorem. Given $u_0 \in H^4$, T^* is the maximal existence time of the corresponding solution to (1.2) with the initial data u_0.

Since $q(t, x)$ is an increasing diffeomorphism of $\mathbb{R}$ as long as $t \in [0, T)$, we deduce

$$\begin{cases} m'(t, x) \leq 0, & if \quad x \leq q(t, x_0), \\ m'(t, x) \geq 0, & if \quad x \geq q(t, x_0). \end{cases} \tag{6.1}$$

Because of $u_x = G \star m_x$ where $G > 0$, we can write $u_x(t, x)$ and $u_{xx}(t, x)$ as

$$u_x(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^{\xi} m_\xi(t, \xi) d\xi + \frac{e^x}{2} \int_x^{\infty} e^{-\xi} m_\xi(t, \xi) d\xi, \tag{6.2}$$

$$u_{xx}(t, x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^{\xi} m_\xi(t, \xi) d\xi + \frac{e^x}{2} \int_x^{\infty} e^{-\xi} m_\xi(t, \xi) d\xi. \tag{6.3}$$

Consequently

$$(u_x + u_{xx})(t, x) = e^x \int_x^{\infty} e^{-\xi} m_\xi(t, \xi) d\xi,$$

$$(u_x - u_{xx})(t, x) = e^{-x} \int_{-\infty}^{x} e^{-\xi} m_\xi(t, \xi) d\xi,$$

for all $t \geq 0$.

Differentiating (4.2) with respect to x by two times, we find

$$u_{txx} - u_x u_{xxx} = \frac{1}{2}u_{xx}^2 - u_x^2 + G * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]. \tag{6.4}$$

Combining the inequality $G * [u_x^2 + \frac{1}{2}u_{xx}^2] \geq \frac{1}{2}u_x^2$ with (6.4), we deduce that

$$u_{txx} - u_x u_{xxx} \geq \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_x^2. \tag{6.5}$$

Defining now $w(t) := u_{xx}(t, q(t, x_0))$, $\frac{dq(t,x)}{dt} = -u_x(t, q(t, x))$, we obtain from the above inequality the relation

$$w_t(t, q(t, x_0)) \geq \frac{1}{2}u_{xx}^2(t, q(t, x_0)) - \frac{1}{2}u_x^2(t, q(t, x_0)). \tag{6.6}$$

Letting

$$V(t) = (u_x - u_{xx})(t, q(t, x_0)) = e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\xi} m_{\xi}(t, \xi) d\xi < 0, \tag{6.7}$$

$$K(t) = (u_x + u_{xx})(t, q(t, x_0)) = e^{q(t,x_0)} \int_{q(t,x_0)}^{\infty} e^{-\xi} m_{\xi}(t, \xi) d\xi > 0. \tag{6.8}$$

Differentiating (6.7) with respect to x, we obtain

$$\frac{d}{dt}V(t) = e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\xi} m_{t\xi}(t, \xi) d\xi - \frac{d}{dt}q(t, x_0)V(t). \tag{6.9}$$

Direct computations show that

$$\begin{aligned}
&\int_{-\infty}^{q(t,x_0)} e^{\xi} m_{t\xi}(t, \xi) d\xi \qquad (6.10)\\
&= \int_{-\infty}^{q(t,x_0)} e^{\xi} (u_{\xi} m_{\xi\xi} + 2u_{\xi\xi} m_{\xi}) d\xi\\
&= \int_{-\infty}^{q(t,x_0)} e^{\xi} (\partial_{\xi}(u_{\xi} m_{\xi}) + u_{\xi\xi} m_{\xi}) d\xi\\
&= \int_{-\infty}^{q(t,x_0)} e^{\xi} (-u_{\xi} m_{\xi} + u_{\xi\xi} m_{\xi}) d\xi\\
&= \int_{-\infty}^{q(t,x_0)} \partial_{\xi} \left[e^{\xi} \left(-\frac{1}{2}u_{\xi\xi}^2 + u_{\xi} u_{\xi\xi}\right)\right] d\xi - \int_{-\infty}^{q(t,x_0)} \left[e^{\xi} \left(\frac{1}{2}u_{\xi\xi}^2 + u_{\xi}^2\right)\right] d\xi\\
&= e^{q(t,x_0)} \left[-\frac{1}{2}u_{xx}^2(t, q(t, x_0)) + u_x u_{xx}(t, q(t, x_0))\right] - \int_{-\infty}^{q(t,x_0)} \left[e^{\xi} \left(\frac{1}{2}u_{\xi\xi}^2 + u_{\xi}^2\right)\right] d\xi,
\end{aligned}$$

which follow with (6.9) and $V(t) = (u_x - u_{xx})(t, q(t, x_0))$ yield

$$\begin{aligned}\frac{d}{dt}V(t) =& -\frac{1}{2}u_{xx}^2(t, q(t, x_0)) + u_x u_{xx})(t, q(t, x_0))\\ &- \int_{-\infty}^{x}\left[e^{\xi}\left(\frac{1}{2}u_{\xi\xi}^2 + u_{\xi}^2\right)d\xi + u_x\right)(t, q(t, x_0))V(t)\\ \leq& -\frac{1}{2}u_{xx}^2(t, q(t, x_0)) + \frac{1}{2}u_x^2(t, q(t, x_0)) < 0. \qquad (6.11)\end{aligned}$$

In an analogous way,we obtain

$$\begin{aligned}\frac{d}{dt}K(t) =& \frac{1}{2}u_{xx}^2(t, q(t, x_0)) + u_x u_{xx})(t, q(t, x_0))\\ &+ \int_{-\infty}^{x}\left[e^{\xi}\left(\frac{1}{2}u_{\xi\xi}^2 + u_{\xi}^2\right)d\xi - u_x\right)(t, q(t, x_0))K(t)\\ \geq& \frac{1}{2}u_{xx}^2(t, q(t, x_0)) - \frac{1}{2}u_x^2(t, q(t, x_0)) > 0. \qquad (6.12)\end{aligned}$$

In view of (6.6)–(6.9) and (6.11)–(6.12), we see that

$$\begin{aligned}w_t(t, q(t, x_0)) \geq& \frac{1}{2}u_{xx}^2(t, q(t, x_0)) - \frac{1}{2}u_x^2(t, q(t, x_0)) \geq -\frac{1}{2}V(t)K(t)\\ \geq& -\frac{1}{2}V(0)K(0) \geq 0. \qquad (6.13)\end{aligned}$$

Assume that the solution exists globally in time. We now show that this leads to a contradiction.

From (6.13), by intergration, we see that

$$w(t) \geq w(0) - \frac{1}{2}V(0)K(0)t. \qquad (6.14)$$

Since $-V(0)W(0) > 0$, and the $H^1(\mathbb{R}) - norm$ of u_x is conservation law, there exists certainly some $t_0 > 0$ such that

$$w^2(t) \geq 2\|u_x\|_{L^\infty(\mathbb{R})}, \qquad t > t_0. \qquad (6.15)$$

Combining (6.6) with (6.15) yields

$$\frac{d}{dt}w(t) \geq \frac{1}{4}w^2(t), \qquad t > t_0. \qquad (6.16)$$

By (6.7), $w(0) > 0$, by (6.16), $w(t) > 0,\ for\ \ t \geq 0$. Then solving the inequality (6.16), we get

$$\frac{1}{w(t)} - \frac{1}{w(t_0)} + \frac{1}{4}(t - t_0) \leq 0. \qquad (6.17)$$

Taking into account that $\frac{1}{w(t)} > 0$ and $\frac{1}{4}(t - t_0) \to \infty$ as $t \to \infty$, we get a contradiction. This proves that the wave $u(t, x)$ breaks in finite time. □

Then we present another blow-up result.

Theorem 6.2 *Assume that* $\varepsilon > 0$, $u_0 \in H^s$, $s > \frac{5}{2}$ *and* $m_0(x_0) < -(1+\varepsilon)[(8\|u_0'\|_{H^1}^2 \ln(1+\frac{2}{\varepsilon}) + \|u_0\|_{H^2}^2)^{\frac{1}{2}} + \|u_0\|_{H^2}]$, *then the corresponding solution of* (1.2) *blows up in finite time.*

Proof By a standard density argument, here we may assume $s = 3$ to prove the theorem.

Given $u_0 \in H^3$, let T be the maximal existence time of the corresponding solution to (1.1) with the initial data $u_0 \in H^3$.

From (1.2), we obtain,

$$\begin{aligned} m_t - u_x m_x &= -\frac{1}{2}m^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u^2 + um \\ &\le -\frac{1}{2}m^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u^2 + \left(\frac{1}{4}m^2 + u^2\right) \\ &\le -\frac{1}{4}m^2 + \frac{1}{2}u_x^2 + \frac{1}{2}u^2 \\ &\le -\frac{1}{4}m^2 + \|u\|_{H^2}^2 \\ &\le -\frac{1}{4}m^2 + (\|u_0\|_{H^2} + 2\|u_0'\|_{H^1}^2 t)^2. \end{aligned} \tag{6.18}$$

Set now $w(t) := \inf_{x\in\mathbb{R}}[\frac{1}{4}m(t, q(t, x))]$, $\frac{dq(t,x)}{dt} = -\frac{1}{4}u_x(t, q(t, x))$, fix $\varepsilon > 0$. and take

$$T_1 = \frac{(8\|u_0'\|_{H^1}^2 \ln(1+\frac{2}{\varepsilon}) + \|u_0\|_{H^2}^2)^{\frac{1}{2}} - \|u_0\|_{H^2}}{4\|u_0'\|_{H^1}^2}, \tag{6.19}$$

$$K(T_1) = \frac{1}{2}\|u_0\|_{H^2} + \|u_0'\|_{H^1}^2 T_1, \tag{6.20}$$

which satisfying

$$2K(T_1)T_1 - \ln\left(1 + \frac{2}{\varepsilon}\right) \ge 0,$$

Then we obtain from the above inequality the relation

$$\frac{dw}{dt} \le -w^2 + K^2(T_1), \quad \forall t \in [0, T] \cap [0, T_1]. \tag{6.21}$$

In view of $w(0) < -(1+\varepsilon)K_T$, we obtain

$$w(t) < -(1+\varepsilon)K(T_1), \quad \forall t \in [0, T] \cap [0, T_1].$$

By solving the inequality (6.21), we get

$$0 \geq \frac{2K(T_1)}{w(t) - K(T_1)} \geq -1 + \frac{w(0) + K(T_1)}{w(0) - K(T_1)} e^{2tK(T_1)}. \tag{6.22}$$

Since

$$0 < \frac{w(0) - K(T_1)}{w(0) + K(T_1)} = 1 - \frac{2K(T_1)}{w(0) + K(T_1)} \leq 1 + \frac{2}{\varepsilon}. \tag{6.23}$$

then it follows that there exists T^*

$$0 < T^* \leq \frac{1}{2K(T_1)} \ln \frac{w(0) - K(T_1)}{w(0) + K(T_1)} \leq \frac{1}{2K(T_1)} \ln \left(1 + \frac{2}{\varepsilon}\right) \leq T_1, \tag{6.24}$$

such that

$$w(t) \leq -K(T_1) + \frac{2K(T_1)}{1 - \frac{w(0) - K(T_1)}{w(0) + K(T_1)} e^{-2K(T_1)t}} \to -\infty, \tag{6.25}$$

as $t \to T^\star$. This proves that the wave $u(t, x)$ breaks in finite time. □

Remark 6.3 The Eq. (1.2) and the CH equation are truly different for the blow-up phenomena. The solution to the CH equation with the particular condition will blow up in finite time. Similarly, the solution to (1.2) will blow up in finite time with the corresponding condition. However, the corresponding solution of (1.2) with $m_0(x_0) < -(1+\varepsilon)[(8\|u_0'\|_{H^1}^2 \ln(1 + \frac{2}{\varepsilon}) + \|u_0\|_{H^2}^2)^{\frac{1}{2}} + \|u_0\|_{H^2}]$ blows up in finite time. But we can't deduce the blow up phenomena to the CH equation with the corresponding condition.

Finally, we prove the exact blow-up rate for blowing-up solutions $m(t, x)$ to (1.2) guaranteed by Theorem 6.2. In order to establish this result, we need the following useful lemma.

Lemma 6.4 [13] *Let $T > 0$ and $u \in C^1([0, T); H^2)$. Then for every $t \in [0, T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$a(t) \triangleq \sup_{x \in \mathbb{R}} (v_x(t, x)) = v_x(t, \xi(t)).$$

The function $a(t)$ is absolutely continuous on $(0, T)$ with

$$\frac{da}{dt} = v_{tx}(t, \xi(t)) \quad a.e.\ on \quad (0, T).$$

Theorem 6.5 *Let $u_0 \in H^s$, $s > \frac{5}{2}$, $m_0(x_0) < -(\|u_0\|_{H^2} + \|u_0\|_{H^2}^2 T)$ and T be the blow-up time of the corresponding solution u to (1.2). Then*

$$\lim_{t \to T} \left(\sup_{x \in \mathbb{R}} [m(t, x)](T - t) \right) = 2. \tag{6.26}$$

Proof As mentioned earlier, we only need to prove the theorem for $s = 3$.

Note that $G(x) = \frac{1}{2}e^{-|x|}$ and $G(x)\star f = (1-\partial_x^2)^{-1}f$ for all $f \in L^2(R)$ and $G\star m = u$. Then we can rewrite (1.2) as follows:

$$u_t = \frac{1}{2}u_x^2 + G\star\left[u_x^2 + \frac{1}{2}u_{xx}^2\right]. \tag{6.27}$$

Defining now $\frac{dq(t,x)}{dt} = -u_x(t, q(t,x))$. In view of (1.2) and (6.27), we obtain

$$\frac{d(m-u)(t,q(t,x))}{dt} + \frac{1}{2}(m-u)^2 = -u_x^2 - G(x) * \left(u_x^2 + \frac{1}{2}u_{xx}^2\right). \tag{6.28}$$

Thanks to (6.28), Lemmas 4.1, we have

$$\begin{aligned}\left|\frac{d(m-u)(t,q(t,x))}{dt} + \frac{1}{2}(m-u)^2\right| &= u_x^2 + G(x) * \left(u_x^2 + \frac{1}{2}u_{xx}^2\right),\\ &\leq 2\|u_x\|_{H^1}^2\\ &\leq 2\|u_0'\|_{H^1}^2.\end{aligned}$$

Defining now $w(t) := \inf_{x\in\mathbb{R}}[\frac{1}{2}(m-u)(t,q(t,x))]$, we obtain from the above inequality the relation

$$\left|\frac{dw}{dt} + w^2\right| \leq 2\|u_0'\|_{H^1}^2, \quad \forall t \in (0, T). \tag{6.29}$$

For every $\varepsilon \in (0, \frac{1}{2})$, in view of (6.25), we can find a $t_0 \in (0, T)$ such that

$$w(t_0) < -\sqrt{2\|u_0'\|_{H^1}^2 + \frac{2\|u_0'\|_{H^1}^2}{\varepsilon}} < -\sqrt{2}\|u_0'\|_{H^1}.$$

Thanks to (6.25) and (6.29), we have $w(t) < -\widetilde{\mathbf{C}_\mathbf{T}}$. This implies that $w(t)$ is decreasing on $[t_0, T)$, hence,

$$w(t) < -\sqrt{2\|u_0'\|_{H^1}^2 + \frac{2\|u_0'\|_{H^1}^2}{\varepsilon}} < -\sqrt{\frac{2\|u_0'\|_{H^1}^2}{\varepsilon}}, \quad \forall t \in [t_0, T).$$

Noticing that $-w^2 - 2\|u_0'\|_{H^1}^2 \leq \frac{dw(t)}{dt} \leq -w^2 + 2\|u_0'\|_{H^1}^2, \quad a.e.\ t \in (t_0, T)$, we get

$$-1-\varepsilon \leq \frac{d}{dt}\left(-\frac{1}{w(t)}\right) \leq -1+\varepsilon, \quad a.e.\ t \in (t_0, T). \tag{6.30}$$

Integrating (6.30) with respect to $t \in [t_0, T)$ on (t, T) and applying $\lim_{t\to T} w(t) = -\infty$ again, we deduce that

$$(-1-\varepsilon)(T-t) \le \frac{1}{w(t)} \le (-1+\varepsilon)(T-t). \tag{6.31}$$

Since $\varepsilon \in (0, \frac{1}{2})$ is arbitrary, it then follows from (6.31) that (6.26) holds. Noting that Lemma 4.2, we get $\lim_{t\to T}(\inf_{x\in\mathbb{R}} u(t,x)(T-t)) = 0$.

This completes the proof of the theorem. □

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