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A note on commutator subgroups in groups of large cardinality

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Abstract
If G is an uncountable group of regular cardinality $\aleph$, we shall denote by $L_\aleph(G)$ the set of all subgroups of G of cardinality $\aleph$. The aim of this paper is to describe the behaviour of groups G for which the set $\mathcal{C}_\aleph(G) = \{X' \mid X \in L_\aleph(G)\}$ is finite, at least when G is locally graded and has no simple sections of cardinality $\aleph$. Among other results, it is proved that such a group has a finite commutator subgroup, provided that it contains an abelian subgroup of cardinality $\aleph$.

Keywords Uncountable group · Commutator subgroup · Normality

Mathematics Subject Classification 20F14

1 Introduction

The behaviour of finite-by-abelian groups (i.e. groups with a finite commutator subgroup) has been investigated by several authors. In particular, Neumann [12] proved that a group is finite-by-abelian if and only if it has boundedly finite conjugacy classes, and he also showed that groups with a finite commutator subgroup can be characterized as those groups in which every subgroup has finite index in its normal closure

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(see [13]). Moreover, it was shown by Phlip Hall [10] that if the commutator subgroup G' of a group G is finite, then the second centre $\zeta_2(G)$ has finite index. A further interesting result on this topic states that any countable finite-by-abelian group can be generated by finitely many abelian subgroups (see [1]). A sufficient condition for a group to have a finite commutator subgroup was found by the second author and Robinson [7], who proved that a locally graded group G is finite-by-abelian if and only if the set

$$\{X' \mid X \leq G\}$$

is finite. Recall here that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. The requirement that the group is locally graded cannot be removed from the above statement, since Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) have precisely two commutator subgroups. On the other hand, the class of locally graded groups is very large, and contains in particular all locally (soluble-by-finite) groups. In [7], the authors studied also groups with finitely many commutator subgroups of infinite subgroups, and proved that locally graded groups with this property either have a finite commutator subgroup or are Černikov groups (and in particular they are countable in the latter case).

In a series of recent papers it has been shown that in many cases the structure of an uncountable group is strongly influenced by the behaviour of its uncountable subgroups (see for instance [2–5,8,9]). The results in these papers suggest that the behaviour of *small* subgroups in a *large* group can often be ignored. In particular, uncountable groups of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal have been described in [3,4]. Groups in which all large subgroups have the same commutator subgroup naturally occur in this description, and they are studied in [4]. These groups may have an infinite commutator subgroup, even in the nilpotent case. In fact, Ehrenfeucht and Faber [6] constructed a nilpotent group G of class 2 and cardinality $\aleph_1$ whose commutator subgroup is isomorphic to the additive group of rational numbers, but $X' = G'$ for each uncountable subgroup X.

If G is a group, and $\aleph$ is any cardinal number, we shall denote by $L_\aleph(G)$ the set of all subgroups of G of cardinality $\aleph$, by $\mathcal{C}_\aleph(G)$ the set of all commutator subgroups of elements of $L_\aleph(G)$ and by $I_\aleph(G)$ the intersection of all members of the set $\mathcal{C}_\aleph(G)$. The aim of this paper is to describe the behaviour of uncountable groups G of regular cardinality $\aleph$ for which the set $\mathcal{C}_\aleph(G)$ is finite. The main obstacle here is the existence of the so-called *Jónsson groups*, i.e. uncountable groups which have cardinality strictly larger than all their proper subgroups. Relevant examples of Jónsson groups of cardinality $\aleph_1$ have been constructed by Shelah [16] and Obraztsov [14]. It is known that if G is any Jónsson group of cardinality $\aleph$, then $G/Z(G)$ is a simple group of cardinality $\aleph$ (see for instance [8], Corollary 2.6). Thus, in order to avoid Jónsson groups and other similar pathologies, we will restrict ourselves to the case of groups which have no large simple sections.

Our main result is the following statement, which characterizes the groups under consideration in terms of the characteristic subgroup $I_\aleph(G)$.

Theorem A *Let G be an uncountable locally graded group of regular cardinality $\aleph$ which has no simple sections of cardinality $\aleph$. Then the set $\mathcal{C}_\aleph(G)$ is finite if and only if the subgroup $I_\aleph(G)$ has finite index in G'. Moreover, in this case either $I_\aleph(G)$ is an abelian group of prime exponent or it is divisible abelian and $I_\aleph(G) = M'$ for a suitable subgroup M of G of cardinality $\aleph$.*

The example of Ehrenfeucht and Faber shows that under the hypotheses of Theorem A the commutator subgroup G' of G may be infinite. On the other hand, if the group G contains an abelian subgroup of cardinality $\aleph$, then $I_\aleph(G) = \{1\}$ and so the above statement admits the following special case.

Corollary B *Let G be an uncountable locally graded group of regular cardinality $\aleph$ such that the set $\mathcal{C}_\aleph(G)$ is finite, and suppose that G has no simple sections of cardinality $\aleph$. If G contains an abelian subgroup of cardinality $\aleph$, then G' is finite.*

Most of our notation is standard and can be found in [15].

2 Statements and proofs

We consider first the case of groups in which all large non-abelian subgroups have the same commutator subgroup. For our purposes, we will need the following two results that have been proved in [3].

Lemma 1 *Let $\aleph$ be an uncountable regular cardinal number, and let R be a principal ideal domain of cardinality strictly smaller than $\aleph$. If M is an R-module of cardinality $\aleph$, then M contains an R-submodule which is the direct sum of a collection of cardinality $\aleph$ of non-trivial R-submodules.*

Lemma 2 *Let G be an uncountable group of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal. If G has no Jónsson normal subgroups of cardinality $\aleph$, then G' has cardinality strictly smaller than $\aleph$. Moreover, if G contains an abelian subgroup of cardinality $\aleph$, then it is a Dedekind group.*

Lemma 3 *Let G be an uncountable locally graded group of regular cardinality $\aleph$ which has no simple sections of cardinality $\aleph$. If $X' = G'$ for each non-abelian subgroup X of G of cardinality $\aleph$, then G' has cardinality strictly smaller than $\aleph$. Moreover, if G' is infinite, then G is nilpotent of class 2 and it has no abelian subgroups of cardinality $\aleph$.*

Proof It can obviously be assumed that G' is infinite, so that in particular G cannot be metahamiltonian. Of course, each non-abelian subgroup of G of cardinality $\aleph$ is normal, and hence even all subgroups of cardinality $\aleph$ must be normal in G (see [3], Theorem 4.3). Thus Lemma 2 shows that G has no abelian subgroups of cardinality $\aleph$ and G' has cardinality strictly smaller than $\aleph$. Thus $|G : C_G(g)| < \aleph$ for each element g of G, and so each centralizer $C_G(g)$ has cardinality $\aleph$. Therefore

$$G' = C_G(g)' \leq C_G(g)$$

for all g, so that $G' \leq Z(G)$ and G is nilpotent of class 2. □

Lemma 4 *Let G be an uncountable locally graded group of regular cardinality $\aleph$ such that the set $\mathcal{C}_{\aleph}(G)$ is finite. If G contains an abelian normal subgroup A of cardinality $\aleph$, then the commutator subgroup G' of G is finite.*

Proof Let g be any element of G. It follows from Lemma 1 that A contains two $\langle g\rangle$-invariant subgroups B_1 and B_2 of cardinality $\aleph$ such that $B_1 \cap B_2 = \{1\}$. For each $i = 1, 2$, the factor group $A\langle g\rangle/B_i$ is locally graded (see [11]) and has only finitely many derived subgroups, so that its commutator subgroup is finite (see [7]). Thus also the commutator subgroup $(A\langle g\rangle)'$ of $A\langle g\rangle$ is finite. Moreover, the set

$$\{(A\langle g\rangle)' \mid g \in G\}$$

is finite, because it is contained in $\mathcal{C}_{\aleph}(G)$, and each subgroup $(A\langle g\rangle)'$ obviously has only finitely many conjugates in G. Therefore

$$[A, G] = \langle [A, g] \mid g \in G\rangle = \langle (A\langle g\rangle)' \mid g \in G\rangle$$

is a finite normal subgroup of G. Put $\overline{G} = G/[A, G]$. Then $\overline{A}$ is contained in the centre of $\overline{G}$, and so $Z(\overline{G})$ has cardinality $\aleph$. If $\overline{X}$ is any subgroup of $\overline{G}$, the product $\overline{X}Z(\overline{G})$ is a subgroup of cardinality $\aleph$ and $\left(\overline{X}Z(\overline{G})\right)' = \overline{X}'$. It follows that the set

$$\{\overline{X}' \mid \overline{X} \leq \overline{G}\}$$

is finite, so that the locally graded group $\overline{G}$ has a finite commutator subgroup (see [7]), and hence G' itself is finite. □

As we mentioned in the introduction, it was proved by Neumann that a group is finite-by-abelian if and only if each of its subgroups has finite index in its normal closure. Actually, in some cases it is enough to impose this condition only on large subgroups (see [4]).

Lemma 5 *Let G be an uncountable group of regular cardinality $\aleph$ such that every subgroup of G of cardinality $\aleph$ has finite index in its normal closure. If G contains an abelian subgroup of cardinality $\aleph$, then the commutator subgroup G' of G is finite.*

Let G be a group, and let $\aleph$ be any cardinal number. As the set $\mathcal{C}_{\aleph}(G)$ is obviously fixed by all automorphisms of G, the intersection

$$D_{\aleph}(G) = \bigcap_{X \in L_{\aleph}(G)} \left(N_G(X')_G\right)$$

is a characteristic subgroup of G. Moreover, if $\mathcal{C}_{\aleph}(G)$ is finite, the subgroup X' has only finitely many conjugates in G for each $X \in L_{\aleph}(G)$, so that its normalizer has finite index in G and hence also the index $|G : D_{\aleph}(G)|$ is finite.

We are now in a position to prove the following statement, which is the crucial step in the proof of Theorem A, and contains Corollary B.

Lemma 6 *Let G be an uncountable locally graded group of regular cardinality $\aleph$ such that the set $\mathcal{C}_\aleph(G)$ is finite, and suppose that G has no simple sections of cardinality $\aleph$. Then G is soluble-by-finite and its commutator subgroup G' has cardinality strictly smaller than $\aleph$. Moreover, if G contains an abelian subgroup of cardinality $\aleph$, then G' is finite.*

Proof Obviously, it can be assumed that G is not abelian. Let m be the number of non-trivial elements in the finite set $\mathcal{C}_\aleph(G)$, and suppose first that $m = 1$. Then $X' = G'$ for each non-abelian subgroup X of G of cardinality $\aleph$, so that it follows from Lemma 3 that G' has cardinality strictly smaller than $\aleph$, and it is even finite, provided that G contains an abelian subgroup of cardinality $\aleph$.

Assume now that $m > 1$. As the characteristic subgroup $D = D_\aleph(G)$ has finite index in G, it has cardinality $\aleph$, and so D' belongs to $\mathcal{C}_\aleph(G)$. If D is abelian, the commutator subgroup G' of G is finite by Lemma 4 and the statement holds. Suppose that D is not abelian, and let M be a non-abelian subgroup of cardinality $\aleph$ of D such that M' is minimal among the non-trivial elements of $\mathcal{C}_\aleph(D)$. Then $Y' = M'$ for every non-abelian subgroup Y of M of cardinality $\aleph$, and so it follows from Lemma 3 that either M' is finite or M is nilpotent and M' has cardinality strictly smaller than $\aleph$. As D is contained in the normalizer $N_G(M')$, the subgroup M' is normal in D, and in any case we obtain that D/M' is a locally graded group of cardinality $\aleph$. Moreover, the set $\mathcal{C}_\aleph(D/M')$ has order strictly smaller than m, and so it can be assumed by induction that D/M' is soluble-by-finite and D'/M' has cardinality strictly smaller than $\aleph$. Thus D is soluble-by-finite and D' has cardinality strictly smaller than $\aleph$, so that G itself is soluble-by-finite and D/D' is an abelian normal subgroup of G/D' of cardinality $\aleph$. As the set $\mathcal{C}_\aleph(G/D')$ is obviously finite, it follows from Lemma 4 that G'/D' is finite, and hence G' has cardinality strictly smaller than $\aleph$.

Suppose now that G contains an abelian subgroup A of cardinality $\aleph$, and let

$$E = \langle x_1, \ldots, x_k \rangle$$

be any finitely generated subgroup of G. For each $i = 1, \ldots, k$ the conjugacy class of x_i in G has cardinality strictly smaller than $\aleph$, so that $|G : C_G(x_i)| < \aleph$, and in particular $|A : C_A(x_i)| < \aleph$. It follows that also the index of the centralizer

$$C_A(E) = \bigcap_{i=1}^{k} C_A(x_i)$$

in A is strictly smaller than $\aleph$, so that $C_A(E)$ has cardinality $\aleph$ and hence by Lemma 1 it contains two subgroups B_1 and B_2 of cardinality $\aleph$ such that $B_1 \cap B_2 = \{1\}$. Clearly, the groups EA/B_1 and EA/B_2 have finitely many derived subgroups, so that they have a finite commutator subgroup, and it follows that also E' is finite. Therefore the commutator subgroup G' of G is locally finite and every finitely generated subgroup of G satisfies the maximal condition on subgroups.

Since G'/D' is finite, it is enough to show that D' is finite, whence the replacement of G by D allows to suppose that X' is normal in G for each element X of $L(G)$. If X is any non-abelian subgroup of G of cardinality $\aleph$, we have that X/X' is an abelian group of cardinality $\aleph$, and the set $\mathcal{C}_\aleph(G/X')$ has less than m non-trivial elements, so that by induction on m it can be assumed that G'/X' is finite. Thus each non-abelian subgroup of G of cardinality $\aleph$ has finite index in its normal closure. Let u and v be elements of G such that $uv \neq vu$, and put $U = \langle u, v\rangle$. As above, the index $|A : C_A(U)|$ is strictly smaller than $\aleph$, so that the centralizer $C_A(U)$ has cardinality $\aleph$ and Lemma 1 shows that it contains two subgroups C_1 and C_2 of cardinality $\aleph$ such that

$$C_1 \cap C_2 = \langle C_1, C_2\rangle \cap U = \{1\}.$$

Then $U = UC_1 \cap UC_2$, and hence U has finite index in its normal closure U^G. Moreover, each subgroup of G/U^G of cardinality $\aleph$ has finite index in its normal closure, and the group G/U^G contains an abelian subgroup of cardinality $\aleph$, so that Lemma 5 yields that G/U^G has a finite commutator subgroup. Therefore the index $|G' : G' \cap U|$ is finite, so that G' is finitely generated and hence even finite. □

Proof of Theorem A If the index $|G' : I_\aleph(G)|$ is finite, it is clear that the set $\mathcal{C}_\aleph(G)$ is finite. Conversely, suppose that $\mathcal{C}_\aleph(G)$ is finite. In order to prove that $I_\aleph(G)$ has finite index in G', we may obviously assume that G' is infinite, so that it follows from Lemma 6 that G has no abelian subgroups of cardinality $\aleph$.

As the normal subgroup $D = D_\aleph(G)$ has cardinality $\aleph$, also the normal subgroup D/D' of G/D' has cardinality $\aleph$, and hence G'/D' is finite by Lemma 4. Let X be any subgroup of G of cardinality $\aleph$, and put $Y = X \cap D$. Clearly, Y has cardinality $\aleph$, so that Y' is normalized by D and Y/Y' is an abelian subgroup of D/Y' of cardinality $\aleph$. Thus D'/Y' is finite by Lemma 6, and hence also the index $|G' : X'|$ is finite. As the set $\mathcal{C}_\aleph(G)$ is finite, it follows that the subgroup $I_\aleph(G)$ has finite index in G'.

Let M be a subgroup of G of cardinality $\aleph$ such that M' is a minimal element of the set $\mathcal{C}_\aleph(G)$. Then $U' = M'$ for each subgroup U of M cardinality $\aleph$, and hence M' is either abelian of prime exponent or divisible abelian (see [4], Lemma 2.6). In the first case, we obtain of course that also $I_\aleph(G)$ is abelian of prime exponent. On the other hand, if M' is divisible abelian, then $I_\aleph(G) = M'$ because the index $|M' : I_\aleph(G)|$ is finite. The statement is proved. □

Our main results have a number of interesting consequences.

Corollary 7 *Let G be an uncountable locally graded group of regular cardinality $\aleph$ such that the set $\mathcal{C}_\aleph(G)$ is finite, and suppose that G has no simple sections of cardinality $\aleph$. Then the factor group $G/Z(G)$ has a finite commutator subgroup.*

Proof The commutator subgroup G' of G has cardinality strictly smaller than $\aleph$ by Lemma 6, so that $|G : C_G(g)| < \aleph$ for each element g of G, and hence each centralizer $C_G(g)$ has cardinality $\aleph$. As the set $\mathcal{C}_\aleph(G)$ is finite, there exist finitely many elements $g_1, \ldots, g_k$ of G such that

$$\{C_G(g)' \mid g \in G\} = \{C_G(g_1)', \ldots, C_G(g_k)'\}.$$

The intersection

$$C = \bigcap_{i=1}^{k} C_G(g_i)$$

is a subgroup of G of cardinality $\aleph$, because $|G : C| < \aleph$. Moreover,

$$C' \leq \bigcap_{i=1}^{k} C_G(g_i)' \leq \bigcap_{g \in G} C_G(g) = Z(G),$$

and so $CZ(G)/Z(G)$ is an abelian subgroup of $G/Z(G)$. If $CZ(G)/Z(G)$ has cardinality $\aleph$, it follows from Lemma 6 that $G'Z(G)/Z(G)$ is finite, while if $CZ(G)/Z(G)$ has cardinality strictly smaller than $\aleph$, the centre $Z(G)$ must have cardinality $\aleph$, and so G' is finite by Lemma 4. The statement is proved. □

The following result shows in particular that under the assumptions of Theorem A, the elements of finite order of G' form a subgroup.

Corollary 8 *Let G be an uncountable locally graded group of regular cardinality $\aleph$ such that the set $\mathcal{C}_\aleph(G)$ is finite, and suppose that G has no simple sections of cardinality $\aleph$. Then the second commutator subgroup G'' of G is finite.*

Proof It follows from Corollary 7 that $G'Z(G)/Z(G)$ is finite, so that G' is finite over its centre and hence G'' is finite by the celebrated theorem of Issai Schur (see for instance [15] Part 1, Theorem 4.12). □

Corollary 9 *Let G be an uncountable locally graded group of regular cardinality $\aleph$ such that the set $\mathcal{C}_\aleph(G)$ is finite, and suppose that G has no simple sections of cardinality $\aleph$. If T is the subgroup consisting of all elements of finite order of G', then $\mathcal{C}_\aleph(G/T) = \{G'/T\}$, and in particular all subgroups of cardinality $\aleph$ of G/T are normal.*

Proof Since the subgroup G' has cardinality strictly smaller than $\aleph$ by Lemma 6, the factor group G/T has cardinality $\aleph$. Moreover, G'' is finite by Corollary 8, and so the replacement of G by G/T allows to suppose that G' is torsion-free abelian. As Corollary 7 shows that $G'Z(G)/Z(G)$ is finite, of order m say, we have

$$[G', G]^m = [(G')^m, G] = \{1\},$$

and so G' is contained in $Z(G)$.

Let X be a subgroup of G of cardinality $\aleph$ such that X' is a minimal element of the finite set $\mathcal{C}_\aleph(G)$. Then $Y' = X'$ for each subgroup Y of X of cardinality $\aleph$. In particular, all subgroups of X of cardinality $\aleph$ are normal, and hence X' is a divisible subgroup of the torsion-free abelian group G' (see [3], Theorem 3.6). On the other hand, X/X' is an abelian subgroup of cardinality $\aleph$ of the group G/X', so that G'/X' is finite by Lemma 6 and hence $G' = X'$. □

Notice that the above statement proves in particular that under the hypotheses of Theorem A, if the commutator subgroup G' is torsion-free, then $\mathcal{C}_\aleph(G) = \{G'\}$.

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