Monatshefte für Mathematik (2019) 190:735–753
https://doi.org/10.1007/s00605-019-01300-x

The Hörmander multiplier theorem, III: the complete bilinear case via interpolation

Loukas Grafakos[1] · **Hanh Van Nguyen**[2]

Received: 15 November 2018 / Accepted: 22 May 2019 / Published online: 3 June 2019

Abstract
We develop a special multilinear complex interpolation theorem that allows us to prove an optimal version of the bilinear Hörmander multiplier theorem concerning symbols that lie in the Sobolev space $L^r_s(\mathbb{R}^{2n})$, $2 \le r < \infty$, $rs > 2n$, uniformly over all annuli. More precisely, given such a symbol with smoothness index s, we find the largest open set of indices $(1/p_1, 1/p_2)$ for which we have boundedness for the associated bilinear multiplier operator from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1/p = 1/p_1 + 1/p_2$, $1 < p_1, p_2 < \infty$.

Keywords Multilinear operator · Multiplier operator · Interpolation

Mathematics Subject Classification 42B15 · 42B30

1 Introduction

Multipliers are linear operators of the form

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\sigma(\xi)e^{2\pi i x\cdot\xi}\, d\xi\ ,$$

Communicated by G. Teschl.

The first author would like to thank the Simons Foundation.

✉ Loukas Grafakos
grafakosl@missouri.edu

Hanh Van Nguyen
hvnguyen@ua.edu

1 Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

2 Department of Mathematics, The University of Alabama, Tuscaloosa, AL 35487, USA

where f is a Schwartz function on $\mathbb{R}^n$ and $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$ is its Fourier transform.

Let Ψ be a Schwartz function whose Fourier transform is supported in the annulus of the form $\{\xi : 1/2 < |\xi| < 2\}$ which satisfies $\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. We denote by Δ the Laplacian and by $(I-\Delta)^{s/2}$ the operator given on the Fourier transform by multiplication by $(1+4\pi^2|\xi|^2)^{s/2}$; also for $s > 0$, and we denote by L^r_s the Sobolev space of all functions h on $\mathbb{R}^n$ with norm $\|h\|_{L^r_s} := \|(I-\Delta)^{s/2}h\|_{L^r} < \infty$. Extending an earlier result of Mikhlin [15], the optimal version of the Hörmander multiplier theorem says that if

$$\sup_{k\in\mathbb{Z}} \left\| \widehat{\Psi}\sigma(2^k\cdot)\right\|_{L^r_s} < \infty \tag{1}$$

and

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{s}{n}, \tag{2}$$

then T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$. Hörmander's [13] original version of this theorem stated boundedness in the entire interval $1 < p < \infty$ provided $s > n/2$. A restriction on the indices first appeared in Calderón and Torchinsky [1], while condition (2) appeared in [5]; this condition is sharp as examples are given in [5] indicating that the theorem fails in general when $\left|\frac{1}{p} - \frac{1}{2}\right| > \frac{s}{n}$. Moreover, recently Slavíková [19] provided an example showing that boundedness may also fail even on the critical line $\left|\frac{1}{p} - \frac{1}{2}\right| = \frac{s}{n}$.

In this paper we provide bilinear analogues of these results. The study of the Hörmander multiplier theorem in the multilinear setting was initiated by Tomita [21] and was further studied by Fujita, Grafakos, Miyachi, Nguyen, Si, Tomita (see [2,7,8,11,17,18]) among others. For a given function σ on $\mathbb{R}^{2n}$ we define a bilinear operator

$$T_\sigma(f_1, f_2)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\sigma(\xi_1,\xi_2)e^{2\pi i x\cdot(\xi_1+\xi_2)}d\xi_1 d\xi_2$$

originally defined on pairs of $\mathcal{C}_0^\infty$ functions f_1, f_2 on $\mathbb{R}^n$. We fix a Schwartz function Ψ on $\mathbb{R}^{2n}$ whose Fourier transform is supported in the annulus $1/2 \leq |(\xi_1,\xi_2)| \leq 2$ and satisfies

$$\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}(\xi_1,\xi_2)) = 1, \qquad (\xi_1,\xi_2) \neq 0.$$

The following theorem is the main result of this paper:

Theorem 1.1 *Let* $2 \leq r < \infty$, $s > \frac{2n}{r}$, $1 < p_1, p_2 \leq \infty$ *and let* $1/p = 1/p_1 + 1/p_2 > 0$.

(a) *Let $n/2 < s \le n$. Suppose that*

$$\frac{1}{p_1} < \frac{s}{n}, \ \frac{1}{p_2} < \frac{s}{n}, \ 1 - \frac{s}{n} < \frac{1}{p} < \frac{s}{n} + \frac{1}{2}. \tag{3}$$

Then for all $\mathcal{C}_0^\infty(\mathbb{R}^n)$ functions f_1, f_2 we have

$$\|T_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \le C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{2n})} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)}. \tag{4}$$

Moreover, if (4) holds for all $f_1, f_2 \in \mathcal{C}_0^\infty$ and all σ satisfying (1), then we must necessarily have

$$\frac{1}{p_1} \le \frac{s}{n}, \ \frac{1}{p_2} \le \frac{s}{n}, \ 1 - \frac{s}{n} \le \frac{1}{p} \le \frac{s}{n} + \frac{1}{2}. \tag{5}$$

(b) *Let $n < s \le 3n/2$ and satisfy*

$$\frac{1}{p} < \frac{s}{n} + \frac{1}{2}. \tag{6}$$

Then (4) holds. Moreover, if (4) holds for all $f_1, f_2 \in \mathcal{C}_0^\infty$ and all σ satisfying (1), then we must necessarily have

$$\frac{1}{p} \le \frac{s}{n} + \frac{1}{2}. \tag{7}$$

(c) *If $s > \frac{3n}{2}$ then (4) holds for all $1 < p_1, p_2 < \infty$ and $\frac{1}{2} < p < \infty$.*

This theorem uses two main tools: First, the optimal $n/2$-derivative result in the local L^2-case contained in [6] and a special type of multilinear interpolation suitable for the purposes of this problem (see Theorem 3.1 below). Figure 1 (Sect. 4), plotted on a slanted $(1/p_1, 1/p_2)$ plane, shows the regions of boundedness for T_σ in the two cases $n/2 < s \le n$ and $n < s \le 3n/2$. Note also that in the former case, the condition $1 - \frac{s}{n} < \frac{1}{p}$ is only needed when $p > 2$.

Finally, we mention that the necessity of conditions (3), (5), and (7) in Theorem 1.1 are consequences of Theorems 2 and 3 in [6]; these say that if boundedness holds, then we must necessarily have

$$\frac{1}{p_1} \le \frac{s}{n}, \quad \frac{1}{p_2} \le \frac{s}{n}, \quad \frac{1}{p} \le \frac{s}{n} + \frac{1}{2}.$$

Also, if T_σ maps $L^{p_1} \times L^{p_2}$ to L^p and $p > 2$, then duality implies that T_σ maps $L^{p'} \times L^{p_2}$ to $L^{p_1'}$. Now p' plays the role of p_1 and so constraint $\frac{1}{p_1} \le \frac{s}{n}$ becomes $1 - \frac{s}{n} \le \frac{1}{p}$. This proves (5). So the main contribution of this work is the sufficiency of the conditions in (3) and (6).

2 Preliminary material for interpolation

In this section we briefly discuss three lemmas needed in our interpolation.

Lemma 2.1 *Let* $0 < p_0 < p < p_1 \leq \infty$ *be related as in* $1/p = (1-\theta)/p_0 + \theta/p_1$ *for some* $\theta \in (0,1)$. *Given* $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *and* $\varepsilon > 0$, *there exist smooth functions* h_j^ε, $j = 1, \dots, N_\varepsilon$, *supported in cubes with pairwise disjoint interiors, and nonzero complex constants* c_j^ε *such that the functions*

$$f^{z,\varepsilon} = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^{\frac{p}{p_0}(1-z)+\frac{p}{p_1}z}\, h_j^\varepsilon \tag{8}$$

satisfy

$$\|f^{\theta,\varepsilon} - f\|_{L^{p_0}} < \varepsilon \quad \text{and} \quad \begin{cases} \|f^{\theta,\varepsilon} - f\|_{L^{p_1}} < \varepsilon & \text{if } p_1 < \infty \\ \|f^{\theta,\varepsilon}\|_{L^\infty} \leq \|f\|_{L^\infty} + \varepsilon & \text{if } p_1 = \infty \end{cases} \tag{9}$$

and

$$\|f^{it,\varepsilon}\|_{L^{p_0}}^{p_0} \leq \|f\|_{L^p}^p + \varepsilon', \quad \|f^{1+it,\varepsilon}\|_{L^{p_1}} \leq \big(\|f\|_{L^p}^p + \varepsilon'\big)^{\frac{1}{p_1}},$$

where ε' *depends on* ε, p_0, p_1, p, $\|f\|_{L^p}$ *and tends to zero as* $\varepsilon \to 0$.

Proof Given $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$, by uniform continuity there are N_ε cubes Q_j^ε (with disjoint interiors) and nonzero complex constants c_j^ε such that

$$\Big\| f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \Big\|_{L^{p_0}}^{\min(1,p_0)} < \frac{\varepsilon^{\min(1,p_0)}}{2}, \qquad \Big\| f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \Big\|_{L^{p_1}}^{\min(1,p_1)} < \frac{\varepsilon^{\min(1,p_1)}}{2},$$

and

$$\Big\| f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \Big\|_{L^p} < \varepsilon. \tag{10}$$

Find smooth functions g_j^ε satisfying $0 \leq g_j^\varepsilon \leq \chi_{Q_j^\varepsilon}$ such that

$$\Big\| f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon g_j^\varepsilon \Big\|_{L^{p_0}}^{\min(1,p_0)} < \frac{\varepsilon^{\min(1,p_0)}}{2} \quad \text{and} \quad \Big\| f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon g_j^\varepsilon \Big\|_{L^{p_1}}^{\min(1,p_1)} < \frac{\varepsilon^{\min(1,p_1)}}{2},$$

where the last estimate is required only when $p_1 < \infty$. We set $h_j^\varepsilon = e^{i\phi_j^\varepsilon} g_j^\varepsilon$, where ϕ_j^ε is the argument of the complex number c_j^ε. Then h_j^ε is that function claimed in (8).

Observe that

$$f^{\theta,\varepsilon} = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon| h_j^\varepsilon = \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon g_j^\varepsilon$$

satisfies (9) when $p_1 < \infty$; in the case $p_1 = \infty$ we have

$$|f^{\theta,\varepsilon}| \le \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon| \chi_{Q_j^\varepsilon} = \left| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \right| \le \left| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} - f \right| + |f| \le \frac{\varepsilon}{2} + |f| \le \varepsilon + \|f\|_{L^\infty}.$$

Now we have

$$\|f^{it,\varepsilon}\|_{L^{p_0}}^{p_0} \le \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^p |Q_j^\varepsilon| = \left\| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \right\|_{L^p}^p \le \left(\varepsilon^{\min(1,p)} + \|f\|_{L^p}^{\min(1,p)} \right)^{\frac{p}{\min(1,p)}},$$

having made use of (10).

Given $a, c > 0$ and $\varepsilon > 0$ set $\varepsilon' = \varepsilon'(\varepsilon, a, c) = (\varepsilon^a + c^a)^{1/a} - c$. Then $(\varepsilon^a + c^a)^{1/a} \le \varepsilon' + c$ and $\varepsilon' \to 0$ as $\varepsilon \to 0$. Then for a suitable ε' that only depends on $\varepsilon, p, p_0, p_1, \|f\|_{L^p}$, the preceding estimate gives: $\|f^{it,\varepsilon}\|_{L^{p_0}}^{p_0} \le \|f\|_{L^p}^p + \varepsilon'$ and analogously $\|f^{1+it,\varepsilon}\|_{L^{p_1}} \le \left(\|f\|_{L^p}^p + \varepsilon'\right)^{1/p_1}$ when $p_1 < \infty$; notice that if $p_1 = \infty$ then $\|f^{1+it,\varepsilon}\|_{L^\infty} \le 1$ and the right hand side of the inequality is equal to 1, thus the inequality is still valid. □

Lemma 2.2 *Given a domain Ω on the complex plane and (M, μ) a measure space, let $V : \Omega \times M \to \mathbb{C}$ be a function such that $V(\cdot, x)$ is analytic on Ω for almost every $x \in M$. If the function*

$$V^*(z, x) = \sup_{w : |w-z| < \frac{1}{2}\mathrm{dist}(z, \partial\Omega)} \left| \frac{dV}{dw}(w, x) \right|, \quad x \in M \tag{11}$$

is integrable over M for each $z \in \Omega$, then the mapping $z \longmapsto V(z, \cdot)$ is an analytic function from Ω to the Banach space $L^1(M, d\mu)$.

Proof Fix $z \in \Omega$ and denote $r_z = \frac{1}{2}\mathrm{dist}(z, \partial\Omega)$. It is enough to show that

$$\lim_{h \to 0} \left\| \frac{V(z+h, \cdot) - V(z, \cdot)}{h} - \frac{dV}{dz}(z, \cdot) \right\|_{L^1(M, d\mu)} = 0. \tag{12}$$

The assumption yields that for some set M_0 with $\mu(M \backslash M_0) = 0$, we have

$$\lim_{h \to 0} \frac{V(z+h, x) - V(z, x)}{h} = \frac{dV}{dz}(z, x)$$

for all $x \in M_0$. Thus for each $x \in M_0$ and $h \in \mathbb{C}$ with $|h| < r_z$ we can write

$$\begin{aligned}\Big|\frac{V(z+h,x)-V(z,x)}{h}-\frac{dV}{dz}(z,x)\Big| &= \Big|\frac{1}{h}\int_0^h \frac{dV}{dw}(w,x)dw-\frac{dV}{dz}(z,x)\Big| \\ &\le 2\sup_{w:|w-z|<r_z}\Big|\frac{dV}{dw}(w,x)\Big| \\ &= 2V^*(z,x).\end{aligned}$$

Since $V^*(z,\cdot)$ is integrable on M_0, the Lebesgue dominated convergence theorem yields

$$\begin{aligned}&\lim_{h\to 0}\int_{M_0}\Big|\frac{V(z+h,x)-V(z,x)}{h}-\frac{dV}{dz}(z,x)\Big|d\mu(x) \\ &\quad=\int_{M_0}\lim_{h\to 0}\Big|\frac{V(z+h,x)-V(z,x)}{h}-\frac{dV}{dz}(z,x)\Big|d\mu(x)=0.\end{aligned}$$

This yields (12) and completes the proof, as the last integral is over the entire space M. □

Lemma 2.3 *Given* $0<a<b<\infty$, $\Omega=\{z\in\mathbb{C}\ :\ a<\Re(z)<b\}$, *and a measure space* (M,μ) *of finite measure, let* $H:\Omega\times\mathbb{R}^d\times M\to\mathbb{C}$ *be a measurable function so that* $H(\cdot,\xi,x)$ *be analytic on* Ω *and continuous on* $\overline{\Omega}$ *for each* $(\xi,x)\in\mathbb{R}^d\times M$. *Suppose that*

$$\sup_{w\in\overline{\Omega}}\Big|H(w,\xi,x)\Big|+\sup_{w\in\Omega}\Big|\frac{dH}{dw}(w,\xi,x)\Big|\le C(1+|\xi|)^{-d-1} \tag{13}$$

for all $(\xi,x)\in\mathbb{R}^d\times M$. *If* φ *be a bounded measurable function on* $\mathbb{R}^d$, *then the mapping* $z\longmapsto V(z,\cdot)$, *defined by*

$$V(z,x)=\int_{\mathbb{R}^d}|\varphi(\xi)|^z e^{iArg(\varphi(\xi))}H(z,\xi,x)d\xi,$$

is an analytic function from Ω *to the Banach space* $L^1(M,d\mu)$ *and is continuous on* $\overline{\Omega}$.

Proof Let $K=\{\xi\in\mathbb{R}^d:\ \varphi(\xi)\neq 0\}$. By assumption, for each $x\in M$ we have

$$\begin{aligned}\frac{dV}{dz}(z,x)&=\int_K|\varphi(\xi)|^z\ln(|\varphi(\xi)|)e^{iArg(\varphi(\xi))}H(z,\xi,x)d\xi \\ &\quad+\int_K|\varphi(\xi)|^z e^{iArg(\varphi(\xi))}\frac{dH}{dz}(z,\xi,x)d\xi.\end{aligned}$$

As for each $z\in\Omega$ we have

$$\Big|\,|\varphi(\xi)|^z\ln(|\varphi(\xi)|)\Big|\le\sup_{|t|\le 1}|t|^a\log\frac{1}{|t|}+(1+\|\varphi\|_{L^\infty})^b\log(1+\|\varphi\|_{L^\infty})=c<\infty$$

and H satisfies assumption (13), the associated function $V^*(z,\cdot)$ defined in (11) is bounded and thus integrable over M. Therefore, using Lemma 2.2 we deduce that $z \longmapsto V(z,\cdot)$ is analytic from Ω to $L^1(M, d\mu)$.

Using Lebesgue's dominated convergence theorem and the fist part of assumption (13) we easily deduce that $V(z,\cdot)$ is continuous up to the boundary of Ω. $\square$

Lemma 2.4 [3] *Let F be analytic on the open strip $S = \{z \in \mathbb{C} \ : \ 0 < \Re(z) < 1\}$ and continuous on its closure. Assume that for all $0 \le \tau \le 1$ there exist functions A_τ on the real line such that*

$$|F(\tau + it)| \le A_\tau(t) \qquad \textit{for all } t \in \mathbb{R},$$

and suppose that there exist constants $A > 0$ and $0 < a < \pi$ such that for all $t \in \mathbb{R}$ we have

$$0 < A_\tau(t) \le \exp\left\{Ae^{a|t|}\right\}.$$

Then for $0 < \theta < 1$ we have

$$|F(\theta)| \le \exp\left\{\frac{\sin(\pi\theta)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|A_0(t)|}{\cosh(\pi t) - \cos(\pi\theta)} + \frac{\log|A_1(t)|}{\cosh(\pi t) + \cos(\pi\theta)}\right]dt\right\}.$$

In calculations it is crucial to note that

$$\frac{\sin(\pi\theta)}{2}\int_{-\infty}^{\infty}\frac{dt}{\cosh(\pi t) - \cos(\pi\theta)} = 1-\theta\,, \quad \frac{\sin(\pi\theta)}{2}\int_{-\infty}^{\infty}\frac{dt}{\cosh(\pi t) + \cos(\pi\theta)} = \theta.$$

3 Multilinear interpolation

In this section we prove the main tool needed to derive Theorem 1.1 by interpolation. We denote by $\vec{\xi} = (\xi_1, \ldots, \xi_m)$ elements of $\mathbb{R}^{mn}$, where $\xi_j \in \mathbb{R}^n$. We fix a Schwartz function Ψ on $\mathbb{R}^{mn}$ whose Fourier transform is supported in the annulus $1/2 \le |\vec{\xi}\,| \le 2$ and satisfies

$$\sum_j \widehat{\Psi}(2^{-j}\vec{\xi}\,) = 1, \qquad 0 \ne \vec{\xi} \in \mathbb{R}^{mn}.$$

Theorem 3.1 *Let $0 < p_1^0, \ldots, p_m^0 \le \infty$, $0 < p_1^1, \ldots, p_m^1 \le \infty$, $0 < q_0, q_1 \le \infty$, $0 \le s_0, s_1 < \infty$, $1 < r_0, r_1 < \infty$, $0 < \theta < 1$, and let*

$$\frac{1}{p_l} = \frac{1-\theta}{p_l^0} + \frac{\theta}{p_l^1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad s = (1-\theta)s_0 + \theta s_1$$

for $l = 1, \dots, m$. Assume $r_0 s_0 > mn$, and $r_1 s_1 > mn$ and that for all $f_l \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $l = 1, \dots, m$, we have

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^{q_k}(\mathbb{R}^n)} \le K_k \sup_{j\in\mathbb{Z}} \left\|\sigma(2^j\cdot)\widehat{\Psi}\right\|_{L^{r_k}_{s_k}(\mathbb{R}^{mn})} \prod_{l=1}^m \|f_l\|_{L^{p_l^k}(\mathbb{R}^n)}$$

for $k = 0, 1$ where K_0, K_1 are positive constants. Then the intermediate estimate holds:

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^q(\mathbb{R}^n)} \le C_* K_0^{1-\theta} K_1^\theta \sup_{j\in\mathbb{Z}} \left\|\sigma(2^j\cdot)\widehat{\Psi}\right\|_{L^r_s(\mathbb{R}^{mn})} \prod_{l=1}^m \|f_l\|_{L^{p_l}(\mathbb{R}^n)} \tag{14}$$

for all $f_l \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, where C_* depends on all the indices, on θ, and on the dimension.

Consequently, if $p_l < \infty$ for all $l \in \{1, \dots, m\}$, then T_σ admits a bounded extension from $L^{p_1} \times \cdots \times L^{p_m}$ to L^q that satisfies (14).

Proof Fix a smooth function $\widehat{\Phi}$ on $\mathbb{R}^{mn}$ such that $\mathrm{supp}(\Phi) \subset \left\{\frac{1}{4} \le |\vec{\xi}\,| \le 4\right\}$ and $\widehat{\Phi} \equiv 1$ on the support of the function $\widehat{\Psi}$. Denote $\varphi_j = (I - \Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\Psi}]$ and define

$$\sigma_z(\vec{\xi}\,) = \sum_{j\in\mathbb{Z}} (I - \Delta)^{-\frac{s_0(1-z)+s_1 z}{2}} \left[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\,\mathrm{Arg}\,(\varphi_j)}\right](2^{-j}\vec{\xi}\,)\widehat{\Phi}(2^{-j}\vec{\xi}\,). \tag{15}$$

This sum has only finitely many terms and we now estimate its L^∞ norm. □

Fix $\vec{\xi} \in \mathbb{R}^{mn}$. Then there is a j_0 such that $|\vec{\xi}\,| \approx 2^{j_0}$ and there are only two terms in the sum in (15). For these terms we estimate the L^∞ norm of $(I - \Delta)^{-\frac{s_0(1-z)+s_1 z}{2}}\left[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\,\mathrm{Arg}\,(\varphi_j)}\right]$. For $z = \tau + it$ with $0 \le \tau \le 1$, let $s_\tau = (1-\tau)s_0 + \tau s_1$ and $1/r_\tau = (1-\tau)/r_0 + \tau/r_1$. By the Sobolev embedding theorem we have

$$\begin{aligned}
&\left\|(I - \Delta)^{-\frac{s_0(1-z)+s_1 z}{2}}\left[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\,\mathrm{Arg}\,(\varphi_j)}\right]\right\|_{L^\infty(\mathbb{R}^{mn})} \\
&\le C(r_\tau, s_\tau, mn)\left\|(I - \Delta)^{-\frac{s_0(1-z)+s_1 z}{2}}\left[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\,\mathrm{Arg}\,(\varphi_j)}\right]\right\|_{L^{r_\tau}_{s_\tau}(\mathbb{R}^{mn})} \\
&\le C(r_\tau, s_\tau, n)\left\|(I - \Delta)^{it\frac{s_0-s_1}{2}}\left[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\,\mathrm{Arg}\,(\varphi_j)}\right]\right\|_{L^{r_\tau}(\mathbb{R}^{mn})} \\
&\le C'(r_\tau, s_\tau, mn)(1 + |s_0 - s_1|\,|t|)^{mn/2+1}\left\||\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\,\mathrm{Arg}\,(\varphi_j)}\right\|_{L^{r_\tau}(\mathbb{R}^{mn})} \\
&\le C''(r_0, r_1, s_0, s_1, \tau, mn)(1 + |t|)^{mn/2+1}\left\||\varphi_j|^{r(\frac{1-\tau}{r_0}+\frac{\tau}{r_1})}\right\|_{L^{r_\tau}(\mathbb{R}^{mn})} \\
&= C''(r_0, r_1, s_0, s_1, \tau, mn)(1 + |t|)^{mn/2+1}\|\varphi_j\|_{L^r(\mathbb{R}^{mn})}^{r/r_\tau}.
\end{aligned}$$

It follows from this that

$$\|\sigma_{\tau+it}\|_{L^\infty(\mathbb{R}^{mn})} \le C''(r_0, r_1, s_0, s_1, \tau, mn)(1+|t|)^{mn/2+1}\Big(\sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^r_s(\mathbb{R}^{mn})}\Big)^{r/r_\tau}. \tag{16}$$

Let T_{σ_z} be the family of operators associated to the multipliers σ_z. Let ε be given.

Suppose first that $\min(p_l^0, p_l^1) < \infty$ for all $l \in \{1, \dots, m\}$. This forces $p_l < \infty$ for all l.

Case I: $\min(q_0, q_1) > 1$ This assumption implies that $q > 1$, hence $q', q_0', q_1' < \infty$. Fix $f_l, g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. For given $\varepsilon > 0$, for every $l \in \{1, \dots, m\}$, by Lemma 2.1 there exist functions $f_l^{z,\varepsilon}$ and $g^{z,\varepsilon}$ of the form (8) such that

$$\|f_l^{\theta,\varepsilon} - f_l\|_{L^{p_l^1}} < \varepsilon, \quad \|f_l^{\theta,\varepsilon} - f_l\|_{L^{p_l^0}} < \varepsilon, \quad \|g^{\theta,\varepsilon} - g\|_{L^{q_0'}} < \varepsilon, \quad \|g^{\theta,\varepsilon} - g\|_{L^{q_1'}} < \varepsilon, \tag{17}$$

when $\max(p_l^0, p_l^1) < \infty$, while one of the first two inequalities is replaced by $\|f_l^{\theta,\varepsilon}\|_{L^\infty} \le \|f_l\|_{L^{p_l^k}} + \varepsilon = \|f_l\|_{L^\infty} + \varepsilon$ when $p_l^k = \max(p_l^0, p_l^1) = \infty$, and that

$$\begin{aligned}
&\|f_l^{it,\varepsilon}\|_{L^{p_l^0}} \le \big(\|f_l\|_{L^{p_l}}^{p_l} + \varepsilon'\big)^{\frac{1}{p_l^0}}, \quad \|f_l^{1+it\varepsilon}\|_{L^{p_l^1}} \le \big(\|f_l\|_{L^{p_l}}^{p_l} + \varepsilon'\big)^{\frac{1}{p_l^1}},\\
&\|g^{it,\varepsilon}\|_{L^{q_0'}} \le \big(\|g\|_{L^{q'}}^{q'} + \varepsilon'\big)^{\frac{1}{q_0'}}, \quad \Big\|g^{1+it,\varepsilon}\Big\|_{L^{q_1'}} \le \big(\|g\|_{L^{q'}}^{q'} + \varepsilon'\big)^{\frac{1}{q_1'}}.
\end{aligned}$$

Define

$$\begin{aligned}
F(z) &= \int_{\mathbb{R}^n} T_{\sigma_z}(f_1^{z,\varepsilon}, \dots, f_m^{z,\varepsilon}) g^{z,\varepsilon}\, dx\\
&= \int_{\mathbb{R}^{mn}} \sigma_z(\vec{\xi}\,)\widehat{f_1^{z,\varepsilon}}(\xi_1)\cdots\widehat{f_m^{z,\varepsilon}}(\xi_m)\widehat{g^{z,\varepsilon}}(-(\xi_1+\dots+\xi_m))\, d\vec{\xi}\\
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{mn}} (I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}} \Big[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\operatorname{Arg}(\varphi_j)}\Big](2^{-j}\xi)\widehat{\Phi}(2^{-j}\vec{\xi}\,)\\
&\quad \times \Big(\prod_{l=1}^m \widehat{f_l^{z,\varepsilon}}(\xi_l)\Big)\widehat{g^{z,\varepsilon}}(-(\xi_1+\dots+\xi_m))\, d\vec{\xi}\\
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{mn}} \Big[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\operatorname{Arg}(\varphi_j)}\Big](2^{-j}\vec{\xi}\,)\\
&\quad \times (I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}}\Big[\widehat{\Phi}(2^{-j}\vec{\xi}\,)\Big(\prod_{l=1}^m \widehat{f_l^{z,\varepsilon}}(\xi_l)\Big)\widehat{g^{z,\varepsilon}}(-(\xi_1+\dots+\xi_m))\Big](\vec{\xi}\,)\, d\vec{\xi}.
\end{aligned}$$

Notice that

$$(I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}}\Big[\widehat{\Phi}(2^{-j}\vec{\xi}\,)\Big(\prod_{l=1}^m \widehat{f_l^{z,\varepsilon}}(\xi_l)\Big)\widehat{g^{z,\varepsilon}}(-(\xi_1+\dots+\xi_m))\Big](\vec{\xi}\,)$$

is equal to a finite sum (over $k_1, \dots, k_m, l$) of terms of the form

$$|c^{\varepsilon}_{k_1}|^{\frac{p_1}{p_1^0}(1-z)+\frac{p_1}{p_1^1}z}\cdots|c^{\varepsilon}_{k_m}|^{\frac{p_m}{p_m^0}(1-z)+\frac{p_m}{p_m^1}z}|d^{\varepsilon}_l|^{\frac{q'}{q_0'}(1-z)+\frac{q'}{q_1'}z}(I-\Delta)^{-\frac{s_0(1-z)+s_1z}{2}}\left[\widehat{\Phi}(2^{-j}\cdot)\zeta_{k_1,\dots,k_m,l}\right](\vec{\xi}\,),$$

which we call $H(z,\vec{\xi}\,)$, where $\zeta_{k_1,\dots,k_m,l}$ are Schwartz functions. Thus $H(z,\vec{\xi}\,)$ is an analytic function in z. Moreover $H(z,\vec{\xi}\,)$ can be thought of as a function of three variables $H(z,\vec{\xi},x_0)$, being constant in the variable x_0, where $\{x_0\}$ is a measure space of one element equipped with counting measure. With this interpretation, it is not hard to verify that $H(z,\vec{\xi},x_0)$ satisfies (13).

Lemma 2.3 guarantees that $F(z)$ is analytic on the strip $0<\Re(z)<1$ and continuous up to the boundary. Furthermore, by Hölder's inequality,

$$|F(it)|\le\left\|T_{\sigma_{it}}(f_1^{it,\varepsilon},\dots,f_m^{it,\varepsilon})\right\|_{L^{q_0}}\left\|g^{\varepsilon}_{it}\right\|_{L^{q_0'}},$$

and noting that only the terms with $j=k-1,k,k+1$ survive in the sum in (15) for $\sigma_{it}(2^k\cdot)\widehat{\Psi}$, the Kato–Ponce inequality [10,14] applied as $\|(I-\Delta)^{s/2}(F\widehat{\Phi})\|_{L^{r_0}}\le C\|(I-\Delta)^{s/2}(F)\|_{L^{r_0}}$ yields

$$\begin{aligned}
&\|T_{\sigma_{it}}(f_1^{it,\varepsilon},\dots,f_m^{it,\varepsilon})\|_{L^{q_0}}\\
&\quad\le K_0\sup_{k\in\mathbb{Z}}\left\|\sigma_{it}(2^k\cdot)\widehat{\Psi}\right\|_{L^{r_0}_{s_0}}\prod_{l=1}^m\|f_l^{it,\varepsilon}\|_{L^{p_l^0}}\\
&\quad\le C_{n,r_0,s_0}K_0\sup_{k\in\mathbb{Z}}\left\|(I-\Delta)^{\frac{s_0}{2}}(I-\Delta)^{-\frac{s_0(1-it)+s_1it}{2}}\left[|\varphi_k|^{r(\frac{1-it}{r_0}+\frac{it}{r_1})}e^{i\operatorname{Arg}(\varphi_k)}\right]\right\|_{L^{r_0}}\\
&\qquad\times\prod_{l=1}^m\|f_l^{it,\varepsilon}\|_{L^{p_l^0}}\\
&\quad\le\ C(m,n,r_0,s_0)(1+|s_1-s_0|\,|t|)^{\frac{mn}{2}+1}K_0\sup_{j\in\mathbb{Z}}\|\varphi_j\|_{L^r}^{\frac{r}{r_0}}\prod_{l=1}^m\left(\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\right)^{\frac{1}{p_l^0}}\\
&\quad=\ C(m,n,r_0,s_0,s_1)(1+|t|)^{\frac{mn}{2}+1}K_0\sup_{j\in\mathbb{Z}}\left\|(I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\Psi}]\right\|_{L^r}^{\frac{r}{r_0}}\\
&\qquad\times\prod_{l=1}^m\left(\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\right)^{\frac{1}{p_l^0}}.
\end{aligned}$$

Thus, for some constant $C=C(m,n,r_0,s_0,s_1)$ we have

$$\begin{aligned}
|F(it)|\le C(1+|t|)^{\frac{mn}{2}+1}K_0\sup_{j\in\mathbb{Z}}\left\|(I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\Psi}]\right\|_{L^r}^{\frac{r}{r_0}}\left(\|g\|_{L^{q'}}^{q'}+\varepsilon'\right)^{\frac{1}{q_0'}}\\
\times\prod_{l=1}^m\left(\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\right)^{\frac{p_l}{p_l^0}}.
\end{aligned}$$

Similarly, we can choose the constant $C = C(m, n, r_1, s_0, s_1)$ above large enough so that

$$|F(1+it)| \le C(1+|t|)^{\frac{mn}{2}+1} K_1 \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\Psi}] \right\|_{L^r}^{\frac{r}{r_1}} \left(\|g\|_{L^{q'}}^{q'} + \varepsilon' \right)^{\frac{1}{q_1'}} \times \prod_{l=1}^{m} \left(\|f_l\|_{L^{p_l}}^{p_l} + \varepsilon' \right)^{\frac{1}{p_l^1}}.$$

Note that $F(z)$ is a combination of finite terms of the form

$$\Lambda_{k_1,\dots,k_m,l}(z) \int_{\mathbb{R}^{mn}} \sigma_z(\vec{\xi}\,) \widehat{h_{j_1}^{1,\varepsilon}}(\xi_1) \cdots \widehat{h_{j_m}^{m,\varepsilon}}(\xi_m) \widehat{g_j^{\varepsilon}}(-(\xi_1+\cdots+\xi_m))\, d\vec{\xi},$$

where

$$\Lambda_{k_1,\dots,k_m,l}(z) = |c_{k_1}^{\varepsilon}|^{\frac{p_1}{p_1^0}(1-z)+\frac{p_1}{p_1^1}z} \cdots |c_{k_m}^{\varepsilon}|^{\frac{p_m}{p_m^0}(1-z)+\frac{p_m}{p_m^1}z} |d_l^{\varepsilon}|^{\frac{q'}{q_0'}(1-z)+\frac{q'}{q_1'}z},$$

and $h_{j_1}^{1,\varepsilon}$, g_j^{ε} are smooth functions with compact support. Thus for $z = \tau + it$, $t \in \mathbb{R}$ and $0 \le \tau \le 1$ it follows from (16) and from the definition of $F(z)$ that

$$|F(z)| \le C(\tau, \epsilon, f_1, \dots, f_m, g, r_l, p_l, q_0, q_1)(1+|t|)^{\frac{mn}{2}+1} \left(\sup_{j\in\mathbb{Z}} \left\| \sigma(2^j\cdot)\widehat{\Psi} \right\|_{L_s^r} \right)^{\frac{r}{r_\tau}} = A_\tau(t).$$

As $A_\tau(t) \le \exp(Ae^{a|t|})$, the admissible growth hypothesis of Lemma 2.4 is satisfied. Applying Lemma 2.4 we obtain

$$|F(\theta)| \le C\, K_0^{1-\theta} K_1^{\theta} \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\psi}\,] \right\|_{L^r} \left(\|g\|_{L^{q'}}^{q'} + \varepsilon' \right)^{\frac{1}{q'}} \prod_{l=1}^{m} \left(\|f_l\|_{L^{p_l}}^{p_l} + \varepsilon' \right)^{\frac{1}{p_l}}. \tag{18}$$

But

$$F(\theta) = \int_{\mathbb{R}^n} T_\sigma(f_1^{\theta,\varepsilon}, \dots, f_m^{\theta,\varepsilon})\, g^{\theta,\varepsilon}\, dx$$

and then we have

$$\int_{\mathbb{R}^n} T_\sigma(f_1,\dots,f_m)\, g\, dx = F(\theta) + \int_{\mathbb{R}^n} \left[T_\sigma(f_1,\dots,f_m) - T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon}) \right] g\, dx + \int_{\mathbb{R}^n} T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon}) \big(g - g^{\theta,\varepsilon} \big)\, dx. \tag{19}$$

A telescoping identity yields

$$|T_\sigma(f_1,\dots,f_m)-T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})|\le\sum_{l=1}^m|T_\sigma(f_1,\dots,f_{l-1},f_l-f_l^{\theta,\varepsilon},f_{l+1}^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})|.$$

For every fixed l, applying the hypothesis that T_σ is bounded from $L^{p_1^k}\times\cdots\times L^{p_m^k}$ to L^{q_k} for $k=0,1$ we obtain

$$\big\|T_\sigma(f_1,\dots,f_{l-1},f_l-f_l^{\theta,\varepsilon},f_{l+1}^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\big\|_{L^{q_k}}\lesssim\big\|f_l-f_l^{\theta,\varepsilon}\big\|_{L^{p_l^k}}\prod_{j\neq l}\big(\|f_j\|_{L^{p_j^k}}^{p_j}+\varepsilon'\big)^{\frac{1}{p_j}}.$$

In view of the inequality $\|h\|_{L^q}\le\|h\|_{L^{q_0}}^{1-\theta}\|h\|_{L^{q_1}}^{\theta}$ these estimates yield

$$\begin{aligned}\big\|T_\sigma(f_1,\dots,f_{l-1},f_l-f_l^{\theta,\varepsilon},f_{l+1}^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\big\|_{L^{q}}\lesssim\big\|f_l-f_l^{\theta,\varepsilon}\big\|_{L^{p_l^0}}^{1-\theta}\big\|f_l-f_l^{\theta,\varepsilon}\big\|_{L^{p_l^1}}^{\theta}\\ \prod_{j\neq l}\big(\|f_j\|_{L^{p_j^k}}^{p_j}+\varepsilon'\big)^{\frac{1}{p_j}}.\end{aligned}$$

As $0<\theta<1$ and one of p_l^0 or p_l^1 is strictly less than infinity, the expression on the right above is bounded by a constant multiple of $\varepsilon^{\min(\theta,1-\theta)}$ and hence it tends to zero as $\varepsilon\to0$ because of (9). This proves that (in fact for all $0<q<\infty$)

$$\big\|T_\sigma(f_1,\dots,f_m)-T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\big\|_{L^q}\le E_\varepsilon, \tag{20}$$

where $E_\varepsilon\to0$ as $\varepsilon\to0$. Returning to (19) and using (18) and Hölder's inequality we write

$$\begin{aligned}&\left|\int T_\sigma(f_1,\dots,f_m)(x)\,g(x)\,dx\right|\\ &\le\ CK_0^{1-\theta}K_1^{\theta}\sup_{j\in\mathbb{Z}}\Big\|(I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\psi}\,]\Big\|_{L^r}\big(\|g\|_{L^{q'}}^{q'}+\varepsilon'\big)^{\frac{1}{q'}}\prod_{l=1}^m\big(\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\big)^{\frac{1}{p_l}}\\ &\quad+E_\varepsilon\|g\|_{L^{q'}}+C\big\|g-g^{\theta,\varepsilon}\big\|_{L^{q_0'}}\prod_{l=1}^m\big\|f_l^{\theta,\varepsilon}\big\|_{L^{p_l^0}}\end{aligned}$$

Recalling (17) and using that each $\|f_l^{\theta,\varepsilon}\|_{L^{p_l^0}}$ remains bounded as $\varepsilon\to0$ we obtain

$$\left|\int T_\sigma(f_1,\dots,f_m)\,g\,dx\right|\le CK_0^{1-\theta}K_1^{\theta}\sup_{j\in\mathbb{Z}}\Big\|(I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\psi}\,]\Big\|_{L^r}\|g\|_{L^{q'}}\prod_{l=1}^m\|f_l\|_{L^{p_l}}$$

by letting $\varepsilon\to0$. Taking the supremum over all functions $g\in L^{q'}$ with $\|g\|_{L^{q'}}=1$ yields the sought estimate (14) in Case I.

Case II: $\min(q_0,q_1)\le1$

Here we will make use of two following lemmas proved by Stein and Weiss [20].

Lemma 3.2 ([20]) *Let $U : \overline{S} \longrightarrow \mathbb{R}$ be an upper semi-continuous function of admissible growth and subharmonic in the unit strip S. Then for $z_0 = x_0 + iy_0 \in S$ we have*

$$U(z_0) \le \int_{-\infty}^{+\infty} U\big(i(y_0+t)\big)\omega(1-x_0,t)dt + \int_{-\infty}^{+\infty} U\big(1+i(y_0+t)\big)\omega(x_0,t)dt,$$

where

$$\omega(x,y) = \frac{1}{2}\frac{\sin \pi x}{\cos \pi x + \cosh \pi y}.$$

Lemma 3.3 ([20]) *Let $0 < c \le 1$ and let (M,μ) be a measure space with finite measure. If a function $V(z,\cdot)$ is analytic from the unit strip S to $L^1(M,\mu)$, then $\log \int_M |V(z,x)|^c \, d\mu$ is subharmonic on S.*

We now continue the proof of the second case. We fix functions f_l as in the previous case. Choose an integer $\rho > 1$ such that $\rho \ge \rho \min(q_0, q_1) > q$. Take an arbitrary positive simple function g with $\|g\|_{L^{\rho'}} = 1$. Assume that $g = \sum_{k=1}^{N} c_k \chi_{E_k}$, where $c_k > 0$ and E_k are pairwise disjoint measurable sets of finite measure and compact support. For $z \in \mathbb{C}$, set

$$\begin{aligned} g^z = \sum_{k=1}^{N} c_k^{\lambda(z)} \chi_{E_k}, \quad \text{where} \quad \lambda(z) \\ = \rho' \left[1 - \frac{q}{\rho}\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right)\right]. \end{aligned}$$

Now consider

$$\begin{aligned} G(z) &= \int_{\mathbb{R}^n} \left|T_{\sigma_z}(f_1^{z,\varepsilon}, \dots, f_m^{z,\varepsilon})(x)\right|^{\frac{q}{\rho}} \left|g^z(x)\right| \, dx \\ &= \sum_{k=1}^{N} \int_{E_k} \left|c_k^{\frac{\rho}{q}\lambda(z)} T_{\sigma_z}(f_1^{z,\varepsilon}, \dots, f_m^{z,\varepsilon})(x)\right|^{\frac{q}{\rho}} dx. \end{aligned}$$

Let $V(z,x) = T_{\sigma_z}(f_1^{z,\varepsilon}, \dots, f_m^{z,\varepsilon})(x)$. Then $V(z,x)$ can be represented as a finite sum of terms of the form

$$\begin{aligned} &\int\limits_{\mathbb{R}^{mn}} e^{P(z)} |\varphi_j(\vec{\xi}\,)|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{i\operatorname{Arg}(\varphi_j)} (I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}} \Big[e^{2\pi i x 2^j \cdot (\sum_{\kappa=1}^{m} \xi_\kappa)} \widehat{\Phi}(\vec{\xi}\,) \\ &\quad \times \prod_{\kappa=1}^{m} \widehat{h_\kappa^\varepsilon}(2^j \xi_\kappa)\Big](\vec{\xi}\,) d\vec{\xi}, \end{aligned}$$

where h_κ^ε are the smooth functions with compact support in (8) and P is a polynomial. Setting

$$H(z,\vec{\xi},x)=(I-\Delta)^{-\frac{s_0}{2}(1-z)-\frac{s_1}{2}z}\Big[e^{2\pi i 2^j x\cdot(\xi_1+\cdots+\xi_n)}\widehat{\Phi}(\vec{\xi}\,)\prod_{\kappa=1}^{m}\widehat{h_\kappa^\varepsilon}(2^j\xi_\kappa)\Big],$$

we note that $H(z,\vec{\xi},x)$ is analytic in z, smooth in ξ and bounded in x, as long as x remains in a compact set. Moreover H satisfies (13). Applying Lemma 2.3 we obtain that for all $(\vec{\xi},x)$ the mapping $H(\cdot,\vec{\xi},x)$ is analytic from S to $L^1(E_k,dx)$ Then Lemma 3.3 applies and yields that $\log G$ is subharmonic on S. Using Hölder's inequality with indices $\frac{\rho q_0}{q}$ and $\big(\frac{\rho q_0}{q}\big)'$ and the fact that the $L^{\rho'}$-norm of g is equal to 1, we have

$$\begin{aligned}G(it)&\le\left\{\int_{\mathbb{R}^n}\Big|T_{\sigma_{it}}(f_1^{it,\varepsilon},\ldots,f_m^{it,\varepsilon})(x)\Big|^{q_0}dx\right\}^{\frac{q}{\rho q_0}}\big\|g^{it}\big\|_{L^{(\frac{\rho q_0}{q})'}}\\&\le C\left((1+|t|)^{\frac{mn}{2}+1}\right)^{\frac{q}{\rho}}\left(K_0\sup_{j\in Z}\Big\|\sigma(2^j\cdot)\widehat{\psi}\Big\|_{L_s^r}^{\frac{r}{r_0}}\prod_{l=1}^{m}\big(\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\big)^{\frac{1}{p_l}}\right)^{\frac{q}{\rho}}.\end{aligned}$$

Similarly, we can estimate

$$\begin{aligned}G(1+it)&\le\left\{\int_{\mathbb{R}^n}\Big|T_{\sigma_{it}}(f_1^{1+it,\varepsilon},\ldots,f_m^{1+it,\varepsilon})(x)\Big|^{q_1}dx\right\}^{\frac{q}{\rho q_1}}\big\|g^{1+it}\big\|_{L^{(\frac{\rho q_1}{q})'}}\\&\le C\left((1+|t|)^{\frac{mn}{2}+1}\right)^{\frac{q}{\rho}}\left(K_1\sup_{j\in Z}\Big\|\sigma(2^j\cdot)\widehat{\psi}\Big\|_{L_s^r}^{\frac{r}{r_1}}\prod_{l=1}^{m}\big(\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\big)^{\frac{1}{p_l}}\right)^{\frac{q}{\rho}}.\end{aligned}$$

Applying Lemma 3.2 to $U=\log G$ (with $y_0=0$ and $x_0=\theta$) and using that for $0<\theta<1$ we have

$$\begin{aligned}\frac{\sin(\pi(1-\theta))}{2}\int_{-\infty}^{+\infty}\frac{1}{\cosh(\pi t)+\cos(\pi(1-\theta))}\,dt&=1-\theta,\\\frac{\sin(\pi\theta)}{2}\int_{-\infty}^{+\infty}\frac{1}{\cosh(\pi t)+\cos(\pi\theta)}\,dt&=\theta,\end{aligned}$$

(see [3, Page 48]) we obtain

$$G(\theta)\le C'_*\left(K_0^{1-\theta}K_1^{\theta}\sup_{j\in Z}\Big\|\sigma(2^j\cdot)\widehat{\psi}\Big\|_{L_s^r}\prod_{l=1}^{m}\big(\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\big)^{\frac{1}{p_l}}\right)^{\frac{q}{\rho}}.\tag{21}$$

Notice that as

$$G(\theta)=\int_{\mathbb{R}^n}\Big|T_\sigma(f_1^{\theta,\varepsilon},\ldots,f_m^{\theta,\varepsilon})(x)\Big|^{\frac{q}{\rho}}g(x)\,dx,$$

inequality (21) implies that

$$\begin{aligned}
&\left\| T_\sigma(f_1^{\theta,\varepsilon}, \dots, f_m^{\theta,\varepsilon}) \right\|_{L^q} \\
&= \left\| \left| T_\sigma(f_1^{\theta,\varepsilon}, \dots, f_m^{\theta,\varepsilon}) \right|^{\frac{q}{\rho}} \right\|_{L^\rho}^{\frac{\rho}{q}} \\
&= \sup \left\{ \int \left| T_\sigma(f_1^{\theta,\varepsilon}, \dots, f_m^{\theta,\varepsilon})(x) \right|^{\frac{q}{\rho}} g(x)\, dx : \; g \geq 0, g \text{ simple}, \|g\|_{L^{\rho'}} = 1 \right\}^{\frac{\rho}{q}} \\
&\leq (C_*')^{\frac{\rho}{q}} K_0^{1-\theta} K_1^\theta \sup_{j \in Z} \left\| \sigma(2^j \cdot)\widehat{\psi} \right\|_{L_s^r} \prod_{l=1}^m \left(\|f_l\|_{L^{p_l}}^{p_l} + \varepsilon' \right)^{\frac{1}{p_l}}.
\end{aligned} \tag{22}$$

Finally, we use

$$\begin{aligned}
&\|T_\sigma(f_1, \dots, f_m)\|_{L^q} \leq (1 + 2^{\frac{1}{q}-1}) \big[\|T_\sigma(f_1, \dots, f_m) - T_\sigma(f_1^{\theta,\varepsilon}, \dots, f_m^{\theta,\varepsilon})\|_{L^q} \\
&\quad + \|T_\sigma(f_1^{\theta,\varepsilon}, \dots, f_m^{\theta,\varepsilon})\|_{L^q} \big]
\end{aligned}$$

and we note that for the second term we use (22), while the first term tends to zero, in view of (20). Letting $\varepsilon \to 0$, we deduce (14).

We now turn to the case where $\min(p_l^0, p_l^1) = \infty$ for some (but not all) l in $\{1, \dots, m\}$. Then we must have $p_l = \infty$ for these l, and for these l we set $f_l^{z,\varepsilon} = f$, while for the remaining l the functions $f_l^{z,\varepsilon}$ are defined as before; we notice that the preceding argument works with only minor modifications.

Finally we consider the case where $p_l^0 = p_l^1 = \infty$ for all $1 \leq l \leq m$. Here we also take $f_l^{z,\varepsilon} = f_l$ for all l in $\{1, \dots, m\}$. Now (19) becomes

$$\int_{\mathbb{R}^n} T_\sigma(f_1, \dots, f_m)\, g\, dx = F(\theta) + \int_{\mathbb{R}^n} T_\sigma(f_1, \dots, f_m) \big(g - g^{\theta,\varepsilon} \big)\, dx. \tag{23}$$

Hence, in Case I, when $\min(q_0, q_1) > 1$, we have

$$\begin{aligned}
&\left| \int T_\sigma(f_1, \dots, f_m)(x)\, g(x)\, dx \right| \\
&\quad \leq C K_0^{1-\theta} K_1^\theta \sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot)\widehat{\psi}\,] \right\|_{L^r} \left(\|g\|_{L^{q'}}^{q'} + \varepsilon' \right)^{\frac{1}{q'}} \prod_{l=1}^m \|f_l\|_{L^\infty} \\
&\qquad + C \left\| g - g^{\theta,\varepsilon} \right\|_{L^{q_0'}} \prod_{l=1}^m \|f_l\|_{L^\infty}.
\end{aligned}$$

Passing the limit as $\varepsilon \to 0$ to obtain

$$\left| \int T_\sigma(f_1, \dots, f_m)\, g\, dx \right| \leq C K_0^{1-\theta} K_1^\theta \sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot)\widehat{\psi}\,] \right\|_{L^r} \|g\|_{L^{q'}} \prod_{l=1}^m \|f_l\|_{L^\infty}.$$

The result in Case II, which is when $\min(q_0, q_1) \le 1$, can be obtained from that in Case I by choosing $\rho > 1$ such that $\rho \min(q_0, q_1) > q$ and by arguing as before, replacing each term $\big(\|f_l\|_{L^{p_l}}^{p_l} + \varepsilon'\big)^{\frac{1}{p_l}}$ by $\|f_l\|_{L^\infty}$. This concludes the proof of the theorem in all cases. □

Note that the proof of Theorem 3.1 is much simpler in the case $r_0 = r_1 = 2$, and this was proved earlier in [8, Theorem 6.1, Step 1]; see also [9, Theorem 2.3]. In this case, the domains can be arbitrary Hardy spaces. We state the theorem in this case (without providing a proof):

Theorem 3.4 ([8]) *Let $p_l^0, p_l^1, p_l, q_0, q_1, q, s_0, s_1, s$ and $\theta \in (0,1)$ be as in Theorem 3.1 for $l = 1, \dots, m$. Assume that $s_0, s_1 > \frac{mn}{2}$, $p_l^0, p_l^1 < \infty$ for all l, and that*

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^{q_k}(\mathbb{R}^n)} \le K_k \sup_{j\in\mathbb{Z}} \Big\|\sigma(2^j\cdot)\widehat{\Psi}\Big\|_{L^2_{s_k}(\mathbb{R}^{mn})} \prod_{l=1}^m \|f_l\|_{H^{p_l^k}(\mathbb{R}^n)}$$

for $k = 0, 1$ where K_0, K_1 are positive constants. Then we have the intermediate estimate:

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^q(\mathbb{R}^n)} \le C_* K_0^{1-\theta} K_1^\theta \sup_{j\in\mathbb{Z}} \Big\|\sigma(2^j\cdot)\widehat{\Psi}\Big\|_{L^2_s(\mathbb{R}^{mn})} \prod_{l=1}^m \|f_l\|_{H^{p_l}(\mathbb{R}^n)}$$

for all Schwartz functions f_l with vanishing moments of all orders, where C_* depends on all the indices, θ, and the dimension.

4 The proof of the main result via interpolation

We now turn to the proof of Theorem 1.1.

Proof (a) Assume $n/2 < s \le n$ and let

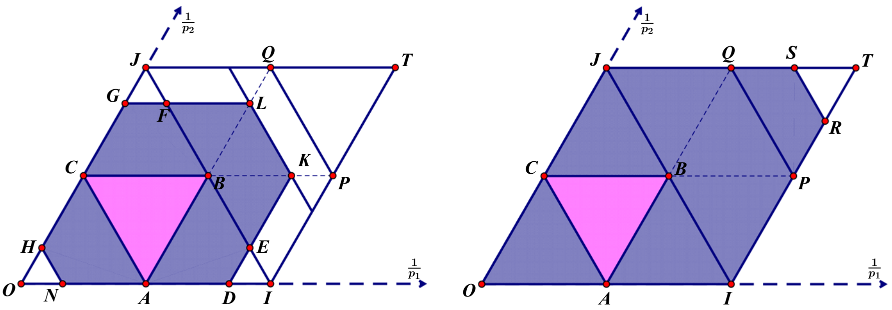
$$\Gamma_1 = \Big\{\Big(\frac{1}{p_1}, \frac{1}{p_2}\Big) : \frac{1}{p_1} < \frac{s}{n}, \frac{1}{p_2} < \frac{s}{n}, 1 - \frac{s}{n} < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < \frac{s}{n} + \frac{1}{2}\Big\}.$$

We will prove that

$$\|T_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \le C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{2n})} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)} \tag{24}$$

for every $(\frac{1}{p_1}, \frac{1}{p_2}) \in \Gamma_1$, which is a convex set with vertices D, K, L, G, H and N (see Fig. 1a below). By multilinear real interpolation [4, Corollary 7.2.4], we only need to verify the boundedness of T_σ at points in Γ_1 near its vertices D, K, L, G, H, N which do not lie in Γ_1.

As showed in [4,11], the Hörmander condition $\sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{2n})}$ is invariant under duality. For $1 \le p < \infty$, by duality, if T_σ maps $L^{p_1} \times L^{p_2} \to L^p$, then



(A) $\frac{n}{2} < s \leq n$ **(B)** $n < s \leq \frac{3n}{2}$

Fig. 1 Boundedness holds in the shaded regions and unboundedness in the white regions. The local L^2 region is shaded in a lighter color

it also maps $L^{p'} \times L^{p_2} \to L^{p_1'}$. Therefore, if T_σ is bounded near D, then T_σ is also bounded near N by duality. By symmetry, if T_σ is bounded near N, D and K then it is bounded near H, G and L as well. From these reductions, it remains to prove (24) at points in Γ_1 near D and K.

With $s_1 > \frac{n}{2}$ and $r_1 s_1 > 2n$, we recall the following [6, Theorem 1]:

$$\|T_\sigma(f_1, f_2)\|_{L^1(\mathbb{R}^n)} \leq C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^{r_1}_{s_1}(\mathbb{R}^{2n})} \|f_1\|_{L^2(\mathbb{R}^n)} \|f_2\|_{L^2(\mathbb{R}^n)}. \tag{25}$$

By duality it follows from (25) that when $s_1 > \frac{n}{2}$ and $r_1 s_1 > 2n$ we have

$$\|T_\sigma(f_1, f_2)\|_{L^2(\mathbb{R}^n)} \leq C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^{r_1}_{s_1}(\mathbb{R}^{2n})} \|f_1\|_{L^2(\mathbb{R}^n)} \|f_2\|_{L^\infty(\mathbb{R}^n)}. \tag{26}$$

Theorem 1.1 in [17] (with $s_1 = s_2$ in [17] being γ below) implies that

$$\begin{aligned} &\|T_\sigma(f_1, f_2)\|_{L^q(\mathbb{R}^n)} \\ &\quad \leq C \sup_{j\in\mathbb{Z}} \|(I - \Delta_{\xi_1})^{\frac{\gamma}{2}}(I - \Delta_{\xi_2})^{\frac{\gamma}{2}}\big[\sigma(2^j\cdot)\widehat{\Psi}\big]\|_{L^2(\mathbb{R}^{2n})} \|f_1\|_{L^{q_1}(\mathbb{R}^n)} \|f_2\|_{L^{q_2}(\mathbb{R}^n)} \end{aligned}$$

for $\gamma > \frac{n}{2}$, where $1 < q_1, q_2 \leq \infty$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < \frac{2\gamma}{n} + \frac{1}{2}$. Given $s_2 > n$, choose $\gamma = \frac{s_2}{2} > \frac{n}{2}$ and observing the trivial estimate

$$\sup_{j\in\mathbb{Z}} \|(I - \Delta_{\xi_1})^{\frac{\gamma}{2}}(I - \Delta_{\xi_2})^{\frac{\gamma}{2}}\big[\sigma(2^j\cdot)\widehat{\Psi}\big]\|_{L^2(\mathbb{R}^{2n})} \leq C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^2_{s_2}(\mathbb{R}^{2n})},$$

we obtain

$$\|T_\sigma(f_1, f_2)\|_{L^q(\mathbb{R}^n)} \leq C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^2_{s_2}(\mathbb{R}^{2n})} \|f_1\|_{L^{q_1}(\mathbb{R}^n)} \|f_2\|_{L^{q_2}(\mathbb{R}^n)} \tag{27}$$

for all $1 < q_1, q_2 \le \infty$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < \frac{s_2}{n} + \frac{1}{2}$.

We now use Theorem 3.1 to interpolate between (26) and (27) (for $q_1 = q$ near 1 and $q_2 = \infty$). We obtain (24) at points $D_1(\frac{1}{p_1}, 0)$ with $\frac{1}{p_1} < \frac{s}{n}$ which are near the point $D(\frac{s}{n}, 0)$. Similarly, interpolating between (25) and (27) (q_1 near 1, $q_2 = 2$) yields (24) at points $K_1(\frac{1}{p_1}, \frac{1}{2})$ with $\frac{1}{p_1} < \frac{s}{n}$ near $K(\frac{s}{n}, \frac{1}{2})$. This yields (24) on Γ_1 and completes part (a).

(b) Assume $n < s \le \frac{3n}{2}$. Since $r \ge 2$, the Kato–Poince inequality [10] implies that

$$\sup_{j\in\mathbb{Z}} \|\sigma(2^j \cdot)\widehat{\Psi}\|_{L^2_s(\mathbb{R}^{2n})} \lesssim \sup_{j\in\mathbb{Z}} \|\sigma(2^j \cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{2n})}. \tag{28}$$

Combining estimates (28) and (27) yields (24) in the open pentagon $OIRSJ$ union the open segments OI and OJ. This completes the second part of Theorem 1.1.

(c) In the last case when $s > \frac{3n}{2}$, notice that condition (7) reduces to $p > \frac{1}{2}$ and since

$$\sup_{j\in\mathbb{Z}} \|\sigma(2^j \cdot)\widehat{\Psi}\|_{L^r_{\frac{3n}{2}}(\mathbb{R}^{2n})} \le \sup_{j\in\mathbb{Z}} \|\sigma(2^j \cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{2n})},$$

the case in part (b) applies and yields (24) for every point in the entire rhombus $OITJ$ union the open segments OI and OJ. The proof of Theorem 1.1 is now complete. □

5 An application

We consider the following multiplier on $\mathbb{R}^{2n}$: $m_{a,b}(\xi_1, \xi_2) = \psi(\xi_1, \xi_2)|(\xi_1, \xi_2)|^{-b} e^{i|(\xi_1,\xi_2)|^a}$ where $a > 0$, $a \ne 1$, $b > 0$, and ψ is a smooth function on $\mathbb{R}^{2n}$ which vanishes in a neighborhood of the origin and is equal to 1 in a neighborhood of infinity. One can verify that $m_{a,b}$ satisfies (1) on $\mathbb{R}^{2n}$ with $s = b/a$ and any $r > 2n/s$.

The range of p's for which $m_{a,b}$ is a bounded bilinear multiplier on $L^p(\mathbb{R}^{2n})$ can be completely described by the equation $|\frac{1}{p} - \frac{1}{2}| \le \frac{b/a}{2n}$ (see Hirschman [12, comments after Theorem 3c], Wainger [22, Part II], and Miyachi [16, Theorem 3]); similar examples of multipliers of limited boundedness are contained in Miyachi and Tomita [17, Section 7].

As a consequence of Theorem 1.1 we obtain that the bilinear multiplier operator associated with $m_{a,b}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ in the following cases:

(i) when $n \ge b/a > n/2$ and

$$\frac{1}{p_1} < \frac{b}{an}, \ \frac{1}{p_2} < \frac{b}{an}, \ 1 - \frac{b}{an} < \frac{1}{p} < \frac{b}{an} + \frac{1}{2}.$$

(ii) when $3n/2 \ge b/a > n$ and

$$\frac{1}{p} < \frac{b}{an} + \frac{1}{2};$$

(iii) when $b/a > 3n/2$ in the entire range of exponents $1 < p_1, p_2 \leq \infty, \frac{1}{2} < p < \infty$.

The boundedness of this specific bilinear multiplier is unknown to us outside the above range of indices.

References

1. Calderón, A.P., Torchinsky, A.: Parabolic maximal functions associated with a distribution, II. Adv. Math. **24**, 101–171 (1977)
2. Fujita, M., Tomita, N.: Weighted norm inequalities for multilinear Fourier multipliers. Trans. Am. Math. Soc. **364**, 6335–6353 (2012)
3. Grafakos, L.: Classical Fourier Analysis, Graduate Texts in Mathematics, GTM 249, 3rd edn. Springer, New York (2014)
4. Grafakos, L.: Modern Fourier Analysis, Graduate Texts in Mathematics, GTM 250, 3rd edn. Springer, New York (2014)
5. Grafakos, L., He, D., Honzík, P., Nguyen, H.V.: The Hörmander multiplier theorem I: the linear case. Ill. J. Math. **61**, 25–35 (2017)
6. Grafakos, L., He, D., Honzík, P.: The Hörmander multiplier theorem II: the local L^2 case. Math. Zeit. **289**, 875–887 (2018)
7. Grafakos, L., Miyachi, A., Nguyen, H.V., Tomita, N.: Multilinear Fourier multipliers with minimal Sobolev regularity, II. J. Math. Soc. Japan **69**, 529–562 (2017)
8. Grafakos, L., Miyachi, A., Tomita, N.: On multilinear Fourier multipliers of limited smoothness. Can. J. Math. **65**, 299–330 (2013)
9. Grafakos, L., Nguyen, H.V.: Multilinear Fourier multipliers with minimal Sobolev regularity, I. Colloq. Math. **144**, 1–30 (2016)
10. Grafakos, L., Oh, S.: The Kato–Ponce inequality. Comm. PDE **39**, 1128–1157 (2014)
11. Grafakos, L., Si, Z.: The Hörmander multiplier theorem for multilinear operators. J. Reine Angew. Math. **668**, 133–147 (2012)
12. Hirschman Jr., I.I.: On multiplier transformations. Duke Math. J. **26**, 221–242 (1959)
13. Hörmander, L.: Estimates for translation invariant operators in L^p spaces. Acta Math. **104**, 93–139 (1960)
14. Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier–Stokes equations. Comm. Pure Appl. Math. **41**, 891–907 (1988)
15. Mikhlin, S.G.: On the multipliers of Fourier integrals. Dokl. Akad. Nauk SSSR (N.S.) **109**, 701–703 (1956)
16. Miyachi, A.: On some Fourier multipliers for $H^p(\mathbb{R}^n)$. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27**, 157–179 (1980)
17. Miyachi, A., Tomita, N.: Minimal smoothness conditions for bilinear Fourier multipliers. Rev. Mat. Iberoam. **29**, 495–530 (2013)
18. Miyachi, A., Tomita, N.: Boundedness criterion for bilinear Fourier multiplier operators. Tohoku Math. J. **66**, 55–76 (2014)
19. Slavíková, L.: On the failure of the Hörmander multiplier theorem in a limiting case. Rev. Mat. Iberoamer **(to appear)**
20. Stein, E.M., Weiss, G.: On the interpolation of analytic families of operators acting on H^p-spaces. Tohoku Math. J. **9**, 318–339 (1957)
21. Tomita, N.: A Hörmander type multiplier theorem for multilinear operators. J. Funct. Anal. **259**, 2028–2044 (2010)
22. Wainger, S.: Special trigonometric series in k-dimensions. Mem. Am. Math. Soc. **59**, 1–102 (1965)