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On the persistence and blow up for the generalized two-component Dullin–Gottwald–Holm system

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Abstract
Considered herein is the persistence property of the solutions to the generalized two-component integrable Dullin–Gottwald–Holm system, which was derived from the Euler equation with nonzero constant vorticity in shallow water waves moving over a linear shear flow. Firstly, the persistence properties of the system are investigated in weighted L^p-spaces for a large class of moderate weights. Then, we establish the new *local-in-space* blow-up results simplifying and extending earlier blow-up criterion for this system.

Keywords Generalized two-component Dullin–Gottwald–Holm system · Persistence property · Blow-up

Mathematics Subject Classification 35G25 · 35B44 · 35Q35

1 Introduction

In 2001, Dullin, Gottwald, and Holm studied the following $1+1$ quadratically non-linear equation,

$$m_t + c_0 u_x + u m_x + 2 m u_x + \gamma u_{xxx} = 0, x \in \mathbb{R}, t > 0, \qquad (1.1)$$

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where $m = u - \alpha^2 u_{xx}$ is a momentum variable. This equation was derived using asymptotic expansions directly in the Hamiltonian for Euler's equation in the shallow water regime, and it is completely integrable with a bi-Hamiltonian as well as with a Lax pair [25].

Using the notating $m = u - \alpha^2 u_{xx}$, Eq. (1.1) can be written as

$$u_t - \alpha^2 u_{txx} + 2\omega u_x + 3u_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), x \in \mathbb{R}, t > 0, \quad (1.2)$$

where ω and α are two positive constants. Formally, when $\alpha^2 = 0$, Eq. (1.2) becomes the Korteweg–de-Vries(KdV) equation,

$$u_t + 2\omega u_x + 3uu_x + \gamma u_{xxx} = 0, x \in \mathbb{R}, t > 0. \quad (1.3)$$

While when $\gamma = 0$ and $\alpha = 1$, Eq. (1.1) turns into the Camassa–Holm equation [8,9],

$$u_t + 2\omega u_x - u_{txx} + 3u_x = 2u_x u_{xx} + uu_{xxx}, x \in \mathbb{R}, t > 0. \quad (1.4)$$

The details concerning the hydrodynamical relevance of Camassa–Holm equation were mathematically rigorously described in [14], where, in addition, authors investigate in what sense model under consideration gives us insight into the wave breaking phenomenon. Alternative derivations of Camassa–Holm equation as a equation for geodesic flow on the diffeomorphism group of the circle were presented by Constantin and Kolev [13] and Ionescu-Kruse [33]. The equation has bi-Hamiltonian structure [27] and is completely integrable [2,9,15,16,24]. Note that local well-posedness for the initial datum $u_0(x) \in H^s$ with $s > \frac{3}{2}$ was proved by several authors (see, for example, [17,37,40]. For the initial data with lower regularity, we refer to papers [7] and [41]. Camassa–Holm equation possesses a solitary wave with discontinuous first derivatives [8], which is named peakon (travelling wave solutions with a corner at their peak). More importantly, the peakons are orbitally stable [19,36], which means that the shape of the peakons is stable so that these wave patterns are physically recognizable. Wave breaking for a large class of initial data has been established in [17,18,37,45,46] and in the recent paper [35], where, in particular, new and direct proof for the result from [39] on the necessary and sufficient condition for wave breaking was presented.

The Camassa–Holm equation also admits many integrable multicomponent generalizations. The most popular one is

$$\begin{cases} m_t - Au_x + um_x + 2mu_x + \rho\rho_x = 0, m = u - u_{xx} \\ \rho_t + (u\rho)_x = 0, \end{cases} \quad (1.5)$$

Notice that the Camassa–Holm equation can be obtained via the obvious reduction $\rho \equiv 0$ and $A = 0$. System (1.5) was derived in 1996 [43]. Recently, Constantin–Ivanov [20] and Ivanov [34] established a rigorous justification of the derivation of system (1.5). Mathematical properties of the system have also been studied further in many works [1,11,12,22,26,28,29,31,32,38,42,44,47–49]. The reciprocal transformation between the two-component Camassa–Holm system and the first negative flow of the Ablowitz–Kaup–Newell–Segur hierarchy is established [12]. Escher, Lechtenfeld, and Yin investigated local well-posedness for the two-component Camassa–Holm system with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 2$ and provided some precise blow-up scenarios for strong solutions to the system (1.5) [26]. The local well-posedness is improved in the Besov spaces (especially in the Sobolev space $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$, and the finite time blow-up is determined by either the slope of the first component u or the slope of the second component ρ [29]. Chen and Liu have derived some conditions of blow-up solutions for the generalized two-component Camassa–Holm system, which was recently derived in [11], following Ivanovs modeling approach [34]. The blow-up criterion is made more precise in [38] where the authors showed that the wave breaking in finite time only depends on the slope u. In other words, the wave breaking in u must occur before that in ρ. This blow-up criterion is further improved [28] to the lowest Sobolev space.

In this paper, we are concerned with the Cauchy problem of the generalized two-component Dullin–Gottwald–Holm (DGH) system [10,30],

$$\begin{cases} m_t - Au_x + \sigma(2mu_x + um_x) + 3(1-\sigma)uu_x + \gamma u_{xxx} + \rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.6}$$

where $m = u - u_{xx}$ and σ is a real parameter. It is a model from the shallow water theory with nonzero constant vorticity, where $u(x,t)$ is the horizontal velocity and $\rho(t,x)$ is related to the free surface elevation from equilibrium with the boundary assumptions, $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$. The scalar $A > 0$ characterizes a linear underlying shear flow and hence the system (1.6) models wave-current interactions. The real dimensionless constant σ is a parameter, which provides the competition, or balance, in fluid convection between nonlinear steeping and amplification due to stretching. System (1.6) can be written in terms of u and ρ,

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_xu_{xx} + uu_{xxx}) + \gamma u_{xxx} + \rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.7}$$

System (1.7) has the following two Hamiltonians:

$$H_1 = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + (\rho-1)^2\right) dx,$$
$$H_2 = \frac{1}{2}\int_{\mathbb{R}} \left(u^3 + \sigma uu_x^2 - Au^2 - \gamma u_x^2 + 2u(\rho-1) + u(\rho-1)^2\right) dx.$$

The goal of the present paper is to study the persistence property of solutions and derive some conditions of blow-up solutions for the initial value problem (1.6). And the main tool to study the persistence property is the method of characteristics, which was initially used by Constantin and Escher [17,21,23] to investigate the question of global existence and the blow-up mechanism for the Camassa–Holm equation. In present paper, working with moderate weight functions that are commonly used in time-frequency analysis [3], we generalized the persistence result on the solution to Eq. (1.6) in the weighted $L^p_\phi = L^p(\mathbb{R}, \phi^p(x)dx)$ space. The blow-up problem for the system (1.6) has been addressed in [10,30]. The conditions on the initial datum u_0 leading to the blow-up typically involves the computation of some global quantities (the Sobolev norm $\|u_0\|_{H^1}$, or some other integral expressions of u_0). For $\sigma = 1$, $\gamma = 0$, motivated by the recent paper [4–6], we establish a *local-in-space* blow-up criterion for the system (1.6), i.e., a blow-up condition involving only the properties of u_0 in a neighborhood of a single point $x_0 \in \mathbb{R}$. Such criterion will be more general (and more natural)than the earlier blow-up results. The details can be found in Theorem 3.1.

The remainder of the paper is organized as follows. In Sect. 2, we established persistence properties and some unique continuous properties of the solution to Eq. (1.6) in weighted $L^p_\phi := L^p(\mathbb{R}, \phi^p(x)dx)$ spaces. Finally, we construct initial data which leads to the *local-in-space* blow-up results.

Notation. In the sequel, we denote by $*$ the convolution. For $1 \le p < \infty$, the norms in the Lebesgue space $L^p(\mathbb{R})$ is $\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{\frac{1}{p}}$, the space $L^\infty(\mathbb{R})$ consists of all essentially bounded, Lebesgue measurable functions f equipped with the norm $\|f\|_\infty = \inf_{\mu(e)=0} \sup_{x\in\mathbb{R}\setminus e} |f(x)|$. For a function f in the classical Sobolev spaces $H^s(\mathbb{R})$ $(s \ge 0)$ the norm is denoted by $\|f\|_{H^s}$. We denote $G(x) = \frac{1}{2}e^{-|x|}$ the fundamental solution of $1-\partial_x^2$ on $\mathbb{R}$, and define the two convolution operators $G_+,\ G_-$ as

$$
\begin{aligned}
G_+ * f(x) &= \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{y} f(y)dy,\\
G_- * f(x) &= \frac{e^{x}}{2}\int_{x}^{\infty} e^{-y} f(y)dy.
\end{aligned}
\tag{1.8}
$$

Then we have the relations $G = G_+ + G_-$, $G_x = G_- - G_+$.

2 Persistence property

In this section, we intend to find a large class of weight functions ϕ such that

$$\sup_{t\in[0,T)} \left(\|u(t)\phi\|_{L^p} + \|u_x(t)\phi\|_{L^p} + \|\rho(t)\phi\|_{L^p}\right) < \infty,$$

this way we obtain a persistence results on solution (u, ρ) to Eq. (1.6) in the weight L^p space $L^p_\phi := L^p(\mathbb{R}, \phi^p(x)dx)$. As a consequence and an application we determine the spatial asymptotic behavior of certain solutions to Eq. (1.6). We will work with moderate weight functions which appear with regularity in the theory of time-frequency analysis and have led to optimal results for the Camassa–Holm equation in [3]. Firstly, we list some knowledge in time frequency analysis for later use, for the details see [3].

Definition 2.1 An admissible weight function for system (1.6) is a local absolutely continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that, for some $A > 0$ and almost all $x \in \mathbb{R}$, $|\phi'(x)| \leq A|\phi(x)|$, and that is $v -$ moderate for some sub-multiplicative function v satisfying $\inf_{\mathbb{R}} v > 0$ and

$$\int_{\mathbb{R}} \frac{v(x)}{e^{|x|}} dx < \infty. \tag{2.1}$$

We can now state our main result on admissible weights.

Theorem 2.1 *Assume that $u_0\phi, u_{0,x}\phi, \rho_0\phi \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$ for an admissible weight function ϕ of Eq. (1.6). Let $(u_0, \rho_{0-1}) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 2$, and $T > 0$ be the maximal existence time of the solution (u, ρ) to system (1.6) with the initial data (u_0, ρ_0). Then, for all $t \in [0, T]$, there is a constant $C > 0$ depending only on weight ϕ such that*

$$\|u(t)\phi\|_{L^p} + \|u_x(t)\phi\|_{L^p} + \|\rho(t)\phi\|_{L^p} \leq (\|u_0\phi\|_{L^p} + \|u_{0,x}\phi\|_{L^p} + \|\rho_0\phi\|_{L^p}) \exp\{\{C(1+M)t\},$$

where

$$M \doteq \sup_{t\in[0,T]} (\|u(t)\|_{L^\infty} + \|u_x(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty}) < \infty.$$

First, we present some standard definitions. In general a weight function is simple a non-negative function. A weight function $v : \mathbb{R}^n \to \mathbb{R}$ is sub-multiplicative if

$$v(x+y) \leq v(x)v(y), \quad \forall x, y \in \mathbb{R}^n. \tag{2.2}$$

Given a sub-multiplicative function $v : \mathbb{R}^n \to \mathbb{R}$, by definition a positive function ϕ is v-moderate if

$$\exists C_0 > 0 : \phi(x+y) \leq C_0 v(x)\phi(y), \forall x, y \in \mathbb{R}^n. \tag{2.3}$$

If ϕ is v-moderate for some sub-multiplicative function v, then we say that ϕ is moderate. Let us recall the most standard examples of such weights.

Example 2.1 ([3]) let $\phi(x) = \phi_{a,b,c,d} = e^{a|x|^b}(1+|x|)^c \log(e+|x|)^d$. Then
(1) For $a, c, d > 0$ and $0 \leq b \leq 1$, such weight is sub-multiplicative.
(2) For $|a| \leq \alpha$, $b \leq \beta$, $|c| \leq \gamma$ and $|d| \leq \delta$, $\phi_{a,b,c,d}$ is $\phi_{\alpha,\beta,\gamma,\delta} -$ moderate.
If v and ϕ are continuous, then they admit the following properties, which may shed some light on Definition 2.1.
(1) If $v \neq 0$ is an even sub-multiplicative weight function, then $inf_{\mathbb{R}} v \geq 1$.
(2) Every nontrivial sub-multiplicative or moderate weight grows and decays not faster than exponentially: there exists $a \geq 0$ such that

$$e^{-a}e^{-a|x|} \leq \phi(x) \leq e^a e^{a|x|}. \tag{2.4}$$

(3) Let ϕ be a locally continuous $v-$ moderate weight such that $C_0 v(0) = 1$ (where C_0 is the constant in (2.8). If v has both left and right derivatives at the origin, then for a.e. $y \in \mathbb{R}$,

$$|\phi'(y)| \leq A\phi(y) \tag{2.5}$$

where $A = C_0 \max\{|v'(0-)|, |v'(0+)|\}$.

The interest of imposing the sub-multiplicativity condition on a weight function is also made clear by the following proposition:

Proposition 2.1 *([3]) Let $v : \mathbb{R}^n \to \mathbb{R}^+$ and $C_0 > 0$. Then the following conditions are equivalent:*
(1) $\forall x, y : v(x+y) \leq C_0 v(x)v(y)$.
(2) For all $1 \leq p, q, r \leq \infty$ and for any measurable function $f_1, f_2 : \mathbb{R}^n \to \mathcal{C}$ the weighted Young inequality hold:

$$\|(f_1 * f_2)v\|_r \leq C_0 \|f_1 v\|_p \|f_2 v\|_q, \, 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{2.6}$$

The moderateness of a weight function is the good condition for weighed Young inequalities with two different weights.

Proposition 2.2 *([3]) Let $1 \leq p \leq \infty$ and v be a sub-multiplicative weight on $\mathbb{R}^n$. Then the following two conditions are equivalent:*
(1) ϕ is $v-$ moderate weight function (with constant C_0).
(2) For all measurable function f_1 and f_2 the weighted Young estimate holds

$$\|(f_1 * f_2)\phi\|_p \leq C_0 \|f_1 v\|_1 \|f_2 \phi\|_p. \tag{2.7}$$

Definition 2.2 An admissible weight function for the problem (1.6) is a locally absolutely continuous function $\phi : \mathbb{R} \to \mathbb{R}$ s.t. for some $A > 0$ and a.e. $x \in \mathbb{R}$, $|\phi'(t)| \leq A|\phi|$, and that is $v - moderate$, for some sub-multiplicative weight function v satisfying $\inf_{x\in\mathbb{R}} v > 0$ and

$$ve^{-|\cdot|} \in L^p(\mathbb{R}). \tag{2.8}$$

Now, we give the needed results to pursue our goal. The DGH system (1.6) can be written in the following "transport" type:

$$\begin{cases} u_t + (\sigma u - \gamma)u_x = -\partial_x G * \left[\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2\right], t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), x \in \mathbb{R}. \end{cases} \tag{2.9}$$

The local well-posedness results for the system (1.6) has been established in the Sobolev space $H^s \times H^{s-1}$, let us recall it as

Theorem 2.2 [30] *If* $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s \geq 2$, *then there exists a maximal time* $T = T(\|(u_0, \rho_{0-1})\|_{H^s \times H^{s-1}}) > 0$ *and a unique solution* $(u, \rho - 1)$ *of system (1.6) in* $C([0, T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$ *with* $(u, \rho - 1)|_{t=0} = (u_0, \rho_0 - 1)$. *Moreover, the solution depends continuously on the initial data, and* T *is independent of* s.

As introduced in the introduction, we consider the following two associated Lagrangian scales of the generalized two component system (1.6):

$$\begin{cases} \frac{\partial q_1}{\partial t} = u(t, q_1), & 0 < t < T, \\ q_1(0, x) = x, & x \in \mathbb{R} \end{cases} \tag{2.10}$$

and

$$\begin{cases} \frac{\partial q_2}{\partial t} = \sigma u(t, q_2) - \gamma, & 0 < t < T, \\ q_2(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{2.11}$$

where $u \in C^1([0, T), H^{s-1})$ is the first component of the solution (u, ρ) to Eq. (1.6) with initial data (u_0, ρ_0) and $T > 0$ is the maximal time of existence.

A direct calculation shows

$$q_{1,tx}(t, x) = u_x(t, q_1(t, x))q_{1,x}(t, x)$$

and

$$q_{2,tx}(t, x) = \sigma u_x(t, q_2(t, x))q_{2,x}(t, x).$$

Thus for $t > 0, x \in \mathbb{R}$

$$q_{1,x}(t, x) = \exp\left(\int_0^t u_x(\tau, q_1(\tau, x))\right) d\tau > 0, (t, x) \in [0, T) \times \mathbb{R}$$

and

$$q_{2,x}(t, x) = \exp\left(\int_0^t \sigma u_x(\tau, q_2(\tau, x))\right) d\tau > 0, (t, x) \in [0, T) \times \mathbb{R},$$

indicating that $q_1(t, \cdot) : \mathbb{R} \to \mathbb{R}$ and $q_2(t, \cdot) : \mathbb{R} \to \mathbb{R}$ are diffeomorphisms of the line for each $t \in [0, T)$. Hence, the L^∞ norm of any function $v(t, \cdot) \in L^\infty(\mathbb{R}), T \in [0, t)$ is preserved under the family of diffeomorphisms $q_1(t, \cdot)$ and $q_2(t, \cdot)$ with $t \in [0, T)$, that is

$$\|v(t, \cdot)\|_{L^\infty(\mathbb{R})} = \|v(t, q_1(t, \cdot))\|_{L^\infty(\mathbb{R})} = \|v(t, q_2(t, \cdot))\|_{L^\infty(\mathbb{R})}, t \in [0, T). \tag{2.12}$$

Similarly, we obtain

$$\inf_{x\in\mathbb{R}} v(t,x) = \inf_{x\in\mathbb{R}} v(t,q_1(t,x)) = \inf_{x\in\mathbb{R}} v(t,q_2(t,x)), t\in[0,T) \tag{2.13}$$

and

$$\sup_{x\in\mathbb{R}} v(t,x) = \sup_{x\in\mathbb{R}} v(t,q_1(t,x)) = \sup_{x\in\mathbb{R}} v(t,q_2(t,x)), t\in[0,T). \tag{2.14}$$

Lemma 2.1 ([26]) *Let* $(u_0,\rho_{0-1}) \in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$ *with* $s\geq 2$*, and* $T>0$ *be the maximal existence time of the solution* (u,ρ) *to system (1.6) with the initial data* (u_0,ρ_0)*.*

$$\rho(t,q(t,x))q_x(t,x) = \rho_0(x), (t,x)\in[0,T)\times\mathbb{R}. \tag{2.15}$$

Moreover, if there exists $x_0\in\mathbb{R}$ *such that* $\rho_0(x_0)=0$*, then* $\rho(t,q(t,x_0))=0$ *for all* $t\in[0,T)$*.*

Proof of Theorem 2.1. Let (u,ρ) be the solution to Eq. (1.6) with the initial data (u_0,ρ_0), and T be the maximal existence times of the solution (u,ρ), which is guaranteed by Theorem 2.2.

From the solution $(u,\rho)\in C([0,T),H^s(\mathbb{R}))\times C([0,T),H^{s-1}), s\geq 2$. The Sobolev's embedding theorem yields

$$M \equiv \sup_{t\in[0,T]} (\|u(t,\cdot)\|_{L^\infty} + \|u_x(t,\cdot)\|_\infty + \|\rho(t,\cdot)\|_{L^\infty}) < \infty. \tag{2.16}$$

For any $N\in Z^+$, let us consider the N-trancations of $\phi(x)$: $f(x)=f_N(x)=\min\{\phi(x),N\}$, then $f:\mathbb{R}\to\mathbb{R}$ is a locally absolutely function such that $\|f\|_\infty\leq N, |f'(x)|\leq A|f(x)|$ a.e on $\mathbb{R}$.

In addition, if $C_1=\max\{C_0,\alpha^{-1}\}$, where $\alpha=\inf_{x\in\mathbb{R}} v(x)>0$, then

$$f(x+y)\leq C_1 v(x) f(y), \quad \forall x,y\in\mathbb{R}.$$

Moreover, as shown in [3], the $N-$trunctions f of a $v-$moderate weight ϕ are uniformly $v-$ moderate with respect to N.

We rewrite Eq. (1.6) as the following form

$$\begin{cases} u_t + (\sigma u-\gamma)u_x = -\partial_x G * P(u,\rho), & t>0, x\in\mathbb{R},\\ \rho_t + (u\rho)_x = 0, & t>0, x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R},\\ \rho(0,x)=\rho_0(x), & x\in\mathbb{R}, \end{cases} \tag{2.17}$$

where $P(u,\rho)\doteq[\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+(\gamma-A)u+\frac{1}{2}\rho^2]$ and $G(x)=\frac{1}{2}e^{-|x|}$.

Let us start from the case $1 \leq p < \infty$, multiplying the first equation in Eq. (2.17) by $|uf|^{p-1}sgn(uf)f$ and integrating it lead to

$$\int_{\mathbb{R}} |uf|^{p-1}sgn(uf)(\partial_t uf)dx = \int_{\mathbb{R}} |uf|^{p-1}sgn(uf)f(\sigma u - \gamma)u_x dx \\ - \int_{\mathbb{R}} |uf|^{p-1}sgn(uf)f\partial_x(G * P(u,\rho))dx. \tag{2.18}$$

The first term on the left hand of (2.18) reads

$$\int_{\mathbb{R}} |uf|^{p-1}sgn(uf)(\partial_t uf)dx = \frac{1}{p}\frac{d}{dt}\|uf\|_{L^p}^{p} = \|uf\|_{L^p}^{p-1} = \|uf\|_{L^p}^{p-1}\frac{d}{dt}\|uf\|_{L^p}.$$

Then, the Hölder inequality is followed by the estimate

$$|\int_{\mathbb{R}} |uf|^{p-1}sgn(uf)(\sigma u - \gamma)u_x f dx| \leq \|uf\|_{L^p}^{p-1}\|(\sigma u - \gamma)u_x f\|_{L^p} \\ \leq (|\sigma M| + |\gamma|)\|uf\|_{L^p}^{p-1}\|u_x f\|_{L^p}.$$

For the nonlocal term, we have

$$|\int_{\mathbb{R}} |uf|^{p-1}sgn(uf)(f\partial_x(G * P(u,\rho)))dx| \\ \leq \|uf\|_{L^p}^{p-1}\|f\partial_x(G * P(u,\rho))\|_{L^p} \\ \leq C_{\alpha,b,k}\|uf\|_{L^p}^{p-1}\{\|(\partial_x G)v\|_{L^1}\|f\cdot(u + u^2 + u_x^2 + \rho^2)\|_{L^p}\} \\ \leq C(1+M)\|uf\|_{L^p}^{p-1}(\|uf\|_{L^p} + \|u_x f\|_{L^p} + \|\rho f\|_{L^p}).$$

where the Hölder's inequality, Propositions 3.1 and 3.2 in [3], and condition (2.1) are applied in the first inequality, the second one, and the last one, respectively, and the constant C only depends on v and ϕ. From Eq. (2.18) one may get

$$\frac{d}{dt}\|uf\|_{L^p} \leq C(1+M)(\|uf\|_{L^p} + \|u_x f\|_{L^p} + \|\rho f\|_{L^p}). \tag{2.19}$$

Let us now give the estimate on $u_x f$. Differentiating the first equation in Eq. (2.17) with respect to the variable x and then multiplying by f, we may arrive at

$$\partial_t(u_x f) + \sigma u_x^2 f + uu_{xx} f + f\partial_x^2(G * P(u,\rho)) = 0.$$

which yields

$$\int_{\mathbb{R}} |u_x f|^{p-1}sgn(u_x f)\partial_t(u_x f)dx = \|u_x f\|_{L^p}^{p-1}\frac{d}{dt}\|u_x f\|_{L^p},$$

$$|\int_{\mathbb{R}} |u_x f|^{p-1} sgn(u_x f) f u_x^2 dx| \leq \|u_x f\|_{L^p}^{p-1} \|u_x f u_x\|_{L^p} \leq M \|u_x f\|_{L^p}^{p-1} \|u_x f\|_{L^p}$$

and

$$\begin{aligned} &|\int_{\mathbb{R}} |u_x f|^{p-1} sgn(u_x f)[f \partial_x^2 (G * P(u, \rho))] dx| \\ &\leq \|u_x f\|_{L^p}^{p-1} \|f \partial_x^2 (G * P(u, \rho))\|_{L^p} \\ &\leq C(1+M) \|u_x f\|_{L^p}^{p-1} (\|uf\|_{L^p} + \|u_x f\|_{L^p} + \|\rho f\|_{L^p}). \end{aligned}$$

For the second order derivative term, we have

$$\begin{aligned} |\int_{\mathbb{R}} |u_x f|^{p-1} sgn(u_x f) u u_{xx} f dx| &= |\int_{\mathbb{R}} |u_x f|^{p-1} sgn(u_x f) u [\partial_x (u_x f) - u_x f_x] dx| \\ &= |\int_{\mathbb{R}} u \partial_x \left(\frac{|u_x f|^p}{p} \right) dx \\ &\quad - \int_{\mathbb{R}} |u_x f|^{p-1} sgn(u_x f) u u_x f_x dx| \\ &\leq M(1+A) \|u_x f\|_{L^p}^p \end{aligned}$$

where the inequality $|f_x(x)| \leq Af(x)$ for a.e. x is applied. Thus, it follows that

$$\frac{d}{dt} \|u_x f\|_{L^p} \leq C_3 (1+M)(\|uf\|_{L^p} + \|u_x f\|_{L^p} + \|\rho f\|_{L^p}). \tag{2.20}$$

We now multiply the second equation in Eq. (2.17) with $|\rho f|^{p-1} sgn(\rho f) f$ and integrate to obtain the identity

$$\frac{1}{p} \frac{d}{dt} \|\rho f\|_{L^p}^p + \int_{\mathbb{R}} |\rho f|^{p-1} sgn(\rho f) f u \rho_x dx + \int_{\mathbb{R}} |\rho f|^{p-1} sgn(\rho f) f u_x \rho dx = 0.$$

As above, we get

$$\int_{\mathbb{R}} |\rho f|^{p-1} sgn(\rho f) f u_x \rho dx| \leq \|\rho f\|_{L^p}^{p-1} \|f u_x \rho\|_{L^p} \leq M \|\rho f\|_{L^p}^{p-1} \|\rho f\|_{L^p},$$

and

$$\begin{aligned} \int_{\mathbb{R}} |\rho f|^{p-1} sgn(\rho f) f u \rho_x dx| &= |\int_{\mathbb{R}} |\rho f|^{p-1} sgn(\rho f) u [\partial_x (\rho f) - \rho f_x] dx| \\ &= |\int_{\mathbb{R}} u \partial_x \left(\frac{|\rho f|^p}{p} \right) dx - \int_{\mathbb{R}} |\rho f|^{p-1} sgn(\rho f) u \rho f_x dx| \\ &\leq M(1+A) \|\rho f\|_{L^p}^p, \end{aligned}$$

this yields

$$\frac{d}{dt}\|\rho f\|_{L^p} \le C_4 M \|\rho f\|_{L^p}. \tag{2.21}$$

Based on the inequalities (2.19)–(2.21), by Gronwall's inequality

$$\begin{aligned}&\|u(t)f\|_{L^p} + \|u_x(t)f\|_{L^p} + \|\rho(t)f\|_{L^p}\\&\quad\le (\|u_0 f\|_{L^p} + \|u_{0,x} f\|_{L^p} + \|\rho_0 f\|_{L^p}) \exp(C(1+M)t),\end{aligned}$$

for all $t \in [0, T)$. Since $f(x) = f_N(x) \uparrow \phi(x)$ as $N \to \infty$ for a.e. $x \in \mathbb{R}$ and $u_0\phi, u_{0,x}\phi, \rho_0\phi \in L^p(\mathbb{R})$ the assertion of the theorem follows for the case $p \in [1, \infty)$. Since $\|\cdot\|_{L^\infty} = \lim_{p\to\infty} \|\cdot\|_{L^p}$ it is clear that the theorem also applies for $p = \infty$. □

Corollary 2.1 *Let $1 \le p \le \infty$ and ϕ be a $v-$ moderate weight function as in Definition 2.1 satisfying $ve^{-|\cdot|} \in L^p(\mathbb{R})$. Assume that $u_0\phi, u_{0,x}\phi, \rho_0\phi \in L^p(\mathbb{R})$ and $u_0\phi^{\frac12}, u_{0,x}\phi^{\frac12}, \rho_0\phi^{\frac12} \in L^2(\mathbb{R})$. Let also $(u, \rho) \in C([0, T), H^s(\mathbb{R})) \times C([0, T), H^{s-1}(\mathbb{R}))$, $s \ge 2$ be the strong solution of the Cauchy problem for Eq. (1.6) emanating from (u_0, ρ_0). Then*

$$\sup_{t\in[0,T)} (\|u(t)\phi\|_{L^p} + \|u_x(t)\phi\|_{L^p} + \|\rho(t)\phi\|_{L^p}) < \infty$$

and

$$\sup_{t\in[0,T)} (\|u(t)\phi^{1/2}\|_{L^p} + \|u_x(t)\phi^{1/2}\|_{L^p} + \|\rho(t)\phi^{1/2}\|_{L^p}) < \infty.$$

Proof As explained in [3], if the function ϕ is a $v-$moderate weight function, then the function $\phi^{\frac12}$ is also a $v^{1/2}-$moderate weight satisfying $|(\phi^{1/2})'(x)| \le \frac{A}{2}\phi^{1/2}(x)$, $\inf v^{1/2} > 0$ and $v^{1/2}e^{-|\cdot|} \in L^1(\mathbb{R})$. We use Theorem 2.1 with $p = 2$ to the weight $\phi^{1/2}$ and obtain

$$\begin{aligned}&\|u(t)\phi^{1/2}\|_{L^2} + \|u_x(t)\phi^{1/2}\|_{L^2} + \|\rho(t)\phi^{1/2}\|_{L^2}\\&\quad\le (\|u_0(t)\phi^{1/2}\|_{L^2} + \|u_{0,x}(t)\phi^{1/2}\|_{L^2} + \|\rho_0(t)\phi^{1/2}\|_{L^2}) \exp(C(1+M)t).\end{aligned} \tag{2.22}$$

In view of Proposition 3.2 in [3], noticing $f(x) = f_N(x) = \min\{\phi(x), N\}$ admits

$$\begin{aligned}&\|f\partial_x(G * P(u, u_x))\|_{L^p}\\&\quad\le C_{\alpha,\kappa,b}\|f\partial_x(G(x) * (u + u^2 + u_x^2 + \rho^2))\|_{L^p}\\&\quad\le C\|f\partial_x G(x)\|_{L^p}\|f(u + u^2 + u^3 + u^4 + u_x^2)\|_{L^1}\\&\quad\le C\|fe^{-|x|}\|_{L^p}(\|fu\|_{L^1} + \|f^{1/2}u\|_{L^2}^2 + \|f^{1/2}u_x\|_{L^2}^2 + \|f^{1/2}\rho\|_{L^2}^2)\\&\quad\le C_1\exp(C_2(1+M)t),\end{aligned} \tag{2.23}$$

where we used (2.22) and Theorem 2.1 with $p = 1$.

Similarly, noticing $\partial_x^2 G = G - \delta$ reveals

$$\begin{aligned}&\|f\partial_x^2(G*P(u,u_x))\|_{L^p}\\&\quad\le C_1\exp(C_2(1+M)t)+C_3(1+M)(\|uf\|_{L^p}+\|fu_x\|_{L^p}+\|f\rho\|_{L^p}),\end{aligned} \tag{2.24}$$

where the constants on the right-hand side of Eqs. (2.23) and (2.24) are independent of N.

By using the procedure as shown in the proof of Theorem 2.1, we can readily obtain

$$\frac{d}{dt}\|uf\|_{L^p}\le C(1+M)\|uf\|_{L^p}+\|f\partial_x(G*P(u,u_x))\|_{L^p} \tag{2.25}$$

and

$$\frac{d}{dt}\|u_xf\|_{L^p}\le C(1+M)\|u_xf\|_{L^p}+\|f\partial_x^2(G*P(u,u_x))\|_{L^p} \tag{2.26}$$

for $1\le p<\infty$. Plugging Eqs. (2.23) and (2.24) into Eqs. (2.25) and (2.26), respectively, and summing up them, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|u(t)f\|_{L^p}+\|u_x(t)f\|_{L^p}+\|\rho(t)f\|_{L^p})\\&\quad\le K_1(1+M)(\|u_0f\|_{L^p}+\|u_{0,x}f\|_{L^p}+\|\rho_0f\|_{L^p})+C_1\exp(C_2(1+M)t),\end{aligned}$$

which is taken integration and limit $N\to\infty$ to get the conclusion in the case $1\le p<\infty$. The constants throughout the proof are independent of p. Therefore, for $p=\infty$ one can rely on the result established for the finite exponents q and then let $q\to\infty$. The argument is fully similar to that of Theorem 2.1. □

3 Blow-up

For $\sigma=1,\gamma=0$, we investigate the precise blow-up scenario of strong solution to system (1.6). Firstly, we present the following convolution estimates, which is the key technical issue for the blow-up analysis.

Lemma 3.1 *Let* $-1\le a\le 3$, $-1\le\beta\le 1$. *Then*

$$(G\pm\beta\partial_xG)*\left(\frac{a}{2}u^2+\frac{3-a}{2}u_x^2-\gamma u\right)\ge\delta_a\left(u-\frac{\gamma}{a}\right)^2-\frac{\gamma^2}{2a}, \tag{3.1}$$

where $\delta_a=\frac{\sqrt{3-a}}{4}(\sqrt{3(1+a)}-\sqrt{3-a})$.

Proof We denote by $+$ and $-$ the characteristic function of R^+ and R^- respectively. Then, we obtain

$$G_+ * \left(\frac{a}{2}u^2 + \frac{3-a}{2}u_x^2 - \gamma u\right) = G_+ * \left(\frac{a}{2}(u^2 - \frac{2\gamma}{a}u) + \frac{3-a}{2}u_x^2\right)$$
$$= G_+ * \left(\frac{a}{2}(u - \frac{\gamma}{a})^2 + \frac{3-a}{2}u_x^2\right) - \frac{\gamma^2}{4a}$$
$$= \frac{3-a}{2}G_+ * \left(\frac{a}{3-a}(u - \frac{\gamma}{a})^2 + u_x^2\right) - \frac{\gamma^2}{4a}.$$

Let $r \in \mathbb{R}$, using Cauchy inequality, we have

$$G_+ * \left(r^2\left(u - \frac{\gamma}{a}\right)^2 + u_x^2\right) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\xi}\left(r^2\left(u - \frac{\gamma}{a}\right)^2 + u_x^2\right)d\xi$$
$$\geq re^{-x}\int_{-\infty}^{x}\left(u - \frac{\gamma}{a}\right)u_x e^{\xi}\,d\xi$$
$$= \frac{r}{2}\left(u - \frac{\gamma}{a}\right)^2 - rG_+ * \left(u - \frac{\gamma}{a}\right)^2.$$

This leads to

$$G_+ * \left((r^2 + r)\left(u - \frac{\gamma}{a}\right)^2 + u_x^2\right) \geq \frac{r}{2}\left(u - \frac{\gamma}{a}\right)^2.$$

Similarly, we get

$$G_- * \left((r^2 + r)\left(u - \frac{\gamma}{a}\right)^2 + u_x^2\right) \geq \frac{r}{2}\left(u - \frac{\gamma}{a}\right)^2.$$

Choose r such that $r^2 + r = \frac{a}{3-a}$. This is indeed possible if $-1 \leq a < 3$(if $a = 3$, the proposition is trivial and there is nothing to prove). So,

$$G_+ * \left(\frac{a}{2}u^2 + \frac{3-a}{2}u_x^2 - \gamma u\right) \geq \frac{\delta_a}{2}\left(u - \frac{\gamma}{a}\right)^2 - \frac{\gamma^2}{4a}, \tag{3.2}$$
$$G_- * \left(\frac{a}{2}u^2 + \frac{3-a}{2}u_x^2 - \gamma u\right) \geq \frac{\delta_a}{2}\left(u - \frac{\gamma}{a}\right)^2 - \frac{\gamma^2}{4a}, \tag{3.3}$$

where $\delta_a = \frac{\sqrt{3-a}}{4}(\sqrt{3(1+a)} - \sqrt{3-a})$.

If $-1 \leq \beta \leq 1$, then from (3.2) and (3.3), we deduce

$$(G \pm \beta\partial_x G) * \left(\frac{a}{2}u^2 + \frac{3-a}{2}u_x^2 - \gamma u\right) \geq \delta_a\left(u - \frac{\gamma}{a}\right)^2 - \frac{\gamma^2}{2a}. \tag{3.4}$$

This completes the proof of Lemma 3.1. □

The blow-up of solution will rely on the following basic property:

Lemma 3.2 ([5]) *Let* $0 < T^* \le \infty$ *and* $f, g \in C^1([0, T^*], \mathbb{R})$ *be such that, for some constant* $c > 0$ *and all* $t \in [0, T^*]$,

$$\frac{df}{dt}(t) \ge cf(t)g(t),$$
$$\frac{dg}{dt}(t) \ge cf(t)g(t).$$

If $f(0) > 0$ *and* $g(0) > 0$, *then*

$$T^* \le \frac{1}{c\sqrt{f(0)g(0)}}.$$

Next, we give the wave-breaking criterion for $\sigma \neq 0$.

Theorem 3.1 [30] (Wave-breaking criterion for $\sigma \neq 0$). *Let* $(u_0, \rho_{0-1}) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ *with* $s \ge 2$, *and* $T > 0$ *be the maximal existence time of the solution* (u, ρ) *to system* (1.6) *with the initial data* (u_0, ρ_0). *Then the solution blows up in finite time if and only if*

$$\lim_{t \to T^-} \inf_{x \in \mathbb{R}} \sigma u_x(t, x) = -\infty.$$

We are now in a position to give the following local-in-space criterion for finite time blow-up mechanism to system (1.6).

Theorem 3.2 *Let* $(u_0, \rho_{0-1}) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ *with* $s \ge 2$, *and* $T > 0$ *be the maximal existence time of the solution* (u, ρ) *to system* (1.6) *with the initial data* (u_0, ρ_0). *Assume that there exists* $x_0 \in \mathbb{R}$, *such that*

$$\rho_0(x_0) = 0$$

and

$$u_{0,x}(x_0) < -|u_0(x_0) - \frac{A}{2}|, \tag{3.5}$$

then the corresponding solution (u, ρ) *of system* (1.6) *arising from* (u_0, ρ_0) *blows up in finite time. More precisely, the following upper bound estimate for* T^* *holds:*

$$T^* \le \frac{2}{\sqrt{(u_{0,x}(x_0))^2 - (u_0(x_0) - \frac{A}{2})^2}}. \tag{3.6}$$

Proof First, differentiating the equation

$$u_t + uu_x = -\partial_x G * \left[u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\rho^2\right], \tag{3.7}$$

with respect to x variable and applying the identity $G * f - \partial_x^2 G * f = f$, we have

$$\begin{aligned} &u_{tx} + uu_{xx} + u_x^2 \\ &= \frac{1}{2}\rho^2 + u^2 + \frac{1}{2}u_x^2 - Au - G * \left(u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\rho^2\right). \end{aligned} \tag{3.8}$$

When $\sigma = 1$ and $\gamma = 0$ the two characteristics $q_1(t, x)$ defined in (2.10) and $q_2(t, x)$ defined in (2.11) coincide, so we can carry out the analysis along the trajectory. Denote

$$\omega(t) = u(t, q(t; x_0)), n(t) = u_x(t, q(t; x_0)), \xi(t) = \rho(t, q(t; x_0)).$$

Then, we use (3.7), (3.8) and the second equation of (3.1) to obtain the following time derivatives along the flow $q(t, x_0)$,

$$\frac{d\omega(t)}{dt} = -\,\partial_x G * \left(u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\rho^2\right), \tag{3.9}$$

$$\frac{dn(t)}{dt} = -\,\frac{1}{2}n(t)^2 + \omega(t)^2 + \frac{1}{2}\xi(t) - A\omega(t) - G * \left(u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\rho^2\right), \tag{3.10}$$

$$\frac{d\xi(t)}{dt} = -\,n(t)\xi(t). \tag{3.11}$$

From the last equation above and the initial conditions on ρ_0, we get

$$\xi(t) = \xi(0)e^{-\int_0^t n(\tau)d\tau} = \rho_0(x_0)e^{-\int_0^t n(\tau)d\tau} = 0. \tag{3.12}$$

Let us introduce the two C^1 functions of the time variable depending on β,

$$A(t) = \left(\beta\left(\omega(t) - \frac{A}{2}\right) - n(t)\right)(t, q_1(t, x_0)) \tag{3.13}$$

and

$$B(t) = \left(-\beta\left(\omega(t) - \frac{A}{2}\right) - n(t)\right)(t, q_1(t, x_0)). \tag{3.14}$$

Differentiating with respect to t and using (3.9), (3.10), we obtain

$$\begin{aligned} A_t(t) &= \beta\omega(t)_t - n(t)_t \\ &= \beta\left[-\partial_x G * \left(u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\rho^2\right)\right] \end{aligned}$$

$$
\begin{aligned}
&-\left[-\frac{1}{2}n(t)^2+\omega(t)^2-A\omega(t)-G*\left(u^2+\frac{1}{2}u_x^2-Au+\frac{1}{2}\rho^2\right)(t,q_1(t;x_0))\right]\\
&=\frac{1}{2}n(t)^2-\omega(t)^2+A\omega(t)+(G-\beta\partial_x G)*\left(u^2+\frac{1}{2}u_x^2-Au+\frac{1}{2}\rho^2\right)(t,q_1(t;x_0))
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
B_t(t)=&-\beta\omega(t)_t-n(t)_t\\
=&-\beta\left[-\partial_x G*\left(u^2+\frac{1}{2}u_x^2-Au+\frac{1}{2}\rho^2\right]\right.\\
&-\left[-\frac{1}{2}n(t)^2+\omega(t)^2-A\omega(t)-G*\left(u^2+\frac{1}{2}u_x^2-Au+\frac{1}{2}\rho^2\right)(t,q_1(t;x_0))\right]\\
=&\frac{1}{2}n(t)^2-\omega(t)^2+A\omega(t)+(G+\beta\partial_x G)*\left(u^2+\frac{1}{2}u_x^2-Au+\frac{1}{2}\rho^2\right)(t,q_1(t;x_0)).
\end{aligned}
\tag{3.16}
$$

Choose $\beta=1$, it follows from Lemma 3.1 and the fact $G_\pm*\frac{1}{2}\rho^2\geq 0$ that

$$
\begin{aligned}
A_t(t)&\geq\frac{1}{2}\left(n(t)^2-\left(\omega(t)-\frac{A}{2}\right)^2\right)+(G-\partial_x G)*\frac{1}{2}\rho^2\\
&\geq\frac{1}{2}\left[n(t)^2-\left(\omega(t)-\frac{A}{2}\right)^2\right]\\
&=\frac{1}{2}(AB)(t,x_0),
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
B_t(t)&\geq\frac{1}{2}\left(n(t)^2-\left(\omega(t)-\frac{A}{2}\right)^2\right)+(G+\partial_x G)*\frac{1}{2}\rho^2\\
&\geq\frac{1}{2}\left[n(t)^2-\left(\omega(t)-\frac{A}{2}\right)^2\right]\\
&=\frac{1}{2}(AB)(t,x_0).
\end{aligned}
\tag{3.18}
$$

By our assumption on the initial conditions made in Theorem 3.2,

$$
u_{0,x}(x_0)<-|u_0(x_0)-\frac{A}{2}|,
$$

or, equivalently, as

$$
A(0,x_0)>0\quad and\quad B(0,x_0)>0.
$$

It follows from Lemma 3.2 that the corresponding solution (u, ρ) of system (1.6) arising from (u_0, ρ_0) blows up in finite time and the following upper bound estimate for T^* holds:

$$T^* \leq \frac{2}{\sqrt{(u_{0,x}(x_0))^2 - (u_0(x_0) - \frac{A}{2})^2}}.$$

□

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References

1. Aratyn, H., Gomes, J.F., Zimerman, A.H.: On a negative flow of the AKNS hierarchy and its relation to a two component Camassa–Holm equation. Symmetry Integr. Geom. Methods Appl. **12**, 70 (2006)
2. Beals, R., Sattinger, D., Szmigielski, J.: Acoustic scattering and the extended Korteweg de Vries hierarchy. Adv. Math. **140**, 190–206 (1998)
3. Brandolese, L.: Breakdown for the Camassa–Holm equation using decay criteria and persistence in weighted spaces. Int. Math. Res. Not. **22**, 5161–5181 (2012)
4. Brandolese, L.: Local-in-space criteria for blowup in shallow water and dispersive rod equation. Commun. Math. Phys. **330**, 401–414 (2014)
5. Brandolese, L., Cortez, M.F.: On permanent and breaking waves in hyperelastic rods and rings. J. Funct. Anal. **266**, 6954–6987 (2014)
6. Brandolese, L., Cortez, M.F.: Blow-up issues for a class of nonlinear dispersive wave equation. J. Differ. Equ. **256**, 3981–3998 (2014)
7. Bressan, A., Constantin, A.: Global conservative solutions of the Cammassa–Holm equation. Arch. Ration. Mech. Anal. **183**, 215–239 (2007)
8. Camassa, R., Holm, D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**, 1661–1664 (1993)
9. Camassa, R., Holm, D., Hyman, J.: A new integrable shallow water equation. Adv. Appl. Mech. **31**, 1–33 (1994)
10. Chen, C.X., Yan, Y.H.: On the Wave breaking phenomena for the generalized periodic two-component Dullin–Gottwald–Holm system. J. Math. Phys. **53**, 103709 (2012)
11. Chen, R.M., Liu, Y.: Wave-breaking and global existence for a generalized two-component Camassa–Holm system. Int. Math. Res. Not. **6**, 1381–1416 (2011)
12. Chen, M., Liu, S., Zhang, Y.: A 2-component generalization of the Camassa–Holm equation and its solutions. Lett. Math. Phys. **75**, 1–15 (2006)
13. Constantin, A., Kolev, B.: On the geometric approach to the motion of inertial mechanical systems. J. Phys. A **35**(32), R51–R79 (2002)
14. Constantin, A., Lannes, D.: The hydrodynamical relevant of the Camassa–Holm and Degasperi–Procesi equations. Arch. Ration. Mech. Anal. **192**, 165–186 (2009)
15. Constantin, A., McKean, H.P.: A shallow water eqaution on the circle. Commun. Pure Appl. Math. **52**, 949–982 (1999)
16. Constantin, A.: On the scattering problem for the Camassa–Holm equation. Proc. R. Soc. Lond. Ser. A **457**, 953–970 (2001)
17. Constantin, A., Escher, J.: Wave breaking for nonlinear nonlocal shallow water equation. Acta. Math. **181**, 229–243 (1998)
18. Constantin, A., Escher, J.: Well-posedness, global existence and blowup phenomena for a periadic quasi-linear hyperbolic equation. Commun. Pure Appl. Math. **51**, 475–504 (1998)
19. Constantin, A., Strauss, W.A.: Stability of peakons. Commun. Pure Appl. Math. **53**, 603–610 (2000)
20. Constantin, A.: On the blow-up of solutions of a periodic shallow water equation. J. Nonlinear Sci. **10**, 391C399 (2000)
21. Constantin, A.: Global existence of solutions and breaking waves for a shallow watere quation:a geometric approach. Ann. Inst. Fourier (Grenoble) **50**, 321–362 (2000)

22. Constantin, A., Escher, J.: Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. **181**, 229–243 (1998)
23. Constantin, A., Escher, J.: Global existence and blow-up for a shallow water equation. Ann. Sc. Norm. Super. Pisa **26**, 303–328 (1998)
24. de Monvel, A.B., Kostenko, A., Shepelsky, D., Teschl, G.: Long-time asymptotic for the Camassa–Holm equation. SIAM J. Math. Anal. **41**, 1559–1588 (2009)
25. Dullin, H.R., Gottwald, G.A., Holm, D.D.: An integrable shallow water equation with linear and nonlinear dispersion. Phys. Rev. Lett. **87**, 4501–4504 (2001)
26. Escher, J., Lechtenfeld, O., Yin, Z.: Well-posedness and blow-up phenomena for the 2-component Camassa–Holm equation. Discrete Contin. Dyn. Syst. **19**, 493–513 (2007)
27. Fuchssteiner, B., Fokas, A.S.: Symplectic structures, their B*ä*cklund transformations and hereditary symmetries. *Phys. D*, **4**, 47–66 (1981/1982)
28. Gui, G., Liu, Y.: On the Cauchy problem for the two-component Camassa-Holm system. Math. Z. **268**, 45–46 (2011)
29. Gui, G., Liu, Y.: On the global existence and wave-breaking criteria for the two-component Camassa–Holm system. J. Funct. Anal. **258**, 4251–78 (2010)
30. Han, Y., Guo, F., Gao, H.: On solitary waves and wave-breaking phenomena for a generalized two-component integrable Dullin–Gottwald–Holm system. J. Nonlinear Sci. **23**, 617–656 (2013)
31. Henry, D.: Infinite propagation speed for a two component Camassa–Holm equation. Discrete Contin. Dyn. Syst. Ser. B **12**, 597–606 (2009)
32. Himonas, A.A., Misiolek, G., Ponce, G., Zhou, Y.: Persistencee properties and unique continuation of solutions of the Camassa–Holm equation. Commun. Math. Phys. **271**, 511C522 (2007)
33. Ionescu-Kruse, D.: Variational derivation of the Camassa–Holm equation. J. Nonlinear Math. Phys. **14**, 303–312 (2007)
34. Ivanov, R.: Two-component integrable systems modelling shallow water waves: the constant vorticity case. Wave Motion **46**, 389–396 (2009)
35. Jiang, Z., Ni, L., Zhou, Y.: Wave breaking of the Camassa–Holm equation. J. Nonlinear Sci **22**(2), 235–245 (2012)
36. Lenells, J.: A variational approach to the stablity of periodic peakons. J. Nonlinear Math. Phys. **11**, 151–163 (2004)
37. Liu, Y., Olver, P.: Well-posedness and blow-up solutions for an integrable nonlinear dispersive model wave equation. J. Differ. Equ. **162**, 27–63 (2000)
38. Liu, Y., Zhang, P.: Stability of solitary waves and wave-breaking phenomena for the two-component Camassa–Holm system. Int. Math. Res. Not. **11**, 1981–2021 (2010)
39. McKean, H.P.: Breakdown of a shallow water equation. Asian J. Math **2**(4), 867–874 (1998)
40. Misiolek, G.: Classical solutions of the periodic Camassa–Holm equation. Geom. Funct. Anal. **12**(5), 1080–1104 (2002)
41. Molinet, L.: On well-posedness results for Camassa–Holm equation on the line: a survey. J. Nonlinear Math. Phys. **11**(4), 521–533 (2004)
42. Mustafa, O.: On smooth traveling waves of an integrable two-component Camassa–Holm shallow water system. Wave Motion **46**, 397–402 (2009)
43. Olver, P., Rosenau, P.: Tri-Hamiltonian duality between solitons and solitary-wave solution having compact support. Phys. Rev. E **53**, 1900–1906 (1996)
44. Wang, Y., Zhu, M.: Blow-up issues for a two-component system modelling water waves with constant vorticity. Nonlinear Anal. **172**, 163–179 (2018)
45. Zhou, Y.: Wave breaking for a periodic shallow water equation. J. Math. Anal. Appl. **290**, 591–604 (2004)
46. Zhou, Y.: Wave breaking for a shallow water equation. Nonlinear Anal. **57**(1), 137–152 (2004)
47. Zhu, M., Junxiang, X.: On the wave-breaking phenomena for the periodic two-component Dullin–Gottwald–Holm system. J. Math. Anal. Appl. **391**, 415–428 (2012)
48. Zhu, M., Junxiang, X.: On the Cauchy problem for the two-component b-family system. Math. Method Appl. Sci. **36**, 2154–2173 (2013)
49. Zhu, M., Junxiang, X.: On the wave-breaking phenomena and global existence for the periodic two-component b-family system. Electron. J. Differ. Equ. **44**, 1–27 (2013)

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