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Bounded Engel elements in residually finite groups

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Abstract

Let q be a prime. Let G be a residually finite group satisfying an identity. Suppose that for every $x \in G$ there exists a q-power $m = m(x)$ such that the element x^m is a bounded Engel element. We prove that G is locally virtually nilpotent. Further, let d, n be positive integers and w a non-commutator word. Assume that G is a d-generator residually finite group in which all w-values are n-Engel. We show that the verbal subgroup $w(G)$ has $\{d, n, w\}$-bounded nilpotency class.

Keywords Engel elements · Residually finite groups · Verbal subgroups · Non-commutator words

Mathematics Subject Classification 20F45 · 20E26

1 Introduction

Given a group G, an element $g \in G$ is called a (left) Engel element if for any $x \in G$ there exists a positive integer $n = n(x, g)$ such that $[x,_n g] = 1$, where the commutator $[x,_n g]$ is defined inductively by the rules

$$[x,_1 g] = [x, g] = x^{-1}g^{-1}xg \quad \text{and, for } n \geq 2, \quad [x,_n g] = [[x,_{n-1} g], g].$$

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If n can be chosen independently of x, then g is called a (left) n-Engel element, or more generally a bounded (left) Engel element. The group G is an Engel group (resp. an n-Engel group) if all its elements are Engel (resp. n-Engel).

A celebrated result due to Zelmanov [24–26] refers to the positive solution of the Restricted Burnside Problem (RBP for short): every residually finite group of bounded exponent is locally finite. The group G is said to have a certain property locally if any finitely generated subgroup of G possesses that property. An interesting result in this context, due to Wilson [21], states that every n-Engel residually finite group is locally nilpotent. Another result that was deduced following the positive solution of the RBP is that given positive integers m, n, if G is a residually finite group in which for every $x \in G$ there exists a positive integer $q = q(x) \leqslant m$ such that x^q is n-Engel, then G is locally virtually nilpotent [1]. We recall that a group possesses a certain property virtually if it has a subgroup of finite index with that property. For more details concerning Engel elements in residually finite groups see [1–3,17,18].

One of the goals of the present article is to study residually finite groups in which some powers are bounded Engel elements. We establish the following result.

Theorem A *Let q be a prime. Let G be a residually finite group satisfying an identity. Suppose that for every $x \in G$ there exists a q-power $m = m(x)$ such that the element x^m is a bounded Engel element. Then G is locally virtually nilpotent.*

A natural question arising in the context of the above theorem is whether the theorem remains valid with m allowed to be an arbitrary natural number rather than q-power. This is related to the conjecture that if G is a residually finite periodic group satisfying an identity, then the group G is locally finite [23, p. 400]. Note that the hypothesis that G satisfies an identity is really needed. For instance, it is well known that there are residually finite p-groups that are not locally finite [5]. In particular, these groups cannot be locally virtually nilpotent. Similar examples have been obtained independently by Grigorchuk, Gupta-Sidki and Sushchansky and are published in [6,7,19], respectively.

Recall that a group-word $w = w(x_1, \ldots, x_s)$ is a nontrivial element of the free group $F = F(x_1, \ldots, x_s)$ on free generators $x_1, \ldots, x_s$. A word is a commutator word if it belongs to the commutator subgroup F'. A non-commutator word u is a group-word such that the sum of the exponents of some variable involved in it is non-zero. A group-word w can be viewed as a function defined in any group G. The subgroup of G generated by the w-values is called the verbal subgroup of G corresponding to the word w. It is usually denoted by $w(G)$. However, if k is a positive integer and $w = x_1^k$, it is customary to write G^k rather than $w(G)$.

There is a well-known quantitative version of Wilson's theorem, that is, if G is a d-generator residually finite n-Engel group, then G has $\{d, n\}$-bounded nilpotency class. As usual, the expression "$\{a, b, ...\}$-bounded" means "bounded from above by some function which depends only on parameters $a, b, ...$". We establish the following related result.

Theorem B *Let d, n be positive integers and w a non-commutator word. Assume that G is a d-generator residually finite group in which all w-values are n-Engel. Then the verbal subgroup $w(G)$ has $\{d, n, w\}$-bounded nilpotency class.*

A non-quantitative version of the above theorem already exists in the literature. It was obtained in [3, Theorem C].

The paper is organized as follows. In the next section we describe some important ingredients of what are often called "Lie methods in group theory". Theorems A and B are proved in Sects. 3 and 4, respectively. The proofs of the main results rely of Zelmanov's techniques that led to the solution of the RBP [24–26], Lazard's criterion for a pro-p group to be p-adic analytic [9], and a result of Nikolov and Segal [13] on verbal width in groups.

2 Associated Lie algebras

Let L be a Lie algebra over a field $\mathbb{K}$. We use the left normed notation: thus if $l_1, l_2, \ldots, l_n$ are elements of L, then

$$[l_1, l_2, \ldots, l_n] = [\ldots[[l_1, l_2], l_3], \ldots, l_n].$$

We recall that an element $a \in L$ is called *ad-nilpotent* if there exists a positive integer n such that $[x, {}_n a] = 0$ for all $x \in L$. When n is the least integer with the above property then we say that a is ad-nilpotent of index n.

Let $X \subseteq L$ be any subset of L. By a commutator of elements in X, we mean any element of L that can be obtained from elements of X by means of repeated operation of commutation with an arbitrary system of brackets including the elements of X. Denote by F the free Lie algebra over $\mathbb{K}$ on countably many free generators $x_1, x_2, \ldots$. Let $f = f(x_1, x_2, \ldots, x_n)$ be a non-zero element of F. The algebra L is said to satisfy the identity $f \equiv 0$ if $f(l_1, l_2, \ldots, l_n) = 0$ for any $l_1, l_2, \ldots, l_n \in L$. In this case we say that L is PI. Now, we recall an important theorem of Zelmanov [23, Theorem 3] that has many applications in group theory.

Theorem 2.1 *Let L be a Lie algebra over a field generated by a finite set. Assume that L is PI and that each commutator in the generators is ad-nilpotent. Then L is nilpotent.*

2.1 On Lie algebras associated with groups

Let G be a group and p a prime. Let us denote by $D_i = D_i(G)$ the i-th dimension subgroup of G in characteristic p. These subgroups form a central series of G known as the *Zassenhaus–Jennings–Lazard series* (see [8, p. 250] for more details). Set $L(G) = \bigoplus D_i/D_{i+1}$. Then $L(G)$ can naturally be viewed as a Lie algebra over the field $\mathbb{F}_p$ with p elements.

The subalgebra of $L(G)$ generated by D_1/D_2 will be denoted by $L_p(G)$. The nilpotency of $L_p(G)$ has strong influence in the structure of a finitely generated pro-p group G. According to Lazard [10] the nilpotency of $L_p(G)$ is equivalent to G being p-adic analytic (for details see [10, A.1 in Appendice and Sections 3.1 and 3.4 in Ch. III] or [4, 1.(k) and 1.(o) in Interlude A]).

Theorem 2.2 *Let G be a finitely generated pro-p group. If $L_p(G)$ is nilpotent, then G is p-adic analytic.*

Let $x \in G$ and let $i = i(x)$ be the largest positive integer such that $x \in D_i$ (here, D_i is a term of the p-dimensional central series to G). We denote by $\tilde{x}$ the element $xD_{i+1} \in L(G)$. We now quote two results providing sufficient conditions for $\tilde{x}$ to be ad-nilpotent. The first lemma was established in [9, p. 131].

Lemma 2.3 *For any $x \in G$ we have $(ad\,\tilde{x})^p = ad\,(\widetilde{x^p})$. Consequently, if x is of finite order t then $\tilde{x}$ is ad-nilpotent of index at most t.*

Corollary 2.4 *Let x be an element of a group G for which there exists a positive integer m such that x^m is n-Engel. Then $\tilde{x}$ is ad-nilpotent.*

The following result was established by Wilson and Zelmanov in [22].

Lemma 2.5 *Let G be a group satisfying an identity. Then for each prime number p the Lie algebra $L_p(G)$ is PI.*

3 Proof of Theorem A

Recall that a group is locally graded if every nontrivial finitely generated subgroup has a proper subgroup of finite index. Interesting classes of groups (e.g., locally finite groups, locally nilpotent groups, residually finite groups) are locally graded (see [11,12] for more details).

It is easy to see that a quotient of a locally graded group need not be locally graded (see for instance [14, 6.19]). However, the next result gives a sufficient condition for a quotient to be locally graded [11].

Lemma 3.1 *Let G be a locally graded group and N a normal locally nilpotent subgroup of G. Then G/N is locally graded.*

In [23], Zelmanov has shown that if G is a residually finite p-group which satisfies a nontrivial identity, then G is locally finite. Next, we extend this result to the class of locally graded groups.

Lemma 3.2 *Let p be a prime. Let G be a locally graded p-group which satisfies an identity. Then G is locally finite.*

Proof Choose arbitrarily a finitely generated subgroup H of G. Let R be the finite residual of H, i.e., the intersection of all subgroups of finite index in H. If $R = 1$, then H is a finitely generated residually finite group. By Zelmanov's result [23, Theorem 4], H is finite. So it suffices to show that H is residually finite. We argue by contradiction and suppose that $R \neq 1$. By the above argument, H/R is finite and thus R is finitely generated. As R is locally graded we have that R contains a proper subgroup of finite index in H, which gives a contradiction. Since H be chosen arbitrarily, we now conclude that G is locally finite, as well. The proof is complete. □

We denote by $\mathcal{N}$ the class of all finite nilpotent groups. The following result is a straightforward corollary of [21, Lemma 2.1] (see [15, Lemma 3.5] for details).

Lemma 3.3 *Let G be a finitely generated residually-$\mathcal{N}$ group. For each prime p, let R_p be the intersection of all normal subgroups of G of finite p-power index. If G/R_p is nilpotent for each prime p, then G is nilpotent.*

We are now in a position to prove Theorem A.

Proof of Theorem A Recall that G is a residually finite group satisfying an identity in which for every $x \in G$ there exists a q-power $m = m(x)$ such that the element x^m is a bounded Engel element. We need to prove that every finitely generated subgroup of G is virtually nilpotent.

Firstly, we prove that all bounded Engel elements (in G) are contained in the Hirsch-Plotkin radical of G. Let H be a subgroup generated by finitely many bounded Engel elements in G, say $H = \langle h_1, \dots, h_t \rangle$, where h_i is a bounded Engel element in G for every $i = 1, \dots, t$. Since finite groups generated by Engel elements are nilpotent [14, 12.3.7], we can conclude that H is residually-$\mathcal{N}$. As a consequence of Lemma 3.3, we can assume that G is residually-(finite p-group) for some prime p. Let $L = L_p(H)$ be the Lie algebra associated with the Zassenhaus–Jennings–Lazard series

$$H = D_1 \geq D_2 \geq \cdots$$

of H. Then L is generated by $\tilde{h}_i = h_i D_2, i = 1, 2, \dots, t$. Let $\tilde{h}$ be any Lie-commutator in $\tilde{h}_i$ and h be the group-commutator in h_i having the same system of brackets as $\tilde{h}$. Since for any group commutator h in $h_1 \dots, h_t$ there is a q-power $m = m(h)$ and a positive integer $n = n(h)$ such that h^m is n-Engel, Corollary 2.4 shows that any Lie commutator in $\tilde{h}_1 \dots, \tilde{h}_t$ is ad-nilpotent. On the other hand, H satisfies an identity and therefore, by Lemma 2.5, L satisfies some non-trivial polynomial identity. According to Theorem 2.1 L is nilpotent. Let $\hat{H}$ denote the pro-p completion of H. Then $L_p(\hat{H}) = L$ is nilpotent and $\hat{H}$ is a p-adic analytic group by Theorem 2.2. By [4, 1.(n) and 1.(o) in Interlude A]), $\hat{H}$ is linear, and so therefore is H. Clearly H cannot have a free subgroup of rank 2 and so, by Tits' Alternative [20], H is virtually soluble. By [14, 12.3.7], H is nilpotent. Since $h_1, \dots, h_t$ have been chosen arbitrarily, we now conclude that all bounded Engel elements are in the Hirsch-Plotkin radical of G.

Let H be a finitely generated subgroup of G, and K be the subgroup generated by all bounded Engel elements (in G) contained in H. Now, we need to prove that K is a nilpotent subgroup of finite index in H. By the previous paragraph, K is locally nilpotent. By Lemma 3.1, H/K is a locally graded q-group. Since G satisfies a nontrivial identity, by Lemma 3.2, H/K is finite and so, K is finitely generated. From this we deduce that K is nilpotent. The proof is complete. □

4 Proof of Theorem B

Combining the positive solution of the RBP with the result [3, Theorem C] one can show that if u is a non-commutator word and G is a finitely generated residually finite group in which all u-values are n-Engel, then the verbal subgroup $u(G)$ is nilpotent. This section is devoted to obtain a quantitative version of the aforementioned result.

The proof of Theorem B require the following lemmas.

Lemma 4.1 *Let d, m, n positive integers. Let G be a d-generator residually finite group in which x^m is n-Engel for every $x \in G$. Then the subgroup G^m has $\{d, m, n\}$-bounded nilpotency class.*

Proof Let $H = G^m$. By [3, Theorem C], H is locally nilpotent. Moreover, Lemma 3.1 ensures us that the quotient group G/H is locally graded. By Zelmanov's solution of the RBP, locally graded groups of finite exponent are locally finite (see for example [12, Theorem 1]), and so G/H is finite of $\{d, m\}$-bounded order. We can deduce from [14, Theorem 6.1.8(ii)] that H has $\{d, m\}$-boundedly many generators. In particular, H is nilpotent. In order to complete the proof, we need to show that H has $\{d, m, n\}$-bounded class yet.

Note that there exists a family of normal and finite index subgroups $\{N_i\}_{i \in \mathcal{I}}$ in G which are all contained in H such that H is isomorphic to a subgroup of the Cartesian product of the finite quotients H/N_i. We show that all quotients have $\{d, m, n\}$-bounded class. Indeed, we have $H/N_i = (G/N_i)^m$. Note that H is $\{d, m\}$-boundedly generated. Thus, by [13, Theorem 1], H/N_i is $\{d, m\}$-boundedly generated where any generator is an m-th power which is an n-Engel element. By [17, Lemma 2.2], there exists a number c depending only on $\{d, m, n\}$ such that each factor H/N_i has nilpotency class at most c. So H is of nipotency class at most c, as well. The proof is complete. □

A well known theorem of Gruenberg says that a soluble group generated by finitely many Engel elements is nilpotent (see [14, 12.3.3]). We will require a quantitative version of this theorem whose proof can be found in [16, Lemma 4.1].

Lemma 4.2 *Let G be a group generated by m elements which are n-Engel and suppose that G is soluble with derived length d. Then G is nilpotent of $\{d, m, n\}$-bounded class.*

For the reader's convenience we restate Theorem B.

Theorem B *Let d, n be positive integers and w a non-commutator word. Assume that G is a d-generator residually finite group in which all w-values are n-Engel. Then the verbal subgroup $w(G)$ has $\{d, n, w\}$-bounded nilpotency class.*

Proof Let $w = w(x_1, ..., x_r)$ be a non-commutator word. We may assume that the sum of the exponents of x_1 is $k \neq 0$. Substitute 1 for $x_2, ..., x_r$ and an arbitrary element $g \in G$ for x_1. We see that g^k is a w-value for every $g \in G$. Thus every k-th power is n-Engel in G. Lemma 4.1 ensures that G^k has $\{d, n, w, \}$-bounded nilpotency class.

Following an argument similar to that used in the proof of Lemma 4.1 we can deduce that the verbal subgroup $w(G)$ is nilpotent. By Zelmanov's solution of the RBP, locally graded groups of finite exponent are locally finite (see for example [12, Theorem 1]), and so G/G^k is finite of $\{d, w\}$-bounded order. Thus, the verbal subgroup $w(G)$ has $\{d, m, w\}$-bounded derived length.

Note that there exists a family of normal and finite index subgroups $\{N_i\}_{i\in\mathcal{I}}$ in G that are all contained in $w(G)$ such that $w(G)$ is isomorphic to a subgroup of the Cartesian product of the finite quotients $w(G)/N_i$. We show that all quotients $w(G)/N_i$ have $\{d, n, w\}$-bounded class. Indeed, we have $w(G)/N_i = w(G/N_i)$. We also have $w(G)$ is $\{d, w\}$-boundedly generated. By [13, Theorem 3] each quotient $w(G)/N_i$ is $\{d, w\}$-boundedly generated by w-values which are n-Engel elements. Since $w(G)$ has $\{d, m, w\}$-bounded derived length, according to Lemma 4.2 we can deduce that $w(G)/N_i$ has $\{d, n, w\}$-bounded nilpotency class Thus, $w(G)$ has $\{d, n, w\}$-bounded nilpotency class, as well. This completes the proof. □

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