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The generalized and modified Halton sequences in Cantor bases

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Abstract
This paper aims to generalize results that have appeared in Atanassov (Math Balk New Ser 18(1–2):15–32, 2004). We consider here variants of the Halton sequences in a generalized numeration system, called the Cantor expansion, with respect to arbitrary sequences of permutations of the Cantor base. We first show that they provide a wealth of low-discrepancy sequences by giving an estimate of (star) discrepancy bound of the generalized Halton sequence in bounded Cantor bases. Then we impose certain conditions on the sequences of permutations of the Cantor base which are analogous, but not straightforward, to the modified Halton sequence introduced by E.I. Atanassov. We show that this modified Halton sequence in Cantor bases attains a better estimate of the (star) discrepancy bound than the generalized Halton sequence in Cantor bases.

Keywords Halton sequence · van der Corput sequence · Hammersley point set · Low-discrepancy sequence · Pseudorandom number · Cantor expansion

Mathematics Subject Classification Primary 11J71 · 11K38 · 11K45; Secondary 65C10

1 Introduction

Let $\omega = (x_n)_{n=1}^{\infty}$ be a sequence in $[0, 1)^s$. A standard problem in numerical analysis is estimating the integral of a function, through a knowledge of its value at a finite number of points of the sequence. This is known as the Monte Carlo method in the case of stochastic sequences $(x_n)_{n=1}^{N}$ or the quasi-Monte Carlo method in the case of deterministic $(x_n)_{n=1}^{N}$. This is encapsulated in the famous Koksma–Hlawka inequality

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$$\left|\int_{[0,1]^s} f(x)\,dx - \frac{1}{N}\sum_{n=1}^{N} f(x_n)\right| \leq V(f)D_N^*(\omega)$$

for any function f on $[0, 1]^s$ with bounded variation $V(f)$ in the sense of Hardy and Krause, see [19], and for any finite set of points $(x_n)_{n=1}^N$ with discrepancy

$$D_N^*(\omega) = \sup_{J=\prod_{i=1}^s [0,z_i)\subseteq[0,1)^s} \left|\frac{A(J;N;\omega)}{N} - \lambda_s(J)\right|.$$

Here $A(J; N; \omega) = \#\{1 \leq n \leq N : x_n \in J\}$ is the counting function, $\lambda_s(J)$ denotes the s-dimensional Lebesgue measure of J, and the above supremum is taken over all rectangular solids $J = \prod_{i=1}^s [0, z_i)$ with $0 < z_i \leq 1$ $(1 \leq i \leq s)$. Note that $\lambda_s(J) = \prod_{i=1}^s z_i$. For more details on numerical integration, the reader can consult [5,15] or [16]. Evidently, to estimate $\int_{[0,1]^s} f(x)\,dx$ sufficiently precisely, what is needed is a good bound for $D_N^*(\omega)$. The discrepancy is nothing other than a quantitative measure of uniformity of distribution. In particular, the sequence ω is uniformly distributed on $[0, 1)^s$, if and only if $D_N^*(\omega) \to 0$ as $N \to \infty$. In a sense, the faster $D_N^*(\omega)$ decays as a function of N, the better uniformly distributed the sequence ω is. One of the fundamental obstructions in nature in this subject is that there is a limit to how well distributed any sequence can be. This is encapsulated in the elementary inequality $D_N^*(\omega) \geq 1/2^s N$ $(N \in \mathbb{N})$ whose proof makes an entertaining exercise. This opens the door to the deep subject of irregularities of distribution which addresses just what limitations there are to the uniformity of distribution of an arbitrary sequence, and the complementary problem of constructing sequences with discrepancy as small as possible. This latter issue is clearly central to the initial issue mentioned in this paper.

Perhaps the most famous example of a low-discrepancy sequence is the van der Corput sequence. In 1935, van der Corput [4] introduced a procedure to generate low-discrepancy sequences on $[0, 1)$. These sequences are considered to be among the best distributed over $[0, 1)$, and no other infinitely generated sequences can have discrepancy of smaller order of magnitude than van der Corput sequences. The technique of van der Corput is based on a very simple idea. Let $b > 1$ be a natural number. Then every nonnegative integer n has a unique b-adic representation of the form

$$n = \sum_{j=1}^{\infty} n_j b^{j-1} = n_1 + n_2 b + n_3 b^2 + n_4 b^3 + \cdots,$$

where $n_j \in \{0, 1, \ldots, b-1\}$ $(j \in \mathbb{N})$ and only finitely many of the n_j's are nonzero. The van der Corput sequence $(\phi_b(n))_{n=0}^{\infty}$ in base b is constructed by reversing the base b representation of the sequence of nonnegative integers, where the radical-inverse function $\phi_b : \mathbb{N}_0 \to [0, 1)$ is defined by

$$\phi_b\left(\sum_{j=1}^{\infty} n_j b^{j-1}\right) = \sum_{j=1}^{\infty} \frac{n_j}{b^j} = \frac{n_1}{b} + \frac{n_2}{b^2} + \frac{n_3}{b^3} + \cdots.$$

In applications, a generalization of the van der Corput sequence to higher dimensions is more likely to be of practical use. In 1960, this was proposed by J.H. Halton [11]. Given pairwise coprime integers $b_1, \ldots, b_s$ all greater than 1, the sequence $(\phi_{b_1}(n), \ldots, \phi_{b_s}(n))_{n=0}^{\infty}$ is called the Halton sequence in bases $b_1, \ldots, b_s$.

It was known for a long time that the discrepancy of the first N elements of the Halton sequence in bases $b_1, \ldots, b_s$ can be bounded by

$$c\frac{(\log N)^s}{N} + O\left(\frac{(\log N)^{s-1}}{N}\right), \tag{1}$$

for some constant $c = c(b_1, \ldots, b_s) > 0$. For example, this was shown in [9,11,18,19]. It is believed that the order $(\log N)^s/N$ is the best possible for an arbitrary infinite sequence. That this is the case when $s = 1$ was proved by Schmidt [21]. For $s > 1$, the question remains open. We shall call an infinite sequence ω in $[0, 1)^s$ a low-discrepancy sequence if $D_N^*(\omega) = O((\log N)^s/N)$.

The question of how small the constant c in (1) can be is interesting from both a theoretical and a practical viewpoint. The articles referred to above show that this constant depends very strongly on the dimension s. The minimal value for this quantity can be obtained if we choose $b_1, \ldots, b_s$ to be the first s prime numbers. But even in this case, c grows very fast to infinity if s increases. This deficiency was overcome by Atanassov [1] who could improve the constant so that

$$c = c(b_1, \ldots, b_s) = \frac{1}{s!}\prod_{i=1}^{s}\frac{b_i - 1}{\log b_i}. \tag{2}$$

This estimate is so impressive that, when $b_1, \ldots, b_s$ are the first s prime numbers, $c(b_1, \ldots, b_s) \to 0$ as $s \to \infty$.

In another direction of effort to improve the behavior of Halton sequences, several researchers have studied various ways of generalizing their definition by including permutations, chosen either deterministically or randomly, in the radical-inverse function. This idea goes back to [2,6]. Let $\Sigma = (\sigma_j)_{j=1}^{\infty}$ be an arbitrary sequence of permutations of $\{0, 1, \ldots, b-1\}$ which fix 0. The generalized radical-inverse function $\phi_b^{\Sigma} : \mathbb{N}_0 \to [0, 1)$ with respect to Σ is defined by

$$\phi_b^{\Sigma}\left(\sum_{j=1}^{\infty} n_j b^{j-1}\right) = \sum_{j=1}^{\infty}\frac{\sigma_j(n_j)}{b^j} = \frac{\sigma_1(n_1)}{b} + \frac{\sigma_2(n_2)}{b^2} + \frac{\sigma_3(n_3)}{b^3} + \cdots.$$

The sequence $(\phi_b^{\Sigma}(n))_{n=0}^{\infty}$ is a low-discrepancy sequence, and it is called the generalized van der Corput sequence in base b with respect to Σ. The generalized Halton sequences can be introduced in a similar way. In parallel to these efforts, Atanassov also showed in [1] that any generalized Halton sequence attains the same constant as in (2); furthermore, he could produce certain generalized Halton sequences, by means of the so-called "admissible integers," for which the constants $c = c(b_1, \ldots, b_s, \Sigma_1, \ldots, \Sigma_s)$ of the discrepancy bounds have an even better asymptotic behavior than (2).

In this paper, we introduce the generalized Halton sequence in Cantor bases, which is induced by the a-adic integers and which is called the Cantor expansion, and give an estimate of its discrepancy by adapting the techniques developed by Atanassov.

Also, we extend the notion of admissible integers so that we can derive a special type of generalized Halton sequences in Cantor bases with a better estimate of discrepancy bounds. Our work is an extension of [10] and can be viewed as a generalization of Atanassov's results. Note that the van der Corput sequence and some other one-dimensional low-discrepancy sequences with respect to the Cantor expansion were studied in [3,8]. In addition, Halton sequences defined in a more generalized numeration system than the Cantor expansion, called the G-expansion, were mentioned in [13]; however, the paper aimed to study the Halton sequence in some fixed non-integer bases. Furthermore, it is worth noting that several uniformly distributed sequences, which can be constructed through the notions of Cantor-base-additive function and strongly Cantor-base-additive function, were studied in [14]. This paper also included our generalized Halton sequence in Cantor bases as an example; nevertheless, it aimed to provide criteria for uniform distribution and it did not study the discrepancy of those sequences obtained by Cantor-base-additive functions.

We now summarize the contents of this paper. In Sect. 2, we introduce the concept of a generalized numeration system, called the Cantor expansion. Then we define the generalized Halton sequence induced by this generalized system and state our first main result on the estimate of discrepancy of the sequence. In Sect. 3, we impose certain conditions on the sequences of permutations of the Cantor base to produce an extension of the concept of admissible integers. Then we define the modified Halton sequence in the Cantor expansion and state our second main result regarding the discrepancy bound of this special type of sequence. In Sects. 4 and 5, we prove the first and the second main results, respectively. Finally, we introduce in Sect. 6 the generalized Hammersley point set in Cantor bases and show that it provides a wealth of low-discrepancy point sets by giving an estimate of its discrepancy.

We list here the notation which will be used repeatedly throughout the paper. For each natural number $b > 1$, we write $\mathbb{Z}_b = \{0, 1, \ldots, b-1\}$ and $\mathbb{Z}_b^* = \{1, 2, \ldots, b-1\}$. It is also important to note that every permutation in this paper fixes 0.

2 The generalized Halton sequence in Cantor bases

Let $b = (b_j)_{j=1}^{\infty}$ be a sequence of natural numbers greater than 1. Then it is clear that every nonnegative integer n has a unique b-adic representation of the form

$$n = \sum_{j=1}^{\infty} n_j b_1 \cdots b_{j-1} = n_1 + n_2 b_1 + n_3 b_1 b_2 + n_4 b_1 b_2 b_3 + \cdots ,$$

where $n_j \in \mathbb{Z}_{b_j}$ $(j \in \mathbb{N})$ and all but finitely many n_j's are zero. This b-adic representation is also called the Cantor expansion of n with respect to the Cantor base b. Moreover, every real number $x \in [0, 1)$ has a b-adic expansion of the form

$$x = \sum_{j=1}^{\infty} \frac{x_j}{b_1 \cdots b_j} = \frac{x_1}{b_1} + \frac{x_2}{b_1 b_2} + \frac{x_3}{b_1 b_2 b_3} + \cdots ,$$

where $x_j \in \mathbb{Z}_{b_j}$ $(j \in \mathbb{N})$. The x_j can be calculated by the greedy algorithm

$$x_1 = [xb_1] \quad \text{and} \quad x_j = [\{xb_1 \cdots b_{j-1}\}b_j],$$

where $[\alpha]$ and $\{\alpha\}$ denote the integer part and the fraction part of α, respectively. The idea of this generalized numeration system stems from the a-adic integers, which is a class of locally compact topological groups and possesses a symbolic dynamical structure. For more details on the a-adic integers, see [12, pp. 106–117].

Suppose that $\Sigma = (\sigma_j)_{j=1}^{\infty}$ is a sequence of permutations of $\mathbb{Z}_{b_1}, \mathbb{Z}_{b_2}, \mathbb{Z}_{b_3}, \ldots,$ where the permutations all fix 0. We define the generalized radical-inverse function $\phi_b^{\Sigma} : \mathbb{N}_0 \to [0, 1)$ by

$$\phi_b^{\Sigma}\left(\sum_{j=1}^{\infty} n_j b_1 \cdots b_{j-1}\right) = \sum_{j=1}^{\infty} \frac{\sigma_j(n_j)}{b_1 \cdots b_j} = \frac{\sigma_1(n_1)}{b_1} + \frac{\sigma_2(n_2)}{b_1 b_2} + \frac{\sigma_3(n_3)}{b_1 b_2 b_3} + \cdots .$$

The generalized van der Corput sequence in base b with respect to Σ is defined as $(\phi_b^{\Sigma}(n))_{n=0}^{\infty}$. This sequence was studied in [3,8], where it was proved to be a low-discrepancy sequence with some restriction on the Cantor base b. Furthermore, the sequence where all the permutations are identity was shown, without any restriction on the Cantor base, to be uniformly distributed mod 1 in [17] and to be a low-discrepancy sequence in [10].

Let $b_1 = (b_{1,j})_{j=1}^{\infty}, \ldots, b_s = (b_{s,j})_{j=1}^{\infty}$ be s sequences of natural numbers greater than 1 such that, for all $1 \le i_1 < i_2 \le s$ and all $j_1, j_2 \in \mathbb{N}$, b_{i_1,j_1} and b_{i_2,j_2} are coprime. For each $1 \le i \le s$, let $\Sigma_i = (\sigma_{i,j})_{j=1}^{\infty}$ be a sequence of permutations of $\mathbb{Z}_{b_{i,1}}, \mathbb{Z}_{b_{i,2}}, \mathbb{Z}_{b_{i,3}}, \ldots$. The generalized Halton sequence in Cantor bases $b_1, \ldots, b_s$ with respect to $\Sigma_1, \ldots, \Sigma_s$ is defined to be $(\phi_{b_1}^{\Sigma_1}(n), \ldots, \phi_{b_s}^{\Sigma_s}(n))_{n=0}^{\infty}$.

The following theorem is our first main result which gives an estimate of discrepancy of the generalized Halton sequence in bounded Cantor bases.

Theorem 1 *Let $b_1 = (b_{1,j})_{j=1}^{\infty}, \ldots, b_s = (b_{s,j})_{j=1}^{\infty}$ be s bounded sequences of natural numbers greater than 1 such that, for all $1 \le i_1 < i_2 \le s$ and all $j_1, j_2 \in \mathbb{N}$, b_{i_1,j_1} and b_{i_2,j_2} are coprime. For each $1 \le i \le s$ and $j \in \mathbb{N}$, let $\sigma_{i,j}$ be a permutation of $\mathbb{Z}_{b_{i,j}}$. For each $1 \le i \le s$, denote $\Sigma_i = (\sigma_{i,j})_{j=1}^{\infty}$. Suppose that ω is the generalized Halton sequence in Cantor bases $b_1, \ldots, b_s$ with respect to $\Sigma_1, \ldots, \Sigma_s$. Then, for any $N \in \mathbb{N}$, we have*

$$N D_N^*(\omega) \le \sum_{l=0}^{s} \frac{M_{l+1}}{l!} \prod_{i=1}^{l} \left(\frac{\lfloor M_i/2 \rfloor \log N}{\log m_i} + l\right),$$

where $M_i = \max(b_{i,j})_{j=1}^{\infty}$ and $m_i = \min(b_{i,j})_{j=1}^{\infty}$ $(1 \le i \le s)$, and where $M_{s+1} = 1$. In particular, for any $N \in \mathbb{N}$, we obtain

$$D_N^*(\omega) \le c \frac{(\log N)^s}{N} + O\left(\frac{(\log N)^{s-1}}{N}\right)$$

with

$$c = c(b_1, \ldots, b_s) = \frac{1}{s!} \prod_{i=1}^{s} \frac{\lfloor M_i/2 \rfloor}{\log m_i}.$$

This theorem says that the generalized Halton sequence in bounded Cantor bases is a low-discrepancy sequence. In particular, it generalizes the main result in [10, Main Theorem 2.1 and Corollary 2.2], where all the permutations are fixed to be identity. Also, the constant $c = c(b_1, \ldots, b_s)$ in the bound here is essentially as good as that established in [10].

When the sequences $b_1, \ldots, b_s$ are of period one, that is, $M_i = m_i$ for all $1 \le i \le s$, the estimated bound $c = c(b_1, \ldots, b_s)$ of Theorem 1 is exactly the same as that given in [1] for the Halton sequences based on coprime bases. Though the generalized Halton sequence in Cantor bases do not attain a lower estimate of discrepancy bound than the classical Halton sequence, it provides more variety of sequences with similar estimated bound, especially when M_i is large and when the difference $M_i - m_i$ is small compared with M_i for each $1 \le i < s$. In fact, let $c = c(b_1, \ldots, b_s)$ and $c' = c'(b'_1, \ldots, b'_s)$ denote the constants of the estimated bound appeared in Theorem 1 for the generalized Halton sequence in Cantor bases $b_1, \ldots, b_s$ with respect to $\Sigma_1, \ldots, \Sigma_s$ and for the classical Halton sequence in bases $b'_1, \ldots, b'_s$, respectively, such that $M_i = \max(b_{i,j})_{j=1}^{\infty} = b'_i$ for each $1 \le i \le s$. Suppose that, for each $1 \le i \le s$, $k_i \in \mathbb{N}_0$ is a fixed integer and that $M_i = m_i + k_i$, where $m_i = \min(b_{i,j})_{j=1}^{\infty}$. Then we have

$$\frac{c(b_1, \ldots, b_s)}{c'(b'_1, \ldots, b'_s)} = \frac{\frac{1}{s!} \prod_{i=1}^{s} \frac{\lfloor M_i/2 \rfloor}{\log m_i}}{\frac{1}{s!} \prod_{i=1}^{s} \frac{\lfloor M_i/2 \rfloor}{\log M_i}} = \prod_{i=1}^{s} \frac{\log M_i}{\log m_i} = \prod_{i=1}^{s} \frac{\log M_i}{\log(M_i - k_i)}.$$

It is not hard to see that $c(b_1, \ldots, b_s)/c'(b'_1, \ldots, b'_s)$ tends to 1 exponentially fast as M_i's go to infinity.

3 The modified Halton sequence in Cantor bases

In this section, we introduce a special class of generalized Halton sequences in Cantor bases that involves some deep periodicity properties. It can be considered as a generalization of Atanassov's modified Halton sequences. We shall show that this kind of sequences satisfies a better estimate of discrepancy bound than the generalized Halton sequences.

Definition 1 Let $a_1, \ldots, a_n \in \mathbb{Z}$. We shall denote

$$(\overline{a_1, \ldots, a_n}) := (a_1, \ldots, a_n, a_1, \ldots, a_n, a_1, \ldots, a_n, \ldots)$$

to be the periodic sequence of the integers $a_1, \ldots, a_n$. In addition, we shall sometimes abuse the following notation

$$(a_1, \dots, a_n, a_{n+1}, a_{n+2}, a_{n+3}, \dots)$$
$$:= (a_1, \dots, a_n, a_{n+1 \bmod n}, a_{n+2 \bmod n}, a_{n+3 \bmod n}, \dots)$$

to mean the periodic sequence $(\overline{a_1, \dots, a_n})$, when it is clear from the context that $a_{n+1}, a_{n+2}, a_{n+3}, \dots$ are not defined. Here, for each $m \in \mathbb{N}$, $m \bmod n$ denotes the remainder of the Euclidean division of m by n, except $m \bmod n = n$ when m is divisible by n.

Next we introduce the notion of admissible sequences of integers which extends Atanassov's notion of admissible integers.

Definition 2 Let $j_1, \dots, j_s \in \mathbb{N}$. Suppose $p_{1,1}, \dots, p_{1,j_1}, \dots, p_{s,1}, \dots, p_{s,j_s}$ are distinct prime numbers such that, for each $1 \le i \le s$, there exists a common primitive root modulo $p_{i,1}, \dots, p_{i,j_i}$. For each $1 \le i \le s$, let $p_i = (\overline{p_{i,1}, \dots, p_{i,j_i}})$. Periodic sequences of integers $k_1 = (\overline{k_{1,1}, \dots, k_{1,j_1}}), \dots, k_s = (\overline{k_{s,1}, \dots, k_{s,j_s}})$ are said to be admissible for $p_1, \dots, p_s$ if, for each $(d_1, \dots, d_s) \in \mathbb{Z}^*_{p_{1,\ell_1}} \times \dots \times \mathbb{Z}^*_{p_{s,\ell_s}}$ $(1 \le \ell_i \le j_i$, $1 \le i \le s)$, there exists $(\alpha_1, \dots, \alpha_s) \in \mathbb{N}^s$ such that

$$k_{i,1} \cdots k_{i,\alpha_i - 1} \prod_{\substack{1 \le i_0 \le s \\ i_0 \ne i}} p_{i_0,1} \cdots p_{i_0,\alpha_{i_0}-1} \equiv d_i \pmod{p_{i,\alpha_i}} \quad \text{and} \quad \alpha_i \equiv \ell_i \pmod{j_i}$$

for all $1 \le i \le s$.

Note that the existence of admissible sequences for such prime sequences $p_1, \dots, p_s$ in Definition 2 will be proved in Lemma 6.

Definition 3 Let $p_1 = (\overline{p_{1,1}, \dots, p_{1,j_1}}), \dots, p_s = (\overline{p_{s,1}, \dots, p_{s,j_s}})$ be periodic sequences of distinct prime numbers such that, for each $1 \le i \le s$, there exists a common primitive root modulo $p_{i,1}, \dots, p_{i,j_i}$. Suppose $k_1, \dots, k_s$ are admissible sequences for $p_1, \dots, p_s$. For each $1 \le i \le s$, let $\Sigma_i = (\sigma_{i,j})_{j=1}^{\infty}$ be the sequence of permutations of $\mathbb{Z}_{p_{i,1}}, \mathbb{Z}_{p_{i,2}}, \mathbb{Z}_{p_{i,3}}, \dots$ such that

$$\sigma_{i,j} : \mathbb{Z}_{p_{i,j}} \to \mathbb{Z}_{p_{i,j}} : x \mapsto x k_{i,1} \cdots k_{i,j-1} \bmod p_{i,j}.$$

The modified Halton sequence in Cantor bases $p_1, \dots, p_s$ with respect to $k_1, \dots, k_s$ is defined to be $(\phi_{p_1}^{\Sigma_1}(n), \dots, \phi_{p_s}^{\Sigma_s}(n))_{n=0}^{\infty}$.

When the sequences $p_1, \dots, p_s$ in Definition 3 are of period one, i.e. $p_1 = (\overline{p_{1,1}}), \dots, p_s = (\overline{p_{s,1}})$, our modified Halton sequence in Cantor bases is exactly the modified Halton sequence introduced by Atanassov [1].

The notion of admissible sequences seems technical and hard to understand, so it is worth noting here that this condition involves some periodic properties and is used to improve the estimate in (3) for Λ_1. In particular, we shall be considering the distribution of the modified Halton sequence in Cantor bases over an elementary interval, which will be divided into $\#(\mathbb{Z}_{p_{1,\alpha_1}} \times \dots \times \mathbb{Z}_{p_{s,\alpha_s}})$ subintervals. The admissibility condition ensures that there will be the same number of elements of the sequence in each subinterval. These periodic properties will be seen in Lemma 8. Due to the fact that the subintervals of the considered elementary interval are small and that the exact number of elements of the sequence in each subinterval is known, it is possible to make a better estimate of the discrepancy bound for the modified Halton sequence in Cantor bases than for the generalized Halton sequence in Cantor bases.

The following statement is our second main result which gives an estimate of discrepancy bound of the modified Halton sequence in Cantor bases.

Theorem 2 *Let ω be the modified Halton sequence in Cantor bases $p_1, \dots, p_s$ with respect to $k_1, \dots, k_s$. Then, for any $N \in \mathbb{N}$, we have*

$$D_N^*(\omega) \le c\frac{(\log N)^s}{N} + O\left(\frac{(\log N)^{s-1}}{N}\right)$$

with

$$c = c(p_1, \dots, p_s) = \frac{1}{s!}\left(\sum_{i=1}^{s} \log M_i\right)\prod_{i=1}^{s}\frac{M_i(1 + \log M_i)}{(m_i - 1)\log m_i},$$

where $M_i = \max(p_{i,1}, \dots, p_{i,j_i})$ and $m_i = \min(p_{i,1}, \dots, p_{i,j_i})$ $(1 \le i \le s)$.

Note that Theorem 2 gives a lower estimate $c = c(p_1, \dots, p_s)$ than the bound $c = c(b_1, \dots, b_s)$ provided by Theorem 1, when the m_i's are large enough. Also, when the sequences $p_1, \dots, p_s$ are of period one, i.e. $M_i = m_i$ for all $1 \le i \le s$, the estimated bound $c = c(p_1, \dots, p_s)$ of our modified Halton sequence in Cantor bases is indeed the same as that of the modified Halton sequence given in [1]. Although the modified Halton sequences in Cantor bases do not attain a lower estimate of discrepancy bound than Atanassov's modified Halton sequences, our method gives more variety of sequences with similar estimated bound, especially when M_i is large and when the difference $M_i - m_i$ is small compared with M_i for each $1 \le i < s$. This follows from the same argument as that at the end of Sect. 2.

4 Proof of Theorem 1

The proof of Theorem 1 is indeed inspired by and closely related to that given by Atanassov [1]. Moreover, it can be seen as an extension of that given by Haddley et al. [10]. Note that Lemma 1 is required to make the extension of the proof provided by Haddley et al. [10] possible.

In order to prove the theorem, we need the following five lemmas.

The first preliminary result is a variant of the Chinese remainder theorem, and it is used to prove Lemma 2.

Lemma 1 *Let $b_1 = (b_{1,j})_{j=1}^{\infty}, \dots, b_s = (b_{s,j})_{j=1}^{\infty}$ be s arbitrary sequences of natural numbers greater than 1 such that, for all $1 \le i_1 < i_2 \le s$ and all $j_1, j_2 \in \mathbb{N}$, b_{i_1,j_1} and b_{i_2,j_2} are coprime. For each $1 \le i \le s$ and $j \in \mathbb{N}$, let $\sigma_{i,j}$ be a permutation of $\mathbb{Z}_{b_{i,j}}$, and let $f_i : \mathbb{N}_0 \to \mathbb{N}_0$ be a function defined by*

$$f_i(n) = f_i\left(\sum_{j=1}^{\infty} n_{i,j} b_{i,1}\cdots b_{i,j-1}\right) = \sum_{j=1}^{\infty} \sigma_{i,j}(n_{i,j}) b_{i,1}\cdots b_{i,j-1},$$

for every $n \in \mathbb{N}_0$ with the b_i-adic expansion $\sum_{j=1}^{\infty} n_{i,j} b_{i,1}\cdots b_{i,j-1}$. For each $1 \le i \le s$, let α_i be a natural number, and let $l_{i,1} \in \mathbb{Z}_{b_{i,1}}, \ldots, l_{i,\alpha_i} \in \mathbb{Z}_{b_{i,\alpha_i}}$. Then there exists a unique $0 \le n < \prod_{i=1}^{s} b_{i,1}\cdots b_{i,\alpha_i}$ such that, for all $1 \le i \le s$,

$$f_i(n) \equiv l_{i,1} + l_{i,2} b_{i,1} + l_{i,3} b_{i,1} b_{i,2} + \cdots l_{i,\alpha_i} b_{i,1}\cdots b_{i,\alpha_i - 1} \pmod{b_{i,1}\cdots b_{i,\alpha_i}}.$$

Proof For each $1 \le i \le s$, let $b_i^* = b_{i,1}\cdots b_{i,\alpha_i}$. We first prove the uniqueness of the n. Suppose that n and n' are two solutions of all the congruences such that $0 \le n, n' < \prod_{i=1}^{s} b_i^*$. It follows that $f_i(n) \equiv f_i(n') \pmod{b_i^*}$ for all $1 \le i \le s$, that is, we have

$$\sum_{j=1}^{\infty} \sigma_{i,j}(n_{i,j}) b_{i,1}\cdots b_{i,j-1} \equiv \sum_{j=1}^{\infty} \sigma_{i,j}(n'_{i,j}) b_{i,1}\cdots b_{i,j-1} \pmod{b_i^*},$$

and this is equivalent to

$$\sum_{j=1}^{\alpha_i} \sigma_{i,j}(n_{i,j}) b_{i,1}\cdots b_{i,j-1} \equiv \sum_{j=1}^{\alpha_i} \sigma_{i,j}(n'_{i,j}) b_{i,1}\cdots b_{i,j-1} \pmod{b_i^*}.$$

We know that $|\sum_{j=1}^{\alpha_i} \sigma_{i,j}(n_{i,j}) b_{i,1}\cdots b_{i,j-1} - \sum_{j=1}^{\alpha_i} \sigma_{i,j}(n'_{i,j}) b_{i,1}\cdots b_{i,j-1}| < b_i^*$ for each $1 \le i \le s$, and hence this difference must be 0. Therefore, we obtain $\sigma_{i,j}(n_{i,j}) = \sigma_{i,j}(n'_{i,j})$ for all $1 \le i \le s$ and $1 \le j \le \alpha_i$. Since $\sigma_{i,j}$ are all bijective, we have $n_{i,j} = n'_{i,j}$ for all such i and j. It follows that $b_i^* \mid n - n'$ for each i. This implies that $b_1^* \cdots b_s^* \mid n - n'$ because $b_1^*, \ldots, b_s^*$ are pairwise coprime. By the choice of n and n', we must have $n = n'$.

Now we show the existence of such n. Define $F : \mathbb{Z}_{b_1^* \cdots b_s^*} \to \mathbb{Z}_{b_1^*} \times \cdots \times \mathbb{Z}_{b_s^*}$ by

$$F(n) = (f_1(n) \bmod b_1^*, \ldots, f_s(n) \bmod b_s^*).$$

It suffices to show that F is a bijection. By the proof of uniqueness, F must be an injection. For each $1 \le i \le s$, it is clear that f_i is a bijection on $\mathbb{Z}_{b_i^*}$. It follows immediately that F is a surjection since the domain and the codomain of F have the same number of elements. This proves the existence of such n. □

The following lemma is a consequence of the so-called "elementary interval property" satisfied by Halton sequences.

Lemma 2 *Let $b_1 = (b_{1,j})_{j=1}^{\infty}, \ldots, b_s = (b_{s,j})_{j=1}^{\infty}$ be s arbitrary sequences of natural numbers greater than 1 such that, for all $1 \le i_1 < i_2 \le s$ and all $j_1, j_2 \in \mathbb{N}$, b_{i_1,j_1} and b_{i_2,j_2} are coprime. For each $1 \le i \le s$ and $j \in \mathbb{N}$, let $\sigma_{i,j}$ be a permutation of $\mathbb{Z}_{b_{i,j}}$. For each $1 \le i \le s$, denote $\Sigma_i = (\sigma_{i,j})_{j=1}^{\infty}$. Suppose that ω is the generalized*

Halton sequence in Cantor bases $b_1, \ldots, b_s$ with respect to $\Sigma_1, \ldots, \Sigma_s$. Let J be an interval of the form

$$J = \prod_{i=1}^{s} \left[\frac{u_i}{b_{i,1} \cdots b_{i,\alpha_i}}, \frac{v_i}{b_{i,1} \cdots b_{i,\alpha_i}} \right)$$

with integers $0 \leq u_i < v_i \leq b_{i,1} \cdots b_{i,\alpha_i}$ and $\alpha_i \in \mathbb{N}$ $(1 \leq i \leq s)$. Then

$$|A(J; N; \omega) - N\lambda_s(J)| \leq \prod_{i=1}^{s} (v_i - u_i)$$

holds for every $N \in \mathbb{N}$. Moreover, for every $N \leq \prod_{i=1}^{s} b_{i,1} \cdots b_{i,\alpha_i}$, we have $A(J; N; \omega) \leq \prod_{i=1}^{s} (v_i - u_i)$.

Proof For each $n \in \mathbb{N}_0$, we denote the b_i-adic expansion of n by

$$n = \sum_{j=1}^{\infty} n_{i,j} b_{i,1} \cdots b_{i,j-1} = n_{i,1} + n_{i,2} b_{i,1} + n_{i,3} b_{i,1} b_{i,2} + n_{i,4} b_{i,1} b_{i,2} b_{i,3} + \cdots,$$

where $n_{i,j} \in \mathbb{Z}_{b_{i,j}}$ $(j \in \mathbb{N})$. Let $\ell = (l_1, \ldots, l_s) \in \mathbb{N}_0^s$ be such that, for all $1 \leq i \leq s$, we have $0 \leq l_i < b_{i,1} \cdots b_{i,\alpha_i}$ with the expansion

$$l_i = l_{i,\alpha_i} + l_{i,\alpha_i - 1} b_{i,\alpha_i} + l_{i,\alpha_i - 2} b_{i,\alpha_i} b_{i,\alpha_i - 1} + \cdots + l_{i,1} b_{i,\alpha_i} \cdots b_{i,2},$$

where $l_{i,\alpha_i - j} \in \mathbb{Z}_{b_{i,\alpha_i - j}}$ $(0 \leq j \leq \alpha_i - 1)$. We consider the interval

$$J_\ell = \prod_{i=1}^{s} \left[\frac{l_i}{b_{i,1} \cdots b_{i,\alpha_i}}, \frac{l_i + 1}{b_{i,1} \cdots b_{i,\alpha_i}} \right).$$

Then the nth element ω_n of the generalized Halton sequence in Cantor bases is contained in J_ℓ if and only if, for all $1 \leq i \leq s$,

$$\begin{aligned} \frac{l_{i,1}}{b_{i,1}} + \cdots + \frac{l_{i,\alpha_i}}{b_{i,1} \cdots b_{i,\alpha_i}} &\leq \frac{\sigma_{i,1}(n_{i,1})}{b_{i,1}} + \frac{\sigma_{i,2}(n_{i,2})}{b_{i,1} b_{i,2}} + \cdots \\ &< \frac{l_{i,1}}{b_{i,1}} + \cdots + \frac{l_{i,\alpha_i}}{b_{i,1} \cdots b_{i,\alpha_i}} + \frac{1}{b_{i,1} \cdots b_{i,\alpha_i}}. \end{aligned}$$

This is however equivalent to $\sigma_{i,1}(n_{i,1}) = l_{i,1}, \ldots, \sigma_{i,\alpha_i}(n_{i,\alpha_i}) = l_{i,\alpha_i}$ which in turn is equivalent to

$$\sum_{j=1}^{\infty} \sigma_{i,j}(n_{i,j}) b_{i,1} \cdots b_{i,j-1} \equiv l_{i,1} + l_{i,2} b_{i,1} + \cdots + l_{i,\alpha_i} b_{i,1} \cdots b_{i,\alpha_i - 1} \pmod{b_{i,1} \cdots b_{i,\alpha_i}}$$

for all $1 \leq i \leq s$. By Lemma 1, every $\prod_{i=1}^{s} b_{i,1} \cdots b_{i,\alpha_i}$ consecutive elements of the generalized Halton sequence in Cantor bases contain exactly one element in J_ℓ, in other words, $A(J_\ell; t\prod_{i=1}^{s} b_{i,1} \cdots b_{i,\alpha_i}; \omega) = t$ for all $t \in \mathbb{N}$, and hence

$$A\left(J_\ell; t\prod_{i=1}^{s} b_{i,1} \cdots b_{i,\alpha_i}; \omega\right) - \left(t\prod_{i=1}^{s} b_{i,1} \cdots b_{i,\alpha_i}\right)\lambda_s(J_\ell) = 0.$$

Therefore, for every $N \in \mathbb{N}$, we obtain

$$|A(J_\ell; N; \omega) - N\lambda_s(J_\ell)| \leq 1.$$

Now we write the interval J as a disjoint union of intervals of the form J_ℓ,

$$J = \bigcup_{l_1=u_1}^{v_1-1} \cdots \bigcup_{l_s=u_s}^{v_s-1} J_\ell,$$

where $\ell = (l_1, \ldots, l_s)$. We then have

$$|A(J; N; \omega) - N\lambda_s(J)| \leq \sum_{l_1=u_1}^{v_1-1} \cdots \sum_{l_s=u_s}^{v_s-1} |A(J_\ell; N; \omega) - N\lambda_s(J_\ell)| \leq \prod_{i=1}^{s}(v_i - u_i),$$

which proves the first assertion.

For every $N \leq \prod_{i=1}^{s} b_{i,1} \cdots b_{i,\alpha_i}$, we always have $A(J_\ell; N; \omega) \leq 1$ for each $\ell = (l_1, \ldots, l_s) \in \mathbb{N}_0^s$ with $0 \leq l_i < b_{i,1} \cdots b_{i,\alpha_i}$ for all $1 \leq i \leq s$, and hence

$$A(J; N; \omega) = \sum_{l_1=u_1}^{v_1-1} \cdots \sum_{l_s=u_s}^{v_s-1} A(J_\ell; N; \omega) \leq \prod_{i=1}^{s}(v_i - u_i).$$

This is the second assertion of the lemma. □

The following lemma, which is borrowed from [10], is important for achieving an $s!$ factor in the bounds for the discrepancy.

Lemma 3 [10, Lemma 3.3] *Let $b_1 = (b_{1,j})_{j=1}^{\infty}, \ldots, b_s = (b_{s,j})_{j=1}^{\infty}$ be s arbitrary sequences of natural numbers greater than 1. Suppose $(a_{1,\alpha})_{\alpha=0}^{\infty}, \ldots, (a_{s,\alpha})_{\alpha=0}^{\infty}$ are s bounded sequences of nonnegative real numbers such that $a_{i,0} \leq 1$ and $a_{i,\alpha} \leq f_i$ for some fixed $f_i > 0$ and for each $\alpha \in \mathbb{N}$ and $1 \leq i \leq s$. Then, for any $N \in \mathbb{N}$, we have*

$$\sum_{\substack{(\alpha_1,\ldots,\alpha_s)\in\mathbb{N}_0^s \\ \prod_{i=1}^{s} b_{i,1}\cdots b_{i,\alpha_i} \leq N}} \prod_{i=1}^{s} a_{i,\alpha_i} \leq \frac{1}{s!}\prod_{i=1}^{s}\left(f_i \frac{\log N}{\log m_i} + s\right),$$

where $m_i = \min(b_{i,j})_{j=1}^{\infty}$ $(1 \leq i \leq s)$.

The proof of this lemma is based on an argument of Diophantine geometry which asserts that the number of positive solutions $(\alpha_1, \ldots, \alpha_s)$ of the inequality $\prod_{i=1}^{s} b_{i,1} \cdots b_{i,\alpha_i} \leq N$ is bounded by $\frac{1}{s!} \prod_{i=1}^{s} \frac{\log N}{\log m_i}$.

Next we need to introduce some notation. Let $J \subseteq \mathbb{R}^s$ be an interval. Then a signed splitting of J is a collection of not necessarily disjoint intervals $J_1, \ldots, J_r$ together with signs $\varepsilon_1, \ldots, \varepsilon_r \in \{-1, 1\}$ such that, for all $x \in J$, we have

$$\sum_{\substack{i=1 \\ x \in J_i}}^{r} \varepsilon_i = 1.$$

A function ν on the class of intervals in $\mathbb{R}^s$ is said to be additive if, for each pair of disjoint intervals A, B in $\mathbb{R}^s$, we have $\nu(A \cup B) = \nu(A) + \nu(B)$. It is not hard to see that the s-dimensional Lebesgue measure λ_s and the counting function $A(\cdot; N; \omega)$ are the examples we are particularly interested in. It is not hard to check that, for any additive function ν on the class of intervals in $\mathbb{R}^s$, we have

$$\nu(J) = \sum_{i=1}^{r} \varepsilon_i \nu(J_i \cap J),$$

where $(J_1, \ldots, J_r; \varepsilon_1, \ldots, \varepsilon_r)$ is a signed splitting of J. The following lemma is borrowed from [1] (see also [5] for a detailed proof).

Lemma 4 [1, Lemma 3.5] *Let* $J = \prod_{i=1}^{s} [0, z_i)$ *be an* s*-dimensional interval. For each* $1 \leq i \leq s$, *let* $(z_{i,\alpha})_{\alpha=1,\ldots,n_i}$ *be an arbitrary finite sequence of numbers in* $[0, 1]$. *Define further* $z_{i,0} = 0$ *and* $z_{i,n_i+1} = z_i$ *for all* $1 \leq i \leq s$. *Then the collection of intervals*

$$\prod_{i=1}^{s} [\min(z_{i,\alpha_i}, z_{i,\alpha_i+1}), \max(z_{i,\alpha_i}, z_{i,\alpha_i+1}))$$

together with the signs $\varepsilon_{\alpha_1,\ldots,\alpha_s} = \prod_{i=1}^{s} \operatorname{sgn}(z_{i,\alpha_i+1} - z_{i,\alpha_i})$, *for* $0 \leq \alpha_i \leq n_i$ *and* $1 \leq i \leq s$, *defines a signed splitting of the interval* J.

The signed splitting technique is interesting here because it will lead to the improvement by a 2^s factor in the bounds for the discrepancy. In order to use it, we need a digit expansion of reals $z \in [0, 1)$ in $(b_j)_{j=1}^{\infty}$-adic base which uses signed digits. The next lemma, from [10], shows that such an expansion exists. Note that signed splittings coupled with signed numeration systems were first introduced in [7,9].

Lemma 5 [10, Lemma 3.5] *Let* $b = (b_j)_{j=1}^{\infty}$ *be an arbitrary sequence of natural numbers greater than* 1. *Then every* $z \in [0, 1)$ *can be written in the form*

$$z = a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \cdots$$

with integer digits $a_0, a_1, a_2, \ldots$ *such that* $a_0 \in \{0, 1\}$ *and* $-\lfloor \frac{b_j - 1}{2} \rfloor \leq a_j \leq \lfloor \frac{b_j}{2} \rfloor$ *for all* $j \in \mathbb{N}$. *This expansion is called the signed b-adic expansion of* z.

Now we are ready to prove our first main theorem.

Proof (Proof of Theorem 1) Let $J = \prod_{i=1}^{s}[0, z_i) \subseteq [0, 1)^s$. According to Lemma 5, for all $1 \leq i \leq s$, we consider the signed b_i-adic expansion of z_i of the form

$$z_i = a_{i,0} + \frac{a_{i,1}}{b_{i,1}} + \frac{a_{i,2}}{b_{i,1}b_{i,2}} + \frac{a_{i,3}}{b_{i,1}b_{i,2}b_{i,3}} + \cdots$$

with $a_{i,0} \in \{0, 1\}$ and $-\lfloor \frac{b_{i,j}-1}{2} \rfloor \leq a_{i,j} \leq \lfloor \frac{b_{i,j}}{2} \rfloor$ $(j \in \mathbb{N})$.

For each $1 \leq i \leq s$, let $n_i = \lfloor \frac{\log N}{\log m_i} \rfloor + 1$, and, for each $1 \leq \alpha \leq n_i$, define the truncation of the expansion

$$z_{i,\alpha} = a_{i,0} + \frac{a_{i,1}}{b_{i,1}} + \frac{a_{i,2}}{b_{i,1}b_{i,2}} + \cdots + \frac{a_{i,\alpha-1}}{b_{i,1}\cdots b_{i,\alpha-1}},$$

and let $z_{i,0} = 0$ and $z_{i,n_i+1} = z_i$.

By Lemma 4, the collection of intervals

$$J_{\alpha_1,\ldots,\alpha_s} = \prod_{i=1}^{s}[\min(z_{i,\alpha_i}, z_{i,\alpha_i+1}), \max(z_{i,\alpha_i}, z_{i,\alpha_i+1}))$$

together with the signs $\varepsilon_{\alpha_1,\ldots,\alpha_s} = \prod_{i=1}^{s} \operatorname{sgn}(z_{i,\alpha_i+1} - z_{i,\alpha_i})$, for $0 \leq \alpha_i \leq n_i$ and $1 \leq i \leq s$, defines a signed splitting of the interval J.

Since both λ_s and $A(\cdot; N; \omega)$ are additive functions on the set of intervals, we obtain that

$$\begin{aligned} A(J; N; \omega) - N\lambda_s(J) &= \sum_{\alpha_1=0}^{n_1} \cdots \sum_{\alpha_s=0}^{n_s} \varepsilon_{\alpha_1,\ldots,\alpha_s}(A(J_{\alpha_1,\ldots,\alpha_s}; N; \omega) - N\lambda_s(J_{\alpha_1,\ldots,\alpha_s})) \\ &= \Lambda_1 + \Lambda_2, \end{aligned} \tag{3}$$

where Λ_1 denotes the sum over all $(\alpha_1, \ldots, \alpha_s)$ such that $\prod_{i=1}^{s} b_{i,1}\cdots b_{i,\alpha_i} \leq N$ and Λ_2 denotes the remaining part of the above sum.

First we deal with the sum Λ_1. For each $1 \leq i \leq s$, the length of the interval $[\min(z_{i,\alpha_i}, z_{i,\alpha_i+1}), \max(z_{i,\alpha_i}, z_{i,\alpha_i+1}))$ is $|a_{i,\alpha_i}/b_{i,1}\cdots b_{i,\alpha_i}|$, and the endpoints of this interval are rationals with denominator $b_{i,1}\cdots b_{i,\alpha_i}$. It is worth noting that, due to the choice of n_i, we have $\alpha_i < n_i$ when $\prod_{i=1}^{s} b_{i,1}\cdots b_{i,\alpha_i} \leq N$. Accordingly, the intervals $J_{\alpha_1,\ldots,\alpha_s}$ are of the form as considered in Lemma 2 from which we obtain

$$|A(J_{\alpha_1,\ldots,\alpha_s}; N; \omega) - N\lambda_s(J_{\alpha_1,\ldots,\alpha_s})| \leq \prod_{i=1}^{s} |a_{i,\alpha_i}|.$$

We have $|a_{i,\alpha_i}| \le \lfloor b_{i,\alpha_i}/2 \rfloor \le \lfloor M_i/2 \rfloor =: f_i$. An application of Lemma 3 yields that

$$\Lambda_1 \le \frac{1}{s!} \prod_{i=1}^{s} \left(\frac{\lfloor M_i/2 \rfloor \log N}{\log m_i} + s \right).$$

It remains to estimate Λ_2. To this end, we split the set of s-tuples $(\alpha_1, \ldots, \alpha_s)$ for which $\prod_{i=1}^{s} b_{i,1} \cdots b_{i,\alpha_i} > N$ into disjoint sets $B_0, B_1, \ldots, B_{s-1}$ where we set $B_0 = \{(\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_0^s \colon b_{1,1} \cdots b_{1,\alpha_1} > N\}$ and, for $1 \le l \le s-1$,

$$B_l = \left\{ (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_0^s \colon \prod_{i=1}^{l} b_{i,1} \cdots b_{i,\alpha_i} \le N \text{ and } \prod_{i=1}^{l+1} b_{i,1} \cdots b_{i,\alpha_i} > N \right\}.$$

Here we abuse the notation $\mathbb{N}_0^s$. The choices of $(\alpha_1, \ldots, \alpha_s)$ must also satisfy $\alpha_i \le n_i$ for each $1 \le i \le s$.

For a fixed $1 \le l \le s-1$ and a fixed l-tuple $(\alpha_1, \ldots, \alpha_l)$ with $\prod_{i=1}^{l} b_{i,1} \cdots b_{i,\alpha_i} \le N$, define r to be the largest integers such that

$$\left(\prod_{i=1}^{l} b_{i,1} \cdots b_{i,\alpha_i} \right) (b_{l+1,1} \cdots b_{l+1,r-1}) \le N.$$

It follows that the tuple $(\alpha_1, \ldots, \alpha_l, \alpha_{l+1}, \ldots, \alpha_s)$ is contained in B_l if and only if $\alpha_{l+1} \ge r$.

Therefore, for each $0 \le l \le s-1$ and fixed $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0$ such that $\prod_{i=1}^{l} b_{i,1} \cdots b_{i,\alpha_i} \le N$, we have

$$\begin{aligned} &\sum_{\substack{\alpha_{l+1}, \ldots, \alpha_s \in \mathbb{N}_0 \\ (\alpha_1, \ldots, \alpha_l, \alpha_{l+1}, \ldots, \alpha_s) \in B_l}} \varepsilon_{\alpha_1, \ldots, \alpha_s} (A(J_{\alpha_1, \ldots, \alpha_s}; N; \omega) - N\lambda_s(J_{\alpha_1, \ldots, \alpha_s})) \\ &\quad = \pm (A(L; N; \omega) - N\lambda_s(L)), \end{aligned}$$

where

$$\begin{aligned} L = &\prod_{i=1}^{l} [\min(z_{i,\alpha_i}, z_{i,\alpha_i+1}), \max(z_{i,\alpha_i}, z_{i,\alpha_i+1})) \\ &\times [\min(z_{l+1,r}, z_{l+1}), \max(z_{l+1,r}, z_{l+1})) \times \prod_{i=l+2}^{s} [0, z_i). \end{aligned}$$

Let $(\alpha_1, \ldots, \alpha_s) \in B_l$. Since we have

$$\begin{aligned}
|z_{l+1} - z_{l+1,r}| &= \left| \frac{a_{l+1,r}}{b_{l+1,1} \cdots b_{l+1,r}} + \frac{a_{l+1,r+1}}{b_{l+1,1} \cdots b_{l+1,r+1}} + \frac{a_{l+1,r+2}}{b_{l+1,1} \cdots b_{l+1,r+2}} + \cdots \right| \\
&= \frac{1}{b_{l+1,1} \cdots b_{l+1,r-1}} \left| \frac{a_{l+1,r}}{b_{l+1,r}} + \frac{a_{l+1,r+1}}{b_{l+1,r} b_{l+1,r+1}} + \frac{a_{l+1,r+2}}{b_{l+1,r} b_{l+1,r+1} b_{l+1,r+2}} + \cdots \right| \\
&\leq \frac{1}{b_{l+1,1} \cdots b_{l+1,r-1}} \left(\frac{\lfloor b_{l+1,r}/2 \rfloor}{b_{l+1,r}} + \frac{\lfloor b_{l+1,r+1}/2 \rfloor}{b_{l+1,r} b_{l+1,r+1}} \right. \\
&\quad \left. + \frac{\lfloor b_{l+1,r+2}/2 \rfloor}{b_{l+1,r} b_{l+1,r+1} b_{l+1,r+2}} + \cdots \right) \\
&\leq \frac{1}{b_{l+1,1} \cdots b_{l+1,r-1}} \left(\frac{1}{2} + \frac{1}{2 b_{l+1,r}} + \frac{1}{2 b_{l+1,r} b_{l+1,r+1}} + \cdots \right) \\
&\leq \frac{1}{b_{l+1,1} \cdots b_{l+1,r-1}} \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) \\
&= \frac{1}{b_{l+1,1} \cdots b_{l+1,r-1}},
\end{aligned}$$

the interval $[\min(z_{l+1,r}, z_{l+1}), \max(z_{l+1,r}, z_{l+1}))$ is contained in some interval

$$\left[\frac{u}{b_{l+1,1} \cdots b_{l+1,r}}, \frac{v}{b_{l+1,1} \cdots b_{l+1,r}} \right)$$

for $u, v \in \mathbb{N}_0$ with $v - u \leq b_{l+1,r}$. Hence, L is contained in the interval

$$\begin{aligned}
L' = \prod_{i=1}^{l} &[\min(z_{i,\alpha_i}, z_{i,\alpha_i+1}), \max(z_{i,\alpha_i}, z_{i,\alpha_i+1})) \\
&\times \left[\frac{u}{b_{l+1,1} \cdots b_{l+1,r}}, \frac{v}{b_{l+1,1} \cdots b_{l+1,r}} \right) \times [0, 1)^{s-l-1}.
\end{aligned}$$

Since $(\alpha_1, \ldots, \alpha_s) \in B_l$, we have $(\prod_{i=1}^{l} b_{i,1} \cdots b_{i,\alpha_i})(b_{l+1,1} \cdots b_{l+1,r}) > N$ and $\prod_{i=1}^{l} b_{i,1} \cdots b_{i,\alpha_i} \leq N$. The latter inequality implies that $\alpha_i < n_i$ for every $1 \leq i \leq l$. Thus, an application of Lemma 2 yields that

$$A(L; N; \omega) \leq A(L'; N; \omega) \leq b_{l+1,r} \prod_{i=1}^{l} |a_{i,\alpha_i}|.$$

But on the other hand, we also have $N\lambda_s(L) \leq b_{l+1,r} \prod_{i=1}^{l} |a_{i,\alpha_i}|$. Hence,

$$|A(L; N; \omega) - N\lambda_s(L)| \leq b_{l+1,r} \prod_{i=1}^{l} |a_{i,\alpha_i}| \leq M_{l+1} \prod_{i=1}^{l} c_{i,\alpha_i},$$

where $c_{i,\alpha_i} = 1$ if $\alpha_i = 0$ and $c_{i,\alpha_i} = \lfloor M_i/2 \rfloor$ otherwise.

Summing up, we obtain

$$
\begin{aligned}
|\Lambda_2| &\leq \sum_{l=0}^{s-1} \sum_{\substack{\alpha_1,\ldots,\alpha_l\in\mathbb{N}_0 \\ \prod_{i=1}^{l} b_{i,1}\cdots b_{i,\alpha_i}\leq N}} \Bigg| \sum_{(\alpha_1,\ldots,\alpha_s)\in B_l} \epsilon_{\alpha_1,\ldots,\alpha_s}(A(J_{\alpha_1,\ldots,\alpha_s};N;\omega) \\
&\quad - N\lambda_s(J_{\alpha_1,\ldots,\alpha_s}))\Bigg| \\
&\leq \sum_{l=0}^{s-1} \sum_{\substack{\alpha_1,\ldots,\alpha_l\in\mathbb{N}_0 \\ \prod_{i=1}^{l} b_{i,1}\cdots b_{i,\alpha_i}\leq N}} M_{l+1} \prod_{i=1}^{l} c_{i,\alpha_i} \\
&\leq \sum_{l=0}^{s-1} \frac{M_{l+1}}{l!} \prod_{i=1}^{l} \left(\frac{\lfloor M_i/2 \rfloor \log N}{\log m_i} + l \right),
\end{aligned}
\tag{4}
$$

where we have used Lemma 3 again. Hence, the result follows. □

5 Proof of Theorem 2

Lemma 6 *Suppose that* $p_1 = (\overline{p_{1,1},\ldots,p_{1,j_1}}),\ldots,p_s = (\overline{p_{s,1},\ldots,p_{s,j_s}})$ *are periodic sequences of distinct prime numbers such that, for each* $1 \leq i \leq s$*, there exists a common primitive root modulo* $p_{i,1},\ldots,p_{i,j_i}$*. Then there exist admissible sequences* $k_1 = (\overline{k_{1,1},\ldots,k_{1,j_1}}),\ldots,k_s = (\overline{k_{s,1},\ldots,k_{s,j_s}})$ *for* $p_1,\ldots,p_s$*.*

Proof For each $1 \leq i \leq s$, let g_i be some fixed common primitive root modulo $p_{i,1},\ldots,p_{i,j_i}$. The congruences in Definition 2 lead to the system $(1 \leq i \leq s)$

$$
\begin{aligned}
&g_i^{(a_{i,i,1}+\cdots+a_{i,i,j_i})x_i+a_{i,i,1}+\cdots+a_{i,i,\ell_i-1}+\sum_{i_0\neq i}((a_{i,i_0,1}+\cdots+a_{i,i_0,j_{i_0}})x_{i_0}+a_{i,i_0,1}+\cdots+a_{i,i_0,\ell_{i_0}-1})} \\
&\quad \equiv g_i^{c_i} \pmod{p_{i,\ell_i}}, \quad i=1,\ldots,s,
\end{aligned}
\tag{5}
$$

where $g_i^{a_{i,i_0,1}} \equiv p_{i_0,1} \pmod{p_{i,\ell_i}},\ldots,g_i^{a_{i,i_0,j_{i_0}}} \equiv p_{i_0,j_{i_0}} \pmod{p_{i,\ell_i}}$ for $i_0 \neq i$, $g_i^{a_{i,i,1}} \equiv k_{i,1} \pmod{p_{i,\ell_i}},\ldots,g_i^{a_{i,i,j_i}} \equiv k_{i,j_i} \pmod{p_{i,\ell_i}}$, $g_i^{c_i} \equiv b_i \pmod{p_{i,\ell_i}}$, and $\alpha_i = j_i x_i + \ell_i$. It is worth noting that the choice of each integer $k_{i,j}$ can be fixed according to the Chinese remainder theorem and the fact that $p_{i,1},\ldots,p_{i,j_i}$ are all distinct primes. The system of congruences in (5) is equivalent to

$$
\begin{aligned}
&(a_{i,i,1}+\cdots+a_{i,i,j_i})x_i + a_{i,i,1}+\cdots+a_{i,i,\ell_i-1} \\
&\quad + \sum_{i_0\neq i}(a_{i,i_0}x_{i_0} + a_{i,i_0,1}+\cdots+a_{i,i_0,\ell_{i_0}-1}) \\
&\equiv c_i \pmod{p_{i,\ell_i}-1}, \quad i=1,\ldots,s,
\end{aligned}
\tag{6}
$$

where $a_{i,i_0} = a_{i,i_0,1} + \cdots + a_{i,i_0,j_{i_0}}$ for $i_0 \neq i$. We introduce s integer variables $y_1, \ldots, y_s$ to change the congruences (6) into a system of Diophantine equations

$$(a_{i,i,1} + \cdots + a_{i,i,j_i})x_i + \sum_{i_0 \neq i} a_{i,i_0} x_{i_0} = c_i' + y_i(p_{i,\ell_i} - 1), \quad i = 1, \ldots, s, \tag{7}$$

where $c_i' = c_i - \sum_{i_0=1}^{s}(a_{i,i_0,1} + \cdots + a_{i,i_0,\ell_{i_0}-1})$.

In order to prove the lemma, it suffices to show, for any given integers a_{i,i_0} with $i \neq i_0$, the existence of integers $a_{1,1,1}, \ldots, a_{1,1,j_1}, \ldots, a_{s,s,1}, \ldots, a_{s,s,j_s}$ such that, for any integers $c_1', \ldots, c_s'$ and any integers $y_1, \ldots, y_s$, the system (7) has a solution in integers $x_1, \ldots, x_s$. Note that we actually require $x_i \in \mathbb{N}_0$ so that $\alpha_i = j_i x_i + \ell_i \in \mathbb{N}_0$, but this nonnegativity of x_i can be achieved by a suitable choice of $y_1, \ldots, y_s$. Let

$$A = \begin{pmatrix} a_{1,1,1} + \cdots + a_{1,1,j_1} & a_{1,2} & \cdots & a_{1,s} \\ a_{2,1} & a_{2,2,1} + \cdots + a_{2,2,j_2} & \cdots & a_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s,1} & a_{s,2} & \cdots & a_{s,s,1} + \cdots + a_{s,s,j_s} \end{pmatrix}.$$

By Cramer's Rule, it is enough to show that the determinant of A can be made to be 1 by a suitable choice of the numbers $a_{1,1,1}, \ldots, a_{1,1,j_1}, \ldots, a_{s,s,1}, \ldots, a_{s,s,j_s}$. This claim follows by induction on s. When $s = 1$, choose $a_{1,1,1} = 1$ and $a_{1,1,2} = \cdots = a_{1,1,j_1} = 0$. Next, expand the determinant of A along the last column, $A_{i,s}$ being the cofactors:

$$\det(A) = a_{1,s}A_{1,s} + a_{2,s}A_{2,s} + \cdots + a_{s-1,s}A_{s-1,s} + (a_{s,s,1} + \cdots + a_{s,s,j_s})A_{s,s}.$$

By the induction hypothesis, we have $A_{s,s} = 1$. Setting

$$a_{s,s,1} + \cdots + a_{s,s,j_s} = 1 - (a_{1,s}A_{1,s} + a_{2,s}A_{2,s} + \cdots + a_{s-1,s}A_{s-1,s})$$

yields $\det(A) = 1$, with an appropriate choice of integers $a_{s,s,1}, \ldots, a_{s,s,j_s}$. This proves the existence of admissible sequences $k_1, \ldots, k_s$. □

Before proceeding, we introduce here some notations for brevity. Suppose $p_1 = (\overline{p_{1,1}, \ldots, p_{1,j_1}}), \ldots, p_s = (\overline{p_{s,1}, \ldots, p_{s,j_s}})$ are periodic sequences of distinct prime numbers such that, for each $1 \leq i \leq s$, there exists a common primitive root modulo $p_{i,1}, \ldots, p_{i,j_i}$. Let $k_1 = (\overline{k_{1,1}, \ldots, k_{1,j_1}}), \ldots, k_s = (\overline{k_{s,1}, \ldots, k_{s,j_s}})$ be admissible sequences for $p_1, \ldots, p_s$. Define

$$P_i(\alpha) := k_{i,1} \cdots k_{i,\alpha_i - 1} \prod_{1 \le i_0 \le s,\ i_0 \ne i} p_{i_0,1} \cdots p_{i_0,\alpha_{i_0}-1} \bmod p_{i,\alpha_i}$$

$$T(N) := \left\{ \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_0^s \colon \prod_{i=1}^{s} p_{i,1} \cdots p_{i,\alpha_i} \le N \right\}$$

$$M(\alpha) := \left\{ \ell = (\ell_1, \ldots, \ell_s) \in \mathbb{Z}_{p_{1,\alpha_1}} \times \cdots \times \mathbb{Z}_{p_{s,\alpha_s}} : \ell_1 + \cdots + \ell_s > 0 \right\}$$

$$R(\alpha; \ell) := \prod_{i=1}^{s} \max(1, \min(2\ell_i, 2(p_{i,\alpha_i} - \ell_i)))$$

for all $1 \le i \le s$, $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_0^s$, $N \in \mathbb{N}$ and $\ell = (\ell_1, \ldots, \ell_s) \in \mathbb{N}_0^s$, where we denote $k_{i,-1} = k_{i,0} = p_{i,-1} = p_{i,0} := 1$.

In the following proposition, we formulate an estimate of discrepancy of the modified Halton sequence in Cantor bases which is the basis for our proof of Theorem 2. It can also be used for computational estimation of $D_N^*(\omega)$, if performing $O((\log N)^s)$ operations is not a problem.

Proposition 1 *Let ω be the modified Halton sequence in Cantor bases $p_1, \ldots, p_s$ with respect to $k_1, \ldots, k_s$. Then, for any $N \in \mathbb{N}$, we have*

$$N D_N^*(\omega) \le \sum_{\alpha \in T(N)} \left(1 + \sum_{\ell \in M(\alpha)} \frac{\left\| \sum_{i=1}^{s} \frac{\ell_i}{p_{i,\alpha_i}} P_i(\alpha) \right\|^{-1}}{2R(\alpha; \ell)} \right) + \sum_{l=0}^{s-1} \frac{M_{l+1}}{l!} \prod_{i=1}^{l} \left(\frac{\lfloor M_i/2 \rfloor \log N}{\log m_i} + l \right),$$

where $M_i = \max(p_{i,j})_{j=1}^{\infty}$ and $m_i = \min(p_{i,j})_{j=1}^{\infty}$ $(1 \le i \le s)$, and where $\| \cdot \|$ is the to-the-nearest-integer function.

The proof of this proposition is based on specific periodicity properties of the modified Halton sequence in Cantor bases. These properties will be studied in Lemma 8. Note that the following lemma appears in [1] in slightly different notation, and it is used to derive some properties in Lemma 8.

Lemma 7 [1, Lemma 4.2] *Let $p_{1,\alpha_1}, \ldots, p_{s,\alpha_s}$ be distinct prime numbers, and let $\xi = (\xi_{1,t}, \ldots, \xi_{s,t})_{t=0}^{\infty}$ be a sequence in $\mathbb{Z}^s$. Let v and w be fixed integer s-tuples such that $0 \le v_i < w_i \le p_{i,\alpha_i}$ $(1 \le i \le s)$. For each $K \in \mathbb{N}$, we denote by*

$$A_K(v, w) = \#\{0 \le n \le K - 1 \colon \forall\, 1 \le i \le s,\ \ v_i \le \xi_{i,t} \bmod p_{i,\alpha_i} \le w_i - 1\}$$

the number of the first K terms of ξ such that, for all $1 \le i \le s$, the remainder of $\xi_{i,n}$ modulo p_{i,α_i} is among the numbers $v_i, \ldots, w_i - 1$. Then, for all $K \in \mathbb{N}$, we have

$$\sup_{v,w} \left| A_K(v, w) - K \prod_{i=1}^{s} \frac{w_i - v_i}{p_{i,\alpha_i}} \right| \le \sum_{\ell \in M(\alpha)} \frac{|S_K(\ell; \xi)|}{R(\alpha; \ell)},$$

where

$$S_K(\ell;\xi) := \sum_{t=0}^{K-1} e\left(\sum_{i=1}^{s} \frac{\ell_i \xi_{i,t}}{p_{i,\alpha_i}}\right),$$

with the usual notation $e(x) := \exp(2\pi i x)$.

Lemma 8 *Let* ω *be the modified Halton sequence in Cantor bases* $p_1, \ldots, p_s$ *with respect to* $k_1, \ldots, k_s$. *Let* I *be an elementary interval of the form*

$$I = \prod_{i=1}^{s}\left[\frac{u_i}{p_{i,1}\cdots p_{i,\alpha_i-1}}, \frac{u_i+1}{p_{i,1}\cdots p_{i,\alpha_i-1}}\right)$$

with integers $0 \le u_i \le p_{i,1}\cdots p_{i,\alpha_i-1} - 1$ *and* $\alpha_i \in \mathbb{N}_0$ $(1 \le i \le s)$, *and let* J *be a subinterval of* I *of the form*

$$J = \prod_{i=1}^{s}\left[\frac{u_i}{p_{i,1}\cdots p_{i,\alpha_i-1}} + \frac{v_i}{p_{i,1}\cdots p_{i,\alpha_i}}, \frac{u_i}{p_{i,1}\cdots p_{i,\alpha_i-1}} + \frac{w_i}{p_{i,1}\cdots p_{i,\alpha_i}}\right)$$

with integers $0 \le v_i < w_i \le p_{i,\alpha_i}$ $(1 \le i \le s)$. *There exists a nonnegative integer* n *with* $\omega_n \in I$. *Let* n_0 *be the smallest integer such that* $\omega_{n_0} \in I$. *Suppose that* ω_{n_0} *drops into the interval*

$$\prod_{i=1}^{s}\left[\frac{u_i}{p_{i,1}\cdots p_{i,\alpha_i-1}} + \frac{x_i}{p_{i,1}\cdots p_{i,\alpha_i}}, \frac{u_i}{p_{i,1}\cdots p_{i,\alpha_i-1}} + \frac{x_i+1}{p_{i,1}\cdots p_{i,\alpha_i}}\right)$$

with $0 \le x_i \le p_{i,\alpha_i} - 1$ $(1 \le i \le s)$. *Then the following statements are true.*

(1) $n_0 < \prod_{i=1}^{s} p_{i,1}\cdots p_{i,\alpha_i-1}$, *and the indices of the terms of* ω *that drop into* I *are of the form* $n = n_0 + t\prod_{i=1}^{s} p_{i,1}\cdots p_{i,\alpha_i-1}$ *for some* $t \in \mathbb{N}_0$.

(2) *Suppose that* $n = n_0 + t\prod_{i=1}^{s} p_{i,1}\cdots p_{i,\alpha_i-1}$ *with* $t \in \mathbb{N}_0$. *Then* $\omega_n \in J$ *if and only if there exist* $l_1 \in \{v_1, \ldots, w_1 - 1\}, \ldots, l_s \in \{v_s, \ldots, w_s - 1\}$ *such that* $x_i + tP_i(\alpha) \equiv l_i \pmod{p_{i,\alpha_i}}$ *for all* $1 \le i \le s$.

(3) *Let* $\xi = (\xi_{1,t}, \ldots, \xi_{s,t})_{t=0}^{\infty}$ *be the sequence in* $\mathbb{Z}^s$ *with* $\xi_{i,t} = x_i + tP_i(\alpha)$ *for each* $1 \le i \le s$. *Let* $N \in \mathbb{N}$, *and let* K *be the largest integer such that* $n_0 + (K - 1)\prod_{i=1}^{s} p_{i,1}\cdots p_{i,\alpha_i-1} < N$. *Then we have*

$$|A(J;N;\omega) - N\lambda_s(J)| < 1 + \sum_{\ell\in M(\alpha)} \frac{S_K(\ell;\xi)}{R(\alpha;\ell)}.$$

Proof For each $n \in \mathbb{N}_0$, we denote the p_i-adic expansion of n by

$$n = \sum_{j=1}^{\infty} n_{i,j}\, p_{i,1}\cdots p_{i,j-1},$$

where $n_{i,j} \in \mathbb{Z}_{p_{i,j}}$ $(j \in \mathbb{N})$. On the other hand, we write the expansion of u_i as

$$u_i = u_{i,\alpha_i-1} + u_{i,\alpha_i-2}p_{i,\alpha_i-1} + u_{i,\alpha_i-3}p_{i,\alpha_i-1}p_{i,\alpha_i-2} + \cdots + u_{i,1}p_{i,\alpha_i-1}\cdots p_{i,2},$$

where $u_{i,\alpha_i-j} \in \mathbb{Z}_{p_{i,\alpha_i-j}}$ $(1 \le j \le \alpha_i - 1)$. We have

$$\omega = \left(\sum_{j=1}^{\infty} \frac{n_{1,j}k_{1,1}\cdots k_{1,j-1} \bmod p_{1,j}}{p_{1,1}\cdots p_{1,j}}, \ldots, \sum_{j=1}^{\infty} \frac{n_{s,j}k_{s,1}\cdots k_{s,j-1} \bmod p_{s,j}}{p_{s,1}\cdots p_{s,j}}\right)_{n=0}^{\infty}$$

and

$$I = \prod_{i=1}^{s} \left[\sum_{j=1}^{\alpha_i-1} \frac{u_{i,j}}{p_{i,1}\cdots p_{i,j}}, \sum_{j=1}^{\alpha_i-1} \frac{u_{i,j}}{p_{i,1}\cdots p_{i,j}} + \frac{1}{p_{i,1}\cdots p_{i,\alpha_i-1}}\right).$$

Then the nth element ω_n of the modified Halton sequence in Cantor bases is contained in I if and only, for all $1 \le i \le s$ and all $1 \le j \le \alpha_i - 1$,

$$n_{i,j}k_{i,1}\cdots k_{i,j-1} \equiv u_{i,j} \pmod{p_{i,j}}.$$

It follows, by Lemma 1, that there exists exactly one n_0 such that $\omega_{n_0} \in I$ and $0 \le n_0 < \prod_{i=1}^{s} p_{i,1}\cdots p_{i,\alpha_i-1}$. For each $1 \le i \le s$ and all $t \in \mathbb{N}_0$, the first $\alpha_i - 1$ digits of $n_0 + t\prod_{i_0=1}^{s} p_{i_0,1}\cdots p_{i_0,\alpha_{i_0}-1}$ in the p_i-adic number system are the same as that of n_0. Therefore, $\omega_n \in I$ is equivalent to $n = n_0 + t\prod_{i_0=1}^{s} p_{i_0,1}\cdots p_{i_0,\alpha_{i_0}-1}$ for some $t \in \mathbb{N}_0$. This proves the first assertion.

Next, suppose that $n = n_0 + t\prod_{i_0=1}^{s} p_{i_0,1}\cdots p_{i_0,\alpha_{i_0}-1}$ with $t \in \mathbb{N}_0$. For all $1 \le i \le s$, we now look at the α_ith digit of the p_i-adic expansion of n. Since the α_ith digit of $t\prod_{i_0=1}^{s} p_{i_0,1}\cdots p_{i_0,\alpha_{i_0}-1}$ in the p_i-adic expansion is

$$\left(t\prod_{i_0=1}^{s} p_{i_0,1}\cdots p_{i_0,\alpha_{i_0}-1}\right)_{i,\alpha_i} = t\prod_{\substack{1\le i_0<1\\ i_0\ne i}} p_{i_0,1}\cdots p_{i_0,\alpha_{i_0}-1} \bmod p_{i,\alpha_i},$$

we have

$$\begin{aligned}\sigma_{i,\alpha_i}\left(\left(t\prod_{i_0=1}^{s} p_{i_0,1}\cdots p_{i_0,\alpha_{i_0}-1}\right)_{i,\alpha_i}\right) &= tk_{i,1}\cdots k_{i,\alpha_i-1}\\ &\quad \prod_{\substack{1\le i_0<s\\ i_0\ne i}} p_{i_0,1}\cdots p_{i_0,\alpha_{i_0}-1} \bmod p_{i,\alpha_i}\\ &= tP_i(\alpha) \bmod p_{i,\alpha_i}.\end{aligned}$$

Moreover, we have $\sigma_{i,\alpha_i}((n_0)_{i,\alpha_i}) = x_i$. Due to the fact that all the permutations $\sigma_{i,j}$ are isomorphisms, we obtain

$$\sigma_{i,\alpha_i}((n)_{i,\alpha_i}) = x_i + tP_i(\alpha) \bmod p_{i,\alpha_i}.$$

It follows immediately that $\omega_n \in J$ if and only if, for all $1 \le i \le s$,

$$v_i \le x_i + tP_i(\alpha) \bmod p_{i,\alpha_i} \le w_i - 1.$$

This proves the second assertion.

By the definition of K, we observe that $A(J; N; \omega) = A_K(v, w)$. Also, it is not hard to check that

$$-1 + K \prod_{i=1}^{s} \frac{w_i - v_i}{p_{i,\alpha_i}} < N\lambda_s(J) < 1 + K \prod_{i=1}^{s} \frac{w_i - v_i}{p_{i,\alpha_i}}.$$

Now it follows that

$$|A(J; N; \omega) - N\lambda_s(J)| < 1 + \left| A_K(v, w) - K \prod_{i=1}^{s} \frac{w_i - v_i}{p_{i,\alpha_i}} \right|.$$

By using Lemma 7, we immediately obtain the last assertion. □

Proof (Proof of Proposition 1) Let $J = \prod_{i=1}^{s}[0, z_i) \subseteq [0, 1)^s$. We expand each z_i in the same way as in the proof of Theorem 1, and obtain the equality (3) for $A(J; N; \omega) - N\lambda_s(J)$. The estimate in (4) for Λ_2 depends only on Lemma 2, so we can use it here too. We now investigate

$$\Lambda_1 = \sum_{\alpha \in T(N)} \epsilon_{\alpha_1,\ldots,\alpha_s}(A(J_{\alpha_1,\ldots,\alpha_s}; N; \omega) - N\lambda_s(J_{\alpha_1,\ldots,\alpha_s})).$$

Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in T(N)$. The interval $J_{\alpha_1,\ldots,\alpha_s}$ is contained inside some elementary interval

$$I = \prod_{i=1}^{s} \left[\frac{u_i}{p_{i,1} \cdots p_{i,\alpha_i - 1}}, \frac{u_i + 1}{p_{i,1} \cdots p_{i,\alpha_i - 1}} \right)$$

with $0 \le u_i \le p_{i,1} \cdots p_{i,\alpha_i - 1} - 1$ $(1 \le i \le s)$. Consider the sequence ξ, defined as in Lemma 8, such that $\xi_{i,t} = x_i + tP_i(\alpha)$ $(1 \le i \le s)$, where the integers x_i are determined by the first term of the modified Halton sequence ω that drops into the interval I fits into the smaller interval

$$\prod_{i=1}^{s} \left[\frac{u_i}{p_{i,1} \cdots p_{i,\alpha_i - 1}} + \frac{x_i}{p_{i,1} \cdots p_{i,\alpha_i}}, \frac{u_i}{p_{i,1} \cdots p_{i,\alpha_i - 1}} + \frac{x_i + 1}{p_{i,1} \cdots p_{i,\alpha_i}} \right)$$

From the last property of Lemma 8, it follows that

$$|A(J_{\alpha_1,\dots,\alpha_s};N;\omega)-N\lambda_s(J_{\alpha_1,\dots,\alpha_s})|<1+\sum_{\ell\in M(\alpha)}\frac{|S_K(\ell;\xi)|}{R(\alpha;\ell)},$$

where K is the number of terms of ω among the first N that drop into the interval I.

Now we have

$$\Lambda_1\le\sum_{\alpha\in T(N)}\left(1+\sum_{\ell\in M(\alpha)}\frac{|S_K(\ell;\xi)|}{R(\alpha;\ell)}\right).$$

In order to accomplish the proof, we claim that

$$|S_K(\ell;\xi)|\le\frac{1}{2}\left\|\sum_{i=1}^{s}\frac{\ell_i}{p_{i,\alpha_i}}P_i(\alpha)\right\|^{-1}. \tag{8}$$

Let $\theta=\sum_{i=1}^{s}\frac{\ell_i}{p_{i,\alpha_i}}P_i(\alpha)$. Since $p_{1,\alpha_1},\dots,p_{s,\alpha_s}$ are pairwise coprime and, for each $1\le i\le s$, $p_{i,\alpha_i}\nmid P_i(\alpha)$, we must have $\|\theta\|\ne 0$. The inequality (8) follows immediately from

$$\left|\sum_{t=0}^{K-1}e(t\theta+\vartheta)\right|=\frac{\sin(\pi\|K\theta\|)}{\sin(\pi\|\theta\|)}\le\frac{1}{2\|\theta\|}\quad(\vartheta\in\mathbb{R}).$$

This completes the proof of Proposition 1. □

The following two lemmas help extend Proposition 1 to Theorem 2. The first result shows that the modified Halton sequence in Cantor bases possesses some particular periodicity properties, while the other one which is borrowed directly from [1] provides some technical estimate to be used with the first lemma.

Lemma 9 *Let $p_1=(\overline{p_{1,1},\dots,p_{1,j_1}}),\dots,p_s=(\overline{p_{s,1},\dots,p_{s,j_s}})$ be periodic sequences of distinct prime numbers such that, for each $1\le i\le s$, there is a common primitive root modulo $p_{i,1},\dots,p_{i,j_i}$. Let $k_1=(\overline{k_{1,1},\dots,k_{1,j_1}}),\dots,k_s=(\overline{k_{s,1},\dots,k_{s,j_s}})$ be admissible sequences for $p_1,\dots,p_s$. Denote*

$$K:=(j_1\cdots j_s)\prod_{\gamma_1=1}^{j_1}\cdots\prod_{\gamma_s=1}^{j_s}\left(\prod_{i=1}^{s}\left(p_{i,\gamma_i}-1\right)\right).$$

For each $\beta=(\beta_1,\dots,\beta_s)\in\mathbb{N}_0^s$, denote

$$U(\beta):=\left\{\alpha=(\alpha_1,\dots,\alpha_s)\in\mathbb{N}_0^s\colon\forall\,1\le i\le s,\ \ \beta_iK\le\alpha_i<(\beta_i+1)K\right\}.$$

Then, for any $\beta=(\beta_1,\dots,\beta_s)\in\mathbb{N}_0^s$, any $1\le\gamma_i\le j_i$ $(1\le i\le s)$ and any $(b_1,\dots,b_s)\in\mathbb{Z}_{p_{1,\gamma_1}}^*\times\cdots\times\mathbb{Z}_{p_{s,\gamma_s}}^*$, we have

$$\#\{\alpha \in U(\beta) \colon \forall\, 1 \le i \le s,\ \ P_i(\alpha) = b_i \ \ \textit{and} \ \ \alpha_i \equiv \gamma_i \pmod{j_i}\}$$
$$= \frac{K^s}{j_1 \cdots j_s \prod_{i=1}^{s}(p_{i,\gamma_i} - 1)}.$$

Proof First, we observe that there are $K^s/(j_1 \cdots j_s)$ elements in $U(\beta)$ such that $\alpha_i \equiv \gamma_i \pmod{j_i}$ for each $1 \le i \le s$, and that there are only $\prod_{i=1}^{s}(p_{i,\gamma_i} - 1)$ distinct elements $(b_1, \ldots, b_s)$ in $\mathbb{Z}^*_{p_{1,\gamma_1}} \times \cdots \times \mathbb{Z}^*_{p_{s,\gamma_s}}$. By Pigeonhole Principle, there exists $(b'_1, \ldots, b'_s) \in \mathbb{Z}^*_{p_{1,\gamma_1}} \times \cdots \times \mathbb{Z}^*_{p_{s,\gamma_s}}$ such that

$$\#\{\alpha \in U(\beta) \colon \forall\, 1 \le i \le s,\ \ P_i(\alpha) = b'_i \ \ \text{and} \ \ \alpha_i \equiv \gamma_i \pmod{j_i}\}$$
$$\ge \frac{K^s}{j_1 \cdots j_s \prod_{i=1}^{s}(p_{i,\gamma_i} - 1)}.$$

For each $1 \le i \le s$, let g_i be some fixed common primitive root modulo $p_{i,1}, \ldots, p_{i,j_i}$. Since $p_{i,\gamma_i} \nmid b'_i$, the congruences $b'_i \equiv g_i^{c_i} \pmod{p_{i,\gamma_i}}$ are fulfilled for some integers c_i. Note from Lemma 6 that the equalities $P_i(\alpha) = b_i$ and the congruences $\alpha_i \equiv \gamma_i \pmod{j_i}$ are possible if and only if α together with some integers $y_1, \ldots, y_s$ form a solution to the system (7). We conclude that if $\alpha', \alpha'' \in U(\beta)$ are two (possibly equal) solutions such that $P_i(\alpha') = b'_i = P_i(\alpha'')$ and $\alpha'_i \equiv \gamma_i \equiv \alpha''_i \pmod{j_i}$ for all $1 \le i \le s$, then the s-tuple α''' defined by

$$\alpha'''_i = \alpha'_i - \alpha''_i + \gamma_i - \left[\frac{\alpha'_i - \alpha''_i + \gamma_i}{K}\right] \cdot K, \quad i = 1, \ldots, s,$$

is a (possibly trivial) solution of the congruences $\alpha'''_i \equiv \gamma_i \pmod{j_i}$ and of the equations $P_i(\alpha''') = b^*_i$, where $b^*_i \equiv g_i^{\sum_{i_0}^{s}(a_{i,i_0,1} + \cdots + a_{i,i_0,\gamma_{i_0}-1})} \pmod{p_{i,\gamma_i}}$ with the notation from the system (7), and it is in $U(0)$. It follows, from the choice of $(b'_1, \ldots, b'_s)$, that

$$\#\{\alpha \in U(0) \colon \forall\, 1 \le i \le s,\ \ P_i(\alpha) = b^*_i \ \ \text{and} \ \ \alpha_i \equiv \gamma_i \pmod{j_i}\}$$
$$\ge \frac{K^s}{j_1 \cdots j_s \prod_{i=1}^{s}(p_{i,\gamma_i} - 1)}.$$

Let $(b_1, \ldots, b_s) \in \mathbb{Z}^*_{p_{1,\gamma_1}} \times \cdots \times \mathbb{Z}^*_{p_{s,\gamma_s}}$. Since $k_1, \ldots, k_s$ are admissible, we have

$$\#\{\alpha \in U(\beta) \colon \forall\, 1 \le i \le s,\ \ P_i(\alpha) = b_i \ \ \text{and} \ \ \alpha_i \equiv \gamma_i \pmod{j_i}\} \ge 1.$$

Let $\alpha \in U(\beta)$ be such that $P_i(\alpha) = b_i$ and $\alpha_i \equiv \gamma_i \pmod{j_i}$ for each $1 \le i \le s$, and let $\alpha^0 \in U(0)$ be such that $P_i(\alpha^0) = b^*_i$ and $\alpha^0_i \equiv \gamma_i \pmod{j_i}$ for each i. Then the s-tuple α^* defined by

$$\alpha^*_i = \alpha_i + \alpha^0_i - \gamma_i - \left[\frac{\alpha_i + \alpha^0_i - \gamma_i}{K}\right] \cdot K + \beta_i K, \quad i = 1, \ldots, s,$$

yields another solution of the congruences $\alpha_i^* \equiv \gamma_i \pmod{j_i}$ and of the equations $P_i(\alpha^*) = b_i$, and it is in $U(x)$. It follows immediately that

$$\#\{\alpha \in U(\beta) \colon \forall\, 1 \le i \le s,\ \ P_i(\alpha) = b_i \ \text{ and } \ \alpha_i \equiv \gamma_i \pmod{j_i}\} \\ \ge \frac{K^s}{j_1 \cdots j_s \prod_{i=1}^s (p_{i,\gamma_i} - 1)}$$

since there are at least $K^s/(j_1 \cdots j_s \prod_{i=1}^s (p_{i,\gamma_i} - 1))$ such α_i^0. Because this is true for all $(b_1, \ldots, b_s)$, it follows that the number of the solutions in $U(\beta)$ is exactly $K^s/(j_1 \cdots j_s \prod_{i=1}^s (p_{i,\gamma_i} - 1))$. This completes the proof of Lemma 9. □

Lemma 10 [1, Lemma 4.4] *Let $p_1, \ldots, p_s$ be distinct prime numbers. Then*

$$\sum_{\ell \in M(p_1,\ldots,p_s)} \sum_{b_1=1}^{p_1-1} \cdots \sum_{b_s=1}^{p_s-1} \frac{\left\| \frac{\ell_1 b_1}{p_1} + \cdots + \frac{\ell_s b_s}{p_s} \right\|^{-1}}{2R(p_1, \ldots, p_s; \ell)} \\ \le \left(\sum_{i=1}^s \log p_i \right) \left(\prod_{i=1}^s p_i \right) \left(-1 + \prod_{i=1}^s (1 + \log p_i) \right),$$

where we denote

$$M(p_1, \ldots, p_s) := \left\{ \ell = (\ell_1, \ldots, \ell_s) \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_s} \colon \ell_1 + \cdots + \ell_s > 0 \right\},$$

$$R(p_1, \ldots, p_s; \ell) := \prod_{i=1}^s \max(1, \min(2\ell_i, 2(p_i - \ell_i))).$$

Now we are in a position to prove our second main theorem.

Proof (Proof of Theorem 2) Our proof is based upon Proposition 1. Let

$$K = (j_1 \cdots j_s) \prod_{\gamma_1=1}^{j_1} \cdots \prod_{\gamma_s=1}^{j_s} \left(\prod_{i=1}^s (p_{i,\gamma_i} - 1) \right).$$

For each $\beta = (\beta_1, \ldots, \beta_s) \in \mathbb{N}_0^s$, denote

$$U(\beta) := \left\{ \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_0^s \colon \forall\, 1 \le i \le s,\ \ \beta_i K \le \alpha_i < (\beta_i + 1)K \right\}.$$

Clearly, each $\alpha = (\alpha_1, \ldots, \alpha_s) \in T(N)$ is inside a unique box $U(\beta)$ such that the s-tuple β satisfies $\prod_{i=1}^s p_{i,1} \cdots p_{i,\beta_i K} \le N$. We apply Lemma 3, for the integers $b_{i,j} := (p_{i,1} \cdots p_{i,j_i})^{K/j_i}$ and the bounds $f_i := 1$, to obtain an estimate of the number of those boxes $U(\beta)$ which contain $T(N)$, i.e. we have

$$\sum_{\substack{\beta=(\beta_1,\dots,\beta_s)\in\mathbb{N}_0^s\\ \prod_{i=1}^s p_{i,1}\cdots p_{i,\beta_i K}\le N}} 1 = \sum_{\substack{\beta=(\beta_1,\dots,\beta_s)\in\mathbb{N}_0^s\\ \prod_{i=1}^s\left((p_{i,1}\cdots p_{i,j_i})^{K/j_i}\right)^{\beta_i}\le N}} 1$$

$$\le \frac{1}{s!}\prod_{i=1}^s\left(\frac{\log N}{(K/j_i)\log(p_{i,1}\cdots p_{i,j_i})}+s\right)$$

$$\le \frac{1}{s!}\prod_{i=1}^s\left(\frac{\log N}{K\log m_i}+s\right).$$

Now we can use Lemma 9 to obtain a partial estimate of the first sum in the inequality in Proposition 1 as follows

$$\sum_{\alpha\in T(N)}\sum_{\ell\in M(\alpha)}\frac{\left\|\sum_{i=1}^s\frac{\ell_i}{p_{i,\alpha_i}}P_i(\alpha)\right\|^{-1}}{2R(\alpha;\ell)}$$

$$\le \sum_{\substack{\beta\in\mathbb{N}_0^s\\ \prod_{i=1}^s p_{i,1}\cdots p_{i,\beta_i K}\le N}}\left(\sum_{\gamma_1=1}^{j_1}\cdots\sum_{\gamma_s=1}^{j_s}\frac{K^s}{j_1\cdots j_s\prod_{i=1}^s(p_{i,\gamma_i}-1)}\left(\sum_{\ell\in M(p_{1,\gamma_1},\dots,p_{s,\gamma_s})}\right.\right.$$

$$\left.\left.\sum_{b_1=1}^{p_{1,\gamma_1}-1}\cdots\sum_{b_s=1}^{p_{s,\gamma_s}-1}\frac{\left\|\sum_{i=1}^s\frac{\ell_i b_i}{p_{i,\gamma_i}}\right\|^{-1}}{2R(p_{1,\gamma_1},\dots,p_{s,\gamma_s};\ell)}\right)\right).$$

We apply Lemma 10 to the rightmost sums of the above inequality to get

$$\sum_{\alpha\in T(N)}\sum_{\ell\in M(\alpha)}\frac{\left\|\sum_{i=1}^s\frac{\ell_i}{p_{i,\alpha_i}}P_i(\alpha)\right\|^{-1}}{2R(\alpha;\ell)} \le \sum_{\substack{\beta\in\mathbb{N}_0^s\\ \prod_{i=1}^s p_{i,1}\cdots p_{i,\beta_i K}\le N}}$$

$$\left(\sum_{\gamma_1=1}^{j_1}\cdots\sum_{\gamma_s=1}^{j_s}\frac{K^s}{j_1\cdots j_s\prod_{i=1}^s(p_{i,\gamma_i}-1)}\left(\left(\sum_{i=1}^s\log p_{i,\gamma_i}\right)\left(\prod_{i=1}^s p_{i,\gamma_i}\right)\right.\right.$$

$$\left.\left.\left(-1+\prod_{i=1}^s(1+\log p_{i,\gamma_i})\right)\right)\right) \le \sum_{\substack{\beta\in\mathbb{N}_0^s\\ \prod_{i=1}^s p_{i,1}\cdots p_{i,\beta_i K}\le N}}$$

$$\frac{K^s}{\prod_{i=1}^s (m_i - 1)} \left(\sum_{i=1}^s \log M_i \right) \left(\prod_{i=1}^s M_i \right) \left(-1 + \prod_{i=1}^s (1 + \log M_i) \right)$$
$$\leq \frac{1}{s!} \left(\prod_{i=1}^s \left(\frac{\log N}{K \log m_i} + s \right) \right) \left(\frac{K^s}{\prod_{i=1}^s (m_i - 1)} \right) \left(\sum_{i=1}^s \log M_i \right) \tag{9}$$
$$\left(\prod_{i=1}^s M_i \right) \left(-1 + \prod_{i=1}^s (1 + \log M_i) \right)$$

To obtain the estimate of the other part of the first sum in the inequality in Proposition 1, we utilize Lemma 3 again, that is, we have

$$\sum_{\alpha \in T(N)} 1 = \sum_{\substack{\alpha \in \mathbb{N}_0^s \\ \prod_{i=1}^s p_{i,1} \cdots p_{i,\alpha_i} \leq N}} 1 \leq \frac{1}{s!} \prod_{i=1}^s \left(\frac{\log N}{\log m_i} + s \right). \tag{10}$$

We finally combine Proposition 1 with (9) and (10) to obtain

$$N D_N^*(\omega) \leq \left(\frac{1}{s!} \left(\frac{1}{\prod_{i=1}^s \log m_i} \right) \right.$$
$$\left. \left(1 + \frac{\sum_{i=1}^s \log M_i \prod_{i=1}^s M_i \left(-1 + \prod_{i=1}^s (1 + \log M_i) \right)}{\prod_{i=1}^s (m_i - 1)} \right) \right) (\log N)^s$$
$$+ O\left((\log N)^{s-1} \right)$$
$$\leq \left(\frac{1}{s!} \left(\sum_{i=1}^s \log M_i \right) \prod_{i=1}^s \frac{M_i (1 + \log M_i)}{(m_i - 1) \log m_i} \right) (\log N)^s + O\left((\log N)^{s-1} \right),$$

where the last inequality follows from $(\sum_{i=1}^s \log M_i)(\prod_{i=1}^s M_i/(m_i - 1)) \geq 1$. This completes the proof of Theorem 2. □

6 The generalized Hammersley point set in Cantor bases

Based on the $(s-1)$-dimensional generalized Halton sequence, we can introduce a finite s-dimensional point set which is called the generalized Hammersley point set.

Let $b_1 = (b_{1,j})_{j=1}^\infty, \ldots, b_{s-1} = (b_{s-1,j})_{j=1}^\infty$ be $s-1$ sequences of natural numbers greater than 1 such that, for all $1 \leq i_1 < i_2 \leq s-1$ and all $j_1, j_2 \in \mathbb{N}$, b_{i_1,j_1} and b_{i_2,j_2} are coprime. For each $1 \leq i \leq s-1$, let $\Sigma_i = (\sigma_{i,j})_{j=1}^\infty$ be an arbitrary sequence of permutations of $\mathbb{Z}_{b_{i,j}}$ $(j \in \mathbb{N})$. The generalized Hammersley point set in Cantor bases $b_1, \ldots, b_{s-1}$ with respect to $\Sigma_1, \ldots, \Sigma_{s-1}$, consisting of N points in $[0, 1)^s$, is defined to be the point set

$$\mathcal{P} = \left\{ \left(\frac{n}{N}, \phi_{b_1}^{\Sigma_1}(n), \ldots, \phi_{b_{s-1}}^{\Sigma_{s-1}}(n) \right) : 0 \leq n \leq N-1 \right\}.$$

We deduce a discrepancy bound for the generalized Hammersley point set in Cantor bases with the help of Theorem 1 in combination with the following general result from [19] that goes back to Roth [20].

Lemma 11 [19, Lemma 3.7] *Let* $\omega = (x_n)_{n=0}^{\infty}$ *be an arbitrary sequence in* $[0,1)^{s-1}$ *with discrepancy* $D_N^*(\omega)$. *For* $N \in \mathbb{N}$, *let* $\mathcal{P}$ *be the point set consisting of* $(n/N, x_n)$ *in* $[0,1)^s$ *for* $n = 0, 1, \ldots, N-1$. *Then we have*

$$N D_N^*(\mathcal{P}) \le \max_{1 \le N' \le N} N' D_{N'}^*(\omega) + 1.$$

Theorem 3 *Let* $b_1 = (b_{1,j})_{j=1}^{\infty}, \ldots, b_{s-1} = (b_{s-1,j})_{j=1}^{\infty}$ *be* $s-1$ *arbitrary sequences of natural numbers greater than* 1 *such that, for all* $1 \le i_1 < i_2 \le s-1$ *and all* $j_1, j_2 \in \mathbb{N}$, b_{i_1,j_1} *and* b_{i_2,j_2} *are coprime. For each* $1 \le i \le s-1$ *and* $j \in \mathbb{N}$, *let* $\sigma_{i,j}$ *be a permutation of* $\mathbb{Z}_{b_{i,j}}$. *For each* $1 \le i \le s$, *denote* $\Sigma_i = (\sigma_{i,j})_{j=1}^{\infty}$. *For each* $N \in \mathbb{N}$, *suppose that* $\mathcal{P}$ *is the generalized Hammersley point set in Cantor bases* $b_1, \ldots, b_{s-1}$ *with respect to* $\Sigma_1, \ldots, \Sigma_{s-1}$ *consisting of* N *points. Then, for any* $N \ge 1$, *we have*

$$N D_N^*(\mathcal{P}) \le \frac{1}{(s-1)!} \prod_{i=1}^{s-1} \left(\frac{\lfloor M_i/2 \rfloor \log N}{\log m_i} + s - 1 \right) + \sum_{l=0}^{s-2} \frac{M_{l+1}}{l!} \prod_{i=1}^{l} \left(\frac{\lfloor M_i/2 \rfloor \log N}{\log m_i} + l \right) + 1,$$

where $M_i = \max\{b_{i,j} \in b_i : b_{i,1} \cdots b_{i,j} \le N\}$ *and* $m_i = \min\{b_{i,j} \in b_i : b_{i,1} \cdots b_{i,j} \le N\}$ $(1 \le i \le s-1)$.

Corollary 1 *Let* $b_1 = (b_{1,j})_{j=1}^{\infty}, \ldots, b_{s-1} = (b_{s-1,j})_{j=1}^{\infty}$ *be* $s-1$ *bounded sequences of natural numbers greater than* 1 *such that, for all* $1 \le i_1 < i_2 \le s-1$ *and all* $j_1, j_2 \in \mathbb{N}$, b_{i_1,j_1} *and* b_{i_2,j_2} *are coprime. For each* $1 \le i \le s-1$ *and* $j \in \mathbb{N}$, *let* $\sigma_{i,j}$ *be a permutation of* $\mathbb{Z}_{b_{i,j}}$. *For each* $1 \le i \le s$, *denote* $\Sigma_i = (\sigma_{i,j})_{j=1}^{\infty}$. *For each* $N \in \mathbb{N}$, *suppose that* $\mathcal{P}$ *is the generalized Hammersley point set in Cantor bases* $b_1, \ldots, b_{s-1}$ *with respect to* $\Sigma_1, \ldots, \Sigma_{s-1}$ *consisting of* N *points. Then, for any* $N \ge 1$, *we have*

$$N D_N^*(\mathcal{P}) \le c \frac{(\log N)^{s-1}}{N} + O\left(\frac{(\log N)^{s-2}}{N} \right)$$

with

$$c = c(b_1, \ldots, b_{s-1}) = \frac{1}{(s-1)!} \prod_{i=1}^{s-1} \frac{\lfloor M_i/2 \rfloor}{\log m_i},$$

where $M_i = \max(b_{i,j})_{j=1}^{\infty}$ *and* $m_i = \min(b_{i,j})_{j=1}^{\infty}$ $(1 \le i \le s-1)$.

A point set $\mathcal{P}$ consisting of N points in $[0,1)^s$ is called a low-discrepancy point set if $D_N^*(\mathcal{P}) = O((\log N)^{s-1}/N)$. In this sense, the generalized Hammersley point set in Cantor bases is a low-discrepancy point set.

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