Monatshefte für Mathematik (2019) 189:487–506
https://doi.org/10.1007/s00605-018-1208-5

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Some properties of universal Dirichlet series

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Received: 14 May 2018 / Accepted: 4 July 2018 / Published online: 10 July 2018

Abstract
We establish some properties of universal Dirichlet series. In particular we give a new estimate on the growth of their coefficients. As a consequence we obtain an information about the admissible size of coefficients of Dirichlet polynomials that approximate a given entire function on a compact set. Moreover we prove that, for all $\alpha > -1$, the sequence of Riesz means $\left(\left(\sum_{k=1}^{n} k^{\alpha}\right)^{-1} \sum_{k=1}^{n} k^{\alpha} D_k(f)\right)$ of partial sums of an universal Dirichlet series f is automatically universal. Finally we show that the Dirichlet series satisfying the universal approximation property with respect to every compact set K (with connected complement) contained in a strip $\{z \in \mathbb{C} : \sigma \leq \Re(z) \leq 0\}$ are not necessarily universal in the left half-plane $\{z \in \mathbb{C} : \Re(z) \leq 0\}$.

Keywords Universal Dirichlet series · Boundary behavior of Dirichlet series · Over-convergence · Approximation in the complex domain

Mathematics Subject Classification 30K10 · 30E10

1 Introduction and notations

For a compact set $K \subset \mathbb{C}$, $A(K)$ will stand for the space of all continuous functions on K which are holomorphic in its interior. The notion of universal Taylor series was independently introduced by Luh [13] and Chui and Parnes [7], and was strengthened by Nestoridis [17], who showed the existence of holomorphic function f on the unit

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Communicated by A. Constantin.

A. Mouze was partly supported by the Grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front)..

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disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ whose Taylor series $\sum_{n \geq 0} a_n z^n$ at 0 satisfies the following universal approximation property: for every compact set $K \subset \{z \in \mathbb{C}; |z| \geq 1\}$ with connected complement and for every function $h \in A(K)$, there exists a subsequence $(\lambda_n) \subset \mathbb{N}$ such that $\sum_{k=0}^{\lambda_n} a_k z^k$ converges to h uniformly on K, as n tends to infinity. Since Nestoridis' theorem, many results on universal Taylor series in the complex plane appeared. We refer the reader to [5] and the references therein. Roughly speaking, this curious phenomenon of universality has the following interpretation: the sequence of partial sums of the Taylor series (at 0) of a holomorphic function on $\mathbb{D}$ can have the worst possible behavior, even on the unit circle. In 2005, Bayart proved that (ordinary) Dirichlet series share this very strange property [3]. For a complex number z, $\Re(z)$ (resp. $\Im(z)$) denotes its real part (resp. imaginary part). Let $\mathbb{C}_+ = \{s \in \mathbb{C} \,;\, \Re(s) > 0\}$ (resp. $\mathbb{C}_- = \{s \in \mathbb{C} \,;\, \Re(s) < 0\}$) be the right (resp. left) half-plane. To any Dirichlet series $f = \sum_{n \geq 1} a_n n^{-s}$, one can associate its abscissa of absolute convergence $\sigma_a(f) = \inf\{\Re(s) : \sum_{n \geq 1} a_n n^{-s}$ is absolutely convergent$\}$ and its n-th partial sum $D_n(f)$, i.e. $D_n(f)(s) = \sum_{k=1}^{n} a_k k^{-s}$. Notice that for all s with $\Re(s) > \sigma_a(f)$, one has $\sum_{n \geq 1} |a_n| n^{-\Re(s)} < +\infty$. Let us consider the space $\mathcal{D}_a(\mathbb{C}_+)$ of absolutely convergent Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$ in $\mathbb{C}_+$ endowed with its natural topology given by the semi-norms $\| \sum_{n \geq 1} a_n n^{-s} \|_\sigma = \sum_{n \geq 1} |a_n| n^{-\sigma}$, $\sigma > 0$. In [3], the author established the existence of *universal Dirichlet series* in $\mathcal{D}_a(\mathbb{C}_+)$ in the following sense: for any "admissible" compact set $K \subset \overline{\mathbb{C}_-}$ with connected complement and for any function $h \in A(K)$, there exists an increasing sequence $(\lambda_n) \subset \mathbb{N}$ such that $\sup_{s \in K} |D_{\lambda_n}(f)(s) - h(s)| \to 0$ as n tends to infinity. Recently in [2] the authors relaxed the assumption that the compact sets be admissible. In the sequel, we will call $\mathcal{U}_d$ the set of such universal Dirichlet series. Several results about these can be found in [2,5,8,9,15]. Observe that obviously they remain valid for universal Dirichlet series with respect to *all* compact sets of $\overline{\mathbb{C}_-}$ with connected complement (not necessarily "admissible" compact sets). In particular in [8] the authors obtain estimates on the growth of coefficients of universal Dirichlet series and, in [15], it is proved that universal Dirichlet series cannot be logarithmically summable at any point of their line of convergence, where the sequence $(\sigma_{L,n}(f))$ of logarithmic means of a Dirichlet series f is given by $\sigma_{L,n}(f)(s) = \frac{1}{\log(n)} \sum_{k=1}^{n} \frac{1}{k} D_k(f)(s)$. Therefore we deduce that universal Dirichlet series cannot be Cesàro summable at any point of their line of convergence. Finally let us introduce the class of universal Dirichlet series with respect to a specific compact set K.

Definition 1.1 Let $K \subset \overline{\mathbb{C}_-}$ be a compact set with connected complement. A Dirichlet series $f = \sum_{k \geq 1} a_k k^{-s} \in \mathcal{D}_a(\mathbb{C}_+)$ is said to be universal with respect to K if the set $\{D_n(f) \;:\; n \in \mathbb{N}\}$ is dense in $A(K)$ endowed with the topology given by the supremum norm on K. We will call $\mathcal{U}_{d,K}$ the set of such universal Dirichlet series.

If $K \cap \{z \in \mathbb{C} : \Re(z) = 0\} = \emptyset$, we know that $\mathcal{U}_{d,K} \neq \mathcal{U}_d$ [8].

In the present short paper, we are going to establish some properties of universal Dirichlet series. First we are interested in the notion of restricted universality. As has been pointed out, an universal Dirichlet series with respect to a specific compact set $K \subset \mathbb{C}_-$ with connected complement is not necessarily an element of $\mathcal{U}_d$. A natural question that arises is whether the same conclusion holds for Dirichlet series that are universal with respect to a compact set $K \subset \overline{\mathbb{C}}_-$ (with connected complement) such that $K \cap \{z \in \mathbb{C} : \Re(z) = 0\} \neq \emptyset$. Actually we are going to establish a stronger result: there exists a Dirichlet series which satisfies the universal approximation property with respect to every compact set K (with connected complement) contained in a strip $\{z \in \mathbb{C} : \sigma < \Re(z) \leq 0\}$ $(\sigma < 0)$ whose partial sums tend to infinity for almost every z (in the sense of Lebesgue measure) in the left half-plane $\{z \in \mathbb{C} : \Re(z) < \sigma - \frac{1}{2}\}$. It is a Dirichlet version of a theorem of Kahane and Melas for universal Taylor series with respect to compact sets that are contained in an annulus [11]. Then we study some boundary behaviors of universal Dirichlet series. First we study the effect of Cesàro summability methods on universal Dirichlet series. The same problem has been studied in the case of universal Taylor series ([4,12, 16]) and in particular it is showed that a Taylor series is universal if and only if the sequence of Cesàro means of its partial sums is universal. In the case of Dirichlet series, we already know that a universal element cannot be Cesàro summable on the boundary. It is easy to prove that the sequence of Cesàro means $\left(n^{-1}\sum_{k=1}^{n} D_k(f)(s)\right)$ of an element $f \in \mathcal{U}_d$ satisfies the universal approximation property with respect to all compact subsets of $\overline{\mathbb{C}}_-$ with connected complement. Furthermore we show that the same result remains true for the sequence of weighted Riesz means of the form $\left(\left(\sum_{k=1}^{n} k^{\alpha}\right)^{-1} \sum_{k=1}^{n} k^{\alpha} D_k(f)(s)\right)$ for $\alpha > -1$. Notice that the case $\alpha = 1$ corresponds to logarithmic means, but we don't know if the universal Dirichlet series remain automatically universal under this summability method. As mentioned above, we only know that they cannot be logarithmically summable on $i\mathbb{R}$. Next we improve the result of [8] on the admissible growth of coefficients of such series. As application, we are interested in the following result of the set of Dirichlet polynomials: let $K \subset \overline{\mathbb{C}}_-$ be a compact set with connected complement and $\delta > 0$, then for every integer N the set $\left\{\sum_{n=N}^{M} a_n n^{-(1-\delta)} n^{-s};\ M \geq N, |a_n| \leq 1\right\}$ is dense in $A(K)$ [2] (see Lemma 4.5 below). We show that this result does not hold anymore if the arbitrarily small real number δ is replaced by the sequence $\delta_n = 1/\log^{1+\varepsilon}(\log(n))$ $(\varepsilon > 0)$ for instance. Finally we deal with an extension of a result of Gauthier [10]: we exhibit universal Dirichlet series whose coefficients are generated by the Riemann zeta-function. To do this, we combine the famous Voronin's theorem with the lemma of approximation by Dirichlet polynomials.

The paper is organized as follows: in Sect. 2 we deal with the results on restricted universality. In Sect. 3 we are interested in Riesz summability methods preserving the universality of Dirichlet series. Section 4 is devoted to the study of the growth of coefficients of universal Dirichlet series and its applications for the approximation by Dirichlet polynomials with a control on the size of coefficients, whereas in Sect. 5 we build universal Dirichlet series thanks to Riemann zeta-function. Section 6 concludes the paper with some open problems.

2 Restricted universality

For power series, the notion of universality with respect to a specific compact set does not coincide with that of universality with respect to all suitable compact sets. We know that a Taylor series of $H(\mathbb{D})$ that is universal with respect to any compact set K with connected complement contained in a closed annulus $\{z \in \mathbb{C} : 1 \leq |z| \leq d\}$ is not necessarily universal in the sense of Nestoridis. We refer the reader to [6, Section 4] for an elementary proof using the fact that all universal Taylor series possess Ostrowski-gaps. But more than this is true: Kahane and Melas showed that there exists a power series $\sum_{k\geq 0} a_k z^k$ having radius of convergence equal to 1, that is universal with respect to any compact set K with connected complement contained in the closed annulus $\{z \in \mathbb{C} : 1 \leq |z| \leq d\}$ and satisfies $\sum_{k=0}^{n} a_k z^k \to \infty$ for almost every (in the sense of Lebesgue measure) z in $\{z \in \mathbb{C} : |z| > d\}$, as n tends to infinity [11]. Thus in the context of Dirichlet series, one can wonder if any Dirichlet series which satisfies the universal approximation property in a given strip is necessarily universal. We give a negative answer. To do this, we establish a stronger result in the spirit of Kahane-Melas theorem: following their main ideas, we show that there exists a Dirichlet series which satisfies the universal approximation property in a given strip but which has a regular behavior in a left half-plane.

Theorem 2.1 *Let $\sigma < 0$. There exists a Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$ in $\mathcal{D}_a(\mathbb{C}_+)$ satisfying the universal approximation property in the strip $\{s \in \mathbb{C} : \sigma < \Re(s) \leq 0\}$ such that $\sum_{j=1}^{n} a_j j^{-s} \to \infty$, as n tends to infinity, for almost every s in the half-plane $\{s \in \mathbb{C} : \Re(s) < \sigma - \frac{1}{2}\}$ (in the sense of the Lebesgue measure).*

Proof Let us consider an universal Dirichlet series $f(s) = \sum_{n\geq 1} b_n n^{-s}$ in $\mathcal{D}_a(\mathbb{C}_+)$. We are going to build inductively the sequence of coefficients (a_n). Set $a_1 = b_1$ and assume that we constructed $a_2, \dots, a_{n-1}$. For every complex number d, we define the sets $E_n(d)$ as follows

$$E_n(d) = \left\{ s \in B_n : \left| \sum_{k=1}^{n-1} a_k k^{-s} + b_n n^{-s} + d n^{-s} \right| \leq \log(\log(n+2)) \right\},$$

where B_n is the compact rectangle

$$B_n = \left\{ s \in \mathbb{C} : \sigma - \frac{1}{2} - \log(\log(n+2)) \right. \\ \left. \leq \Re(s) \leq \sigma - \frac{1}{2} - \frac{\log(\log(n+2))}{\log(n)} \text{ and } |\Im(s)| \leq \log(\log(n+2)) \right\}.$$

Clearly the fact $E_n(d) \cap E_n(d') \neq \emptyset$ implies that

$$|d - d'| n^{-\sigma} \sqrt{n} \log(n) \leq 2 \log(\log(n+2)).$$

Let us choose $\{w_1, \ldots, w_{q_n}\}$ in the closed disc $\{w \in \mathbb{C} : |w + \sum_{k=1}^{n-1}(a_k - b_k)k^{-\sigma}| \leq 1/\log(\log(n+2))\}$ maximal under the requirement

$$|w_j - w_k| \geq 2\frac{\log(\log(n+2))}{\sqrt{n}\log(n)}$$

for $j \neq k$. A simple argument of recovering gives

$$q_n \geq C\frac{n\log^2(n)}{\log^4(\log(n+2))},$$

where C is a constant. We set $d_k = w_k n^\sigma$, for $k = 1, \ldots, q_n$. We conclude that, for $i = 1, \ldots, q_n$, the sets $E_n(d_i)$ are pairwise disjoint. Hence there exists d_{i_n} such that

$$\lambda(E_n(d_{i_n})) \leq \frac{\lambda(B_n)}{q_n} \leq C'\frac{\log^6(\log(n+2))}{n\log^2(n)},$$

where C' depends only on C. We let $a_n = b_n + d_{i_n}$ and $g(s) = \sum_{n\geq 1} a_n n^{-s}$. Then, for every $s \in B_n \backslash E_n(d_{i_n})$, we have $|\sum_{k=1}^n a_k k^{-s}| \geq \log(\log(n+2))$, which implies $\sum_{k=1}^n a_k k^{-s} \to \infty$ as $n \to +\infty$, unless $s \in E_n(d_{i_n})$ for infinitely many values of n. Since $\sum_{n\geq 2}\frac{\log^6(\log(n+2))}{n\log^2(n)} < +\infty$, we have $\sum_{n\geq 1}\lambda(E_n(d_{i_n})) < +\infty$, and the Borel-Cantelli lemma applied to any fixed set B_M ensures that $\sum_{j=1}^n a_j j^{-s} \to \infty$, as $n \to +\infty$, for almost every s in the half-plane $\{s \in \mathbb{C} : \Re(s) < \sigma - \frac{1}{2}\}$. On the other hand we have, for every $n \geq 1$,

$$\left|\sum_{k=1}^n (a_k - b_k)k^{-\sigma}\right| \leq 1/\log(\log(n+2)).$$

Hence the abscissa of convergence $\sigma_c(h)$ of the Dirichlet series $h(s) = \sum_{k\geq 1}(a_k - b_k)k^{-s}$ satisfies $\sigma_c(h) \leq \sigma$. Therefore a classical result (see for instance [1, Theorem 11.11]) ensures that $\sum_{k\geq 1}(a_k - b_k)k^{-s}$ converges uniformly on every compact subset lying interior to the half-plane $\{s \in \mathbb{C} : \Re(s) > \sigma\}$, which implies that $\sum_{k\geq N}(a_k - b_k)k^{-s}$ converges uniformly to 0, as N tends to infinity, on such compact sets. Now, let $K \subset \{s \in \mathbb{C} : \sigma < \Re(s) \leq 0\}$ be a compact set such that K^c is connected and let P be any holomorphic polynomial. Since f is an universal Dirichlet series, there exists an increasing sequence of natural numbers (λ_n) such that

$$\sup_{s\in K}\left|\sum_{k=1}^{\lambda_n} b_k k^{-s} - P(s) + \sum_{k=1}^{+\infty}(a_k - b_k)k^{-s}\right|$$
$$= \sup_{s\in K}\left|\sum_{k=1}^{\lambda_n} a_k k^{-s} - P(s) + \sum_{k=1+\lambda_n}^{+\infty}(a_k - b_k)k^{-s}\right| \to 0$$

as n tends to infinity. Thus by the triangle inequality we get

$$\sup_{s\in K} |\sum_{k=1}^{\lambda_n} a_k k^{-s} - P(s)| \leq \sup_{s\in K} |\sum_{k=1}^{\lambda_n} a_k k^{-s} - P(s) + \sum_{k=1+\lambda_n}^{+\infty} (a_k - b_k)k^{-s}| + \sup_{s\in K} |\sum_{k=1+\lambda_n}^{+\infty} (a_k - b_k)k^{-s}|,$$

and combining the last two inequalities with the fact $\sup_{s\in K} |\sum_{k=N}^{+\infty}(a_k - b_k)k^{-s}| \to 0$, as $N \to +\infty$, we conclude that the Dirichlet series g satisfies the universal approximation property in the strip $\{s \in \mathbb{C} : \sigma < \Re(s) \leq 0\}$. This finishes the proof. □

Remark 2.2 (1) In the above construction, there is a grey zone. What is the behavior of the Dirichlet series g in the strip $\{s \in \mathbb{C} : \sigma - \frac{1}{2} \leq \Re(s) \leq \sigma\}$?
(2) Contrary to the case of Taylor series, we do not know an elementary proof of the simple fact that a Dirichlet series which satisfies the universal approximation property in a given strip is not necessarily universal.

3 Riesz means of universal Dirichlet series

For a Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$, let us consider the Cesàro means of its partial sums:

$$\sigma_n(f)(s) = \frac{D_1(f)(s) + D_2(f)(s) + \cdots + D_n(f)(s)}{n}.$$

Let $K \subset \{s \in \mathbb{C}\ ;\ \Re(s) \leq 0\}$ be a compact set with connected complement and h be an entire function. Let L be a compact set with connected complement such that $K \subset L$ and $\{s - 1, s \in K\} \subset L$. Assume that f is an universal Dirichlet series. Therefore there exists a sequence (λ_n) such that

$$\sup_{s\in L} |D_{\lambda_n}(f)(s) - h(s)| \to 0,\ \text{as } n \to +\infty.$$

Observe that we have $\sigma_{\lambda_n}(f)(s) = \left(1 + \frac{1}{\lambda_n}\right) D_{\lambda_n}(f)(s) - \frac{1}{\lambda_n} D_{\lambda_n}(f)(s - 1)$. Thus the triangle inequality gives

$$\begin{aligned}\sup_{s\in K} |\sigma_{\lambda_n}(f)(s) - h(s)| &\leq \sup_{s\in K} |D_{\lambda_n}(f)(s) - h(s)| + \frac{1}{\lambda_n} \sup_{s\in K} |D_{\lambda_n}(f)(s) - h(s)| \\ &\quad + \frac{1}{\lambda_n} \sup_{s\in K} |D_{\lambda_n}(f)(s - 1) - h(s - 1)| \\ &\quad + \tfrac{1}{\lambda_n}(\sup_{s\in K} |h(s)| + \sup_{s\in K} |h(s - 1)|) \\ &\leq \sup_{s\in L} |D_{\lambda_n}(f)(s) - h(s)| + \frac{2}{\lambda_n} \sup_{s\in L} |D_{\lambda_n}(f)(s) - h(s)| \\ &\quad + \tfrac{2}{\lambda_n} \sup_{s\in L} |h(s)|\end{aligned}$$

and we deduce that $\sup_{s\in K} |\sigma_{\lambda_n}(f)(s) - h(s)| \to 0$, as n tends to infinity. Therefore we proved the following Proposition:

Proposition 3.1 *Let f be a Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$. Assume that f is a universal Dirichlet series, then for every compact set $K \subset \overline{\mathbb{C}}_-$ with connected complement and every entire function h, there exists an increasing sequence $(\lambda_n) \subset \mathbb{N}$ such that $\sigma_{\lambda_n}(f)$ converges to h uniformly on K as n tends to infinity, i.e. the sequence of the Cesàro means of partial sums of an universal Dirichlet series is universal.*

More generally we are interested in the Riesz means related to the summability matrix $A_\alpha = (a_{n,k}(\alpha))$ given by

$$a_{n,k}(\alpha) = \begin{cases} \dfrac{k^{-\alpha}}{\sum_{j=1}^{n} j^{-\alpha}} & \text{for } 1 \le k \le n, \\ 0 & \text{for } k \ge n+1, \end{cases}$$

with $0 < \alpha < 1$. For a Dirichlet series $f(s) = \sum_{n\ge1} a_n n^{-s}$, we define the A_α-Riesz means as follows:

$$\sigma_{A_\alpha,n}(f)(s) = \frac{1}{\sum_{k=1}^{n} k^{-\alpha}} \left(\sum_{k=1}^{n} k^{-\alpha} D_k(f)(s) \right).$$

The case $\alpha = 0$ corresponds to the Cesàro means.

We will need the following lemma, which is an exercise left to the reader.

Lemma 3.2 *Let $0 < \alpha < 1$. For any integer $l \ge 1$, we have*

$$\sum_{k=1}^{n} k^{-\alpha} = \frac{n^{1-\alpha}}{1-\alpha} + l_\alpha + \sum_{j=0}^{l} \frac{C_{\alpha,j}}{n^{\alpha+j}} + \frac{\varepsilon_\alpha(n)}{n^{\alpha+l}},$$

where the numbers l_α and $C_{\alpha,j}$, $j = 0, \dots, l$, don't depend on n ($C_{\alpha,0} = \frac{1}{2}$, $C_{\alpha,1} = \frac{-\alpha}{12}, \dots$) and $\varepsilon_\alpha(n) \to 0$ as $n \to +\infty$.

Theorem 3.3 *Let f be a Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$. Assume that f is a universal Dirichlet series, then for every compact set $K \subset \overline{\mathbb{C}}_-$ with connected complement and every entire function h, there exists an increasing sequence $(\lambda_n) \subset \mathbb{N}$ such that $\sigma_{A_\alpha,\lambda_n}(f)$ converges to h uniformly on K as n tends to infinity, i.e. the sequence of the Riesz means $(\sigma_{A_\alpha,n}(f))$ is universal.*

Proof Let $K \subset \overline{\mathbb{C}}_-$ be a compact set with connected complement and h be an entire function. We set $\sigma_K = \inf\{\Re(s);\, s \in K\}$, $t_K = \sup\{|\Im(s)|;\, s \in K\}$ and $l_K = \min\{l \in \mathbb{N};\, \sigma_K + \alpha + l > 0\}$. Let us define the compact sets K_α, K_α^- and K_α^+ as follows:

$$K_\alpha = \{s \in \mathbb{C};\, \sigma_K + \alpha - 1 \le \Re(s) \le \alpha + l_K \text{ and } |\Im(s)| \le t_K\},$$
$$K_\alpha^- = K_\alpha \cap \overline{\mathbb{C}}_- \text{ and } K_\alpha^+ = K_\alpha \cap \overline{\mathbb{C}}_+.$$

Observe that K_α^- has connected complement and $K \subset K_\alpha^-$. Since $f = \sum_{n\geq 1} a_n n^{-s}$ is an universal Dirichlet series, there exists an increasing sequence $(\lambda_n) \subset \mathbb{N}$ such that

$$\sup_{s\in K_\alpha^-} |D_{\lambda_n}(f)(s) - h(s)| \to 0, \text{ as } n \to +\infty. \tag{1}$$

We are going to prove that $\sigma_{A_\alpha,\lambda_n}(f)$ converges to h uniformly on K as n tends to infinity. Let us write

$$\begin{aligned}\sigma_{A_\alpha,n}(f)(s) &= \frac{\sum_{k=1}^n k^{-\alpha} D_k(f)(s)}{\sum_{k=1}^n k^{-\alpha}}\\ &= \frac{\sum_{j=1}^n a_j \left(\sum_{k=1}^n k^{-\alpha} - (\sum_{k=1}^j k^{-\alpha} - j^{-\alpha})\right) j^{-s}}{\sum_{k=1}^n k^{-\alpha}}\\ &= D_n(f)(s) + \frac{D_n(f)(s+\alpha)}{\sum_{k=1}^n k^{-\alpha}} - \frac{\sum_{j=1}^n a_j (\sum_{k=1}^j k^{-\alpha}) j^{-s}}{\sum_{k=1}^n k^{-\alpha}}\end{aligned} \tag{2}$$

Lemma 3.2 ensures that we can find $N \geq 1$ such that for every $n \geq N$, we have

$$\left| \sum_{k=N+1}^n k^{-\alpha} - \frac{n^{1-\alpha}}{1-\alpha} - l_\alpha - \sum_{j=0}^{l_K} \frac{C_{\alpha,j}}{n^{\alpha+j}} \right| < \frac{1}{n^{\alpha+l_K}}. \tag{3}$$

Then we write, for all n with $\lambda_n \geq N$,

$$\begin{aligned}\sum_{j=1}^{\lambda_n} a_j \left(\sum_{k=1}^j k^{-\alpha}\right) j^{-s} &= \sum_{j=1}^{N} a_j \left(\sum_{k=1}^j k^{-\alpha}\right) j^{-s}\\ &+ \sum_{j=N+1}^{\lambda_n} a_j \left(\sum_{k=1}^N k^{-\alpha}\right) j^{-s} + \sum_{j=N+1}^{\lambda_n} a_j (\sum_{k=N+1}^j k^{-\alpha}) j^{-s}.\end{aligned} \tag{4}$$

Clearly, we have

$$\sup_{s\in K} \left| \frac{\sum_{j=1}^N a_j (\sum_{k=1}^j k^{-\alpha}) j^{-s}}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} \right| \to 0 \text{ as } n \to +\infty.$$

By the triangle inequality, we get

$$\left| \sum_{j=N+1}^{\lambda_n} a_j (\sum_{k=1}^N k^{-\alpha}) j^{-s} \right| \leq \left| \sum_{k=1}^N k^{-\alpha} \right| \left(\left| D_{\lambda_n}(f)(s) - h(s) \right| + \left| \sum_{j=1}^N a_j j^{-s} \right| + |h(s)| \right). \tag{5}$$

Therefore we deduce, using both (1) and $K \subset K_\alpha^-$,

$$\sup_{s\in K}\left|\frac{\sum_{j=N+1}^{\lambda_n} a_j(\sum_{k=1}^{N} k^{-\alpha})j^{-s}}{\sum_{k=1}^{\lambda_n} k^{-\alpha}}\right| \to 0 \text{ as } n \to +\infty. \tag{6}$$

Let us now consider the last term of (4). Using Lemma 3.2 and the estimate (3) we can write, for all positive integer n with $\lambda_n \geq N$,

$$\begin{aligned}\sum_{j=N+1}^{\lambda_n} a_j(\sum_{k=N+1}^{j} k^{-\alpha})j^{-s} = \frac{1}{1-\alpha}\sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha-1)} + l_\alpha \sum_{j=N+1}^{\lambda_n} a_j j^{-s} \\ + \sum_{i=0}^{l_K} C_{\alpha,i} \sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha+i)} \\ + \sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha+l_K)}\varepsilon_\alpha(j),\end{aligned} \tag{7}$$

with $|\varepsilon_\alpha(j)| \leq 1$, for all $j \geq N$. Clearly we have

$$\sup_{s\in K}\left|\sum_{j=N+1}^{\lambda_n} a_j j^{-s}\right| \leq \sup_{s\in K}|D_{\lambda_n}(f)(s) - h(s)| + \sup_{s\in K}|h(s)| + \sup_{s\in K}|D_N(f)(s)|. \tag{8}$$

Combining (1) with (8), we get

$$\frac{\sup_{s\in K}|\sum_{j=N+1}^{\lambda_n} a_j j^{-s}|}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} \to 0 \text{ as } n \to +\infty. \tag{9}$$

In the same way, observe that for $s \in K$, we necessarily have $s + \alpha - 1 \in K_\alpha^-$ and we have

$$\begin{aligned}\sup_{s\in K}\left|\sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha-1)}\right| \leq \sup_{s\in K_\alpha^-}|D_{\lambda_n}(f)(s) - h(s)| + \sup_{s\in K_\alpha^-}|h(s)| \\ + \sup_{s\in K_\alpha^-}|D_N(f)(s)|,\end{aligned} \tag{10}$$

which gives, using (1),

$$\frac{\sup_{s\in K}|\sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha-1)}|}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} \to 0 \text{ as } n \to +\infty. \tag{11}$$

Let us now estimate the sums $\sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha+i)}$, for $i = 0, \dots, l_K - 1$ and $s \in K$. By construction the property $s \in K$ implies the following property $s + \alpha + i \in K_\alpha$, for all $i = 0, \dots, l_K - 1$. For any integer $i \in \{0, \dots, l_K - 1\}$, we have to consider two cases:

Case $s+\alpha+i \in K_\alpha^-$: we have, for any complex number $s \in K$ with $s+\alpha+i \in K_\alpha^-$,

$$\left|\sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha+i)}\right| \leq \sup_{s\in K_\alpha^-} |D_{\lambda_n}(f)(s)-h(s)| + \sup_{s\in K_\alpha^-} |h(s)| + \sup_{s\in K_\alpha^-} |D_N(f)(s)|. \tag{12}$$

Case $s+\alpha+i \in K_\alpha^+$: let us choose $\eta > 0$ with $1-\alpha-\eta > 0$. Thus, for any complex number $s \in K$ with $s+\alpha+i \in K_\alpha^+$, the following estimate holds

$$\begin{aligned}\left|\sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha+i)}\right| &\leq \left|\sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha+i+\eta)} j^\eta\right| \\ &\leq \lambda_n^\eta \sum_{j=1}^{+\infty} |a_j| j^{-\eta}\end{aligned} \tag{13}$$

Thus, combining (12) with (13), we get

$$\begin{aligned}\frac{\sup_{s\in K} |\sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha+i)}|}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} \leq &\frac{\sup_{s\in K_\alpha^-} |D_{\lambda_n}(f)(s) - h(s)|}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} + \frac{\sup_{s\in K_\alpha^-} |h(s)|}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} \\ &+ \frac{\sup_{s\in K_\alpha^-} |D_N(f)(s)|}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} + \frac{\lambda_n^\eta}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} \sum_{j=1}^{+\infty} |a_j| j^{-\eta}\end{aligned} \tag{14}$$

Therefore taking into consideration (1), the estimate $\sum_{k=1}^{\lambda_n} k^{-\alpha} \sim \frac{\lambda_n^{1-\alpha}}{1-\alpha}$ as $n \to +\infty$ and the property $1-\alpha-\eta > 0$, we get

$$\frac{\sup_{s\in K} |\sum_{j=1}^{\lambda_n} a_j j^{-(s+\alpha+i)}|}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} \to 0, \text{ as } n \to +\infty. \tag{15}$$

In the same way, we have, for all $\lambda_n \geq N$,

$$\frac{\sup_{s\in K} \left|\sum_{j=N+1}^{\lambda_n} a_j j^{-(s+\alpha+l_K)} \varepsilon_\alpha(j)\right|}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} \leq \frac{\lambda_n^\eta}{\sum_{k=1}^{\lambda_n} k^{-\alpha}} \sum_{j=N+1}^{+\infty} |a_j| j^{-\eta} \to 0 \text{ as } n \to +\infty. \tag{16}$$

Let us recall that $K \subset K_\alpha^-$. Applying (1), we have also $\sup_{s\in K} |D_{\lambda_n}(f)(s)-h(s)| \to 0$ as $n \to +\infty$. Hence combining (4), (6), (7), (9), (11), (15), (16) with (2), we obtain

$$\sup_{s\in K} |\sigma_{A_\alpha,\lambda_n}(f)(s) - h(s)| \to 0 \text{ as } n \to +\infty.$$

This finishes the proof. □

Remark 3.4 (1) On the other hand, the same method allows to prove that Theorem 3.3 remains true in the case $\alpha < 0$. Therefore if f is an universal Dirichlet series, then for all $\alpha < 1$, the sequence of the Riesz means $(\sigma_{A_\alpha,n}(f))$ is universal.
(2) Let $k \geq 1$ be an integer. Let us consider the Cesàro means of order k

$$\sigma_{(C,k),n}(f)(s) = \sum_{j=1}^{n} \left(1 - \frac{j-1}{n}\right)\left(1 - \frac{j-1}{n+1}\right) \dots \left(1 - \frac{j-1}{n+k-1}\right) a_j j^{-s}$$

of a Dirichlet series $f = \sum_{j \geq 1} a_j j^{-s}$. As in Proposition 3.1, it is easy to check that the universality of f implies the universality of the sequence $(\sigma_{(C,k),n}(f))$ for all integer $k \geq 1$.

4 Growth of the coefficients of universal Dirichlet series and applications

We begin by giving a slight improvement of the estimate of the growth of coefficients of universal Dirichlet series. In [8], the authors showed that for an universal Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$ with respect to a compact set $K \subset i\mathbb{R}$ we have $\limsup(n|a_n|e^{-\sqrt{b_n \log(n)}}) = +\infty$, provided that (b_n) is a decreasing sequence satisfying $\sum_{n \geq 2} b_n/(n \log(n)) < +\infty$. The proof was inspired by that employed by [14] to handle the case of coefficients of universal Taylor series. Using similar ideas, we slightly strengthen this result. In the sequel, for $j \geq 1$, we denote by $\log_j$ the j-th iterated of the function log, i.e. for every positive integer n sufficiently large, we have $\log_1(n) = \log(n)$, $\log_2(n) = \log(\log(n))$, ... We set $q_j = \min(n \in \mathbb{N} : \log_j(n) > 1)$. To proceed further, it is convenient to state the following key-lemma.

Lemma 4.1 *Let $\eta > 0$, $k \geq 1$ be an integer and let $\sum_{n \geq 1} a_n n^{-s}$ be a Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$ that satisfies the universal approximation property on $\{it : -\eta \leq t \leq \eta\}$. Assume that (ε_n) is a decreasing sequence of positive real numbers satisfying $\sum \varepsilon_n/(n \log_1(n) \dots \log_k(n)) < +\infty$ and $\varepsilon_n \log(n) \to +\infty$, as n tends to infinity. Then we have*

$$\sum |a_n| e^{-\frac{\varepsilon_n \log(n)}{\log_2(n) \dots \log_{k+1}(n)}} = +\infty.$$

Proof We follow the main ideas of the proof of [8, Lemma 2.1]. We set $\delta_n = e\varepsilon_n$ for $n \in \mathbb{N}$. There exists $N_0 \geq q_{k+1}$ such that, for every $n \geq N_0$, we have

$$\log_k(\varepsilon_n \log(n)) > 1, \quad \frac{\varepsilon_n \log(n)}{\log_1(\varepsilon_n \log(n)) \dots \log_k(\varepsilon_n \log(n))} \geq q_{k+1} \tag{17}$$

and

$$\sum_{n=N_0}^{\infty} \frac{\delta_n}{n \log_1(n) \dots \log_k(n)} < \eta.$$

We define the functions from $i\mathbb{R}$ which are $2i\pi$-periodic letting

$$\begin{cases} H_n(it) = \dfrac{n \log(n)}{\delta_n}\pi & \text{for } |t| < \dfrac{\delta_n}{n \log_1(n) \dots \log_k(n)}, \\ H_n(it) = 0 & \text{for } \dfrac{\delta_n}{n \log_1(n) \dots \log_k(n)} \leq |t| \leq \pi. \end{cases}$$

For $f \in L^1([-\pi, \pi])$, let us define its Fourier coefficients $\hat{f}(\log(m)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(it)m^{it}dt$. We get

$$\hat{H}_n(\log(m)) = \begin{cases} 1 & \text{for } m = 1, \\ \dfrac{n \log_1(n) \dots \log_k(n)}{\delta_n \log(m)} \sin\left(\dfrac{\delta_n \log(m)}{n \log_1(n) \dots \log_k(n)}\right) & \text{for } m \neq 1. \end{cases}$$

Let $N \geq N_0$ be an integer. By hypothesis we can approximate the Dirichlet polynomial $1 + \sum_{m=1}^{N-1} a_m m^{-z}$ by a subsequence of partial sums of f uniformly on the compact set $\{it : -\eta \leq t \leq \eta\}$. Therefore there exists an integer $M > N$ such that we have, for all $t \in [-\eta, \eta]$, $\left|1 - \sum_{m=N}^{M} a_m m^{-it}\right| < \frac{1}{2}$ which implies $\frac{1}{2} \leq \Re\left(\sum_{m=N}^{M} a_m m^{-it}\right)$. Thus we proceed as in the proof of Lemma 2.1 of [8] to obtain

$$\frac{1}{2} \leq \sum_{m=N}^{M} |a_m| \times |\hat{f}(\log(m))|. \tag{18}$$

with

$$\hat{f}(\log(m)) = \prod_{n=N_0}^{M} \frac{\sin\left(\dfrac{\delta_n \log(m)}{n \log_1(n) \dots \log_k(n)}\right)}{\dfrac{\delta_n \log(m)}{n \log_1(n) \dots \log_k(n)}}.$$

As (δ_n) is a decreasing sequence and the series $\sum \frac{\delta_n}{n \log_1(n) \dots \log_k(n)}$ converges, we get $\delta_n \to 0$, as $n \to +\infty$. Therefore, there exists an integer N such that we have the following two inequalities $\frac{\delta_{N_0}}{N_0 l_1(N_0) \dots l_k(N_0)} \log(N) > e$ and $\delta_N < e$. For every $m \in \{N, \dots, M\}$, we have

$$\frac{\delta_{N_0}}{N_0 \log_1(N_0)\dots\log_k(N_0)}\log(m) > e \text{ and } \frac{\delta_m}{m\log_1(m)\dots\log_k(m)}\log(m) < \delta_m \leq \delta_N < e.$$

Then there exists an integer $l \in \{N_0, \cdots, m-1\}$ satisfying

$$\frac{\delta_l}{l\log_1(l)\dots\log_k(l)}\log(m) \geq e \quad \text{and} \quad \frac{\delta_{l+1}}{(l+1)\log_1(l+1)\dots\log_k(l+1)}\log(m) < e.$$

Since the sequence (δ_n) is decreasing, we get

$$|\hat{f}(\log(m))| \leq \prod_{n=N_0}^{l}\frac{n\log_1(n)\dots\log_k(n)}{\delta_n\log(m)} \leq \left(\frac{l\log_1(l)\dots\log_k(l)}{\delta_l\log(m)}\right)^{l+1-N_0} \leq \left(\frac{1}{e}\right)^{l+1-n_0}. \tag{19}$$

Set $u_m = \frac{\varepsilon_m\log(m)}{\log_1(\varepsilon_m\log(m))\dots\log_k(\varepsilon_m\log(m))}$. Taking into account (17), observe that we have

$$u_m\log_1(u_m)\dots\log_k(u_m) \leq u_m\log_1(\varepsilon_m\log(m))\dots\log_k(\varepsilon_m\log(m)) \leq \varepsilon_m\log(m).$$

We deduce

$$l+1 \geq u_m \geq \frac{\varepsilon_m\log(m)}{\log_2(m)\dots\log_{k+1}(m)}. \tag{20}$$

Combining (18) with (19) and (20) we obtain

$$\sum_{m=N}^{M}|a_m|e^{n_0}e^{-\frac{\varepsilon_m\log(m)}{\log_2(m)\dots\log_{k+1}(m)}} \geq \frac{1}{2}.$$

Since this holds for infinitely many pairs (N, M), we have the conclusion. □

Remark 4.2 The estimate given by (20) seems to be optimal since the unique solution x_n of the equation $x\log(x)\log_2(x)\dots\log_k(x) = n$ has the following behavior: $x_n \sim \frac{n}{\log_2(n)\dots\log_{k+1}(n)}$ as n tends to infinity.

Next Lemma 4.1 leads to the following statement.

Theorem 4.3 *Let $\eta > 0$, $k \geq 1$ be an integer and let $\sum_{n\geq 1} a_n n^{-s}$ be a Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$ satisfying the universal approximation property on $\{it : -\eta \leq t \leq \eta\}$. Let (b_n) be a decreasing sequence satisfying $\sum b_n/(n\log_1(n)\dots\log_k(n)) < +\infty$ and $b_n\log(n) \to +\infty$, as n tends to infinity. Then, we have*

$$\limsup_{n\to+\infty}\left(n|a_n|e^{-\frac{b_n\log(n)}{\log_2(n)\dots\log_{k+1}(n)}}\right) = +\infty.$$

Proof Assume that there exists a real number M such that, for all integer n large enough,

$$|a_n| \leq \frac{M}{n} e^{\frac{b_n \log(n)}{\log_2(n) \dots \log_{k+1}(n)}}. \tag{21}$$

Let $\varepsilon > 0$. We set $\varepsilon_n = \max\left(b_n, \frac{\log_{k+1}(n)}{\log_k^{\varepsilon}(n)}\right) + b_n$ for all integer n large enough. Obviously the sequence (ε_n) is decreasing, $\sum \frac{\varepsilon_n}{n \log_1(n) \dots \log_k(n)}$ converges and $\varepsilon_n \log(n) \to +\infty$, as n tends to infinity. Thus Lemma 4.1 ensures that $\sum |a_n| e^{-\frac{\varepsilon_n \log(n)}{\log_2(n) \dots \log_{k+1}(n)}} = +\infty$. On the other hand, combining (21) with the equality $b_n = \varepsilon_n - \max\left(b_n, \frac{\log_{k+1}(n)}{\log_k^{\varepsilon}(n)}\right)$ and the estimate $\max\left(b_n, \frac{\log_{k+1}(n)}{\log_k^{\varepsilon}(n)}\right) \geq \frac{\log_{k+1}(n)}{\log_k^{\varepsilon}(n)}$, we have for every positive integer B, with $B > A$ and A fixed large enough,

$$\sum_{n=A}^{B} |a_n| e^{-\frac{\varepsilon_n \log(n)}{\log_2(n) \dots \log_{k+1}(n)}} \leq M \sum_{n=A}^{B} \frac{1}{n} e^{-\frac{\log(n)}{\log_2(n) \dots \log_{k-1}(n) \log_k^{1+\varepsilon}(n)}}.$$

However it is easy to check that $\sum \frac{1}{n} e^{-\frac{\log(n)}{\log_2(n) \dots \log_{k-1}(n) \log_k^{1+\varepsilon}(n)}} < +\infty$. We obtain a contradiction and we have the announced conclusion. □

Now let $\varepsilon > 0$ and $k \geq 2$ be an integer. If we apply Theorem 4.3 with the sequence (b_n) given by $b_n = \frac{\log_{k+1}(n)}{\log_k^{\varepsilon}(n)}$, we obtain the following corollary.

Corollary 4.4 *Let $k \geq 2$ be a positive integer and $\varepsilon > 0$. Let $\eta > 0$ and let $\sum_{n \geq 1} a_n n^{-s}$ be a Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$ that satisfies the universal approximation property on $\{it : -\eta \leq t \leq \eta\}$. Then, we have*

$$\limsup_{n \to +\infty} \left(n|a_n| e^{-\frac{\log(n)}{\log_2(n) \dots \log_{k-1}(n) (\log_k(n))^{1+\varepsilon}}} \right) = +\infty.$$

Now we apply this last result to obtain some information on the admissible size of coefficients of Dirichlet polynomials that approximate continuous functions on closed intervals of the imaginary axis. More precisely, a careful examination of the proof of Theorem 3.1 of [2] shows that the following result holds.

Lemma 4.5 *Let $K \in \overline{\mathbb{C}}_-$ be a compact set with connected complement and $\delta > 0$. Then, for every $N \in \mathbb{N}$, the set $\left\{ \sum_{n=N}^{M} a_n n^{-(1-\delta)} n^{-s};\ M \geq N, |a_n| \leq 1 \right\}$ is dense in $A(K)$.*

On the other hand, Theorem 4.3 allows to obtain the following result. We shall use the following notation: for every integer $k \geq 2$ and every $\tau > 0$, we set

$$\delta_{k,\tau}(n) = \frac{1}{\log_2(n)\ldots\log_{k-1}(n)\log_k^{1+\tau}(n)} \text{ for all } n \text{ large enough} (n \geq q_k)$$

and

$$A_{k,\tau} = \left\{(a_n) \in \mathbb{C}^{\mathbb{N}} : |a_n| \leq n^{-(1-\tau)} \text{ for } n < q_k \text{ and } |a_n| \leq n^{-(1-\delta_{k,\tau}(n))} \text{ for } n \geq q_k\right\}.$$

Corollary 4.6 *Let $K \in \overline{\mathbb{C}}_-$ be a compact set with connected complement such that K contains a set of the following type $\{it : a \leq t \leq b\}$, $a, b \in \mathbb{R}$. Let $k \geq 2$ be a positive integer and $\tau > 0$. Then the set $\left\{\sum_{n=1}^M a_n n^{-s};\, M \geq 1,\, (a_n) \in A_{k,\tau}\right\}$ is not dense in $A(K)$.*

Up to do a translation on the imaginary axis we can assume that there exists $\eta > 0$ such that $\{it : -\eta \leq t \leq \eta\} \subset K$. Hence to prove Corollary 4.6, it suffices to establish the following statement.

Proposition 4.7 *Let $k \geq 2$ be a positive integer and $\eta, \tau > 0$. Then the set of Dirichlet polynomials defined by $\left\{\sum_{n=1}^M a_n n^{-it};\, M \geq 1,\, (a_n) \in A_{k,\tau}\right\}$ is not dense in $C([-\eta, \eta])$.*

Proof Set $E_N = \left\{\sum_{n=N}^M a_n n^{-it};\, M \geq 1,\, |a_n| \leq n^{-(1-\delta_{k,\tau}(n))}\right\}$ for $N \geq q_k$. We argue by contradiction. Assume that
Hypothesis (H): for every integer $N \geq q_k$, the set E_N is dense in $C([-\eta, \eta])$.
Let also P be a Dirichlet polynomial, $g \in A(K)$, $\sigma > 0$ and $\varepsilon > 0$. We choose $N \geq q_k$ bigger than the degree of P such that $\sum_{n \geq N} n^{-1-\sigma+\delta_{k,\tau}(n)} < \varepsilon$. In particular, by hypothesis there exists $u = \sum_{n=N}^M a_n n^{-(1-\delta_{k,\tau}(n))} n^{-it}$, such that

$$|a_n| \leq 1 \text{ and } \sup_{t \in [-\eta,\eta]} |u(it) - g(t) + P(it)| < \varepsilon.$$

Setting $h(t) = u(t) + P(it)$, we deduce

$$\sup_{t \in [-\eta,\eta]} |u(it) - g(t) + P(it)| < \varepsilon \text{ and } \|h - P\|_\sigma = \|u(s)\|_\sigma \leq \sum_{n \geq N} n^{-1-\sigma+\delta_{k,\tau}(n)} < \varepsilon.$$

Thus we have shown that, under the hypothesis (H), for every Dirichlet polynomial P, for every continuous function $g \in C([-\eta, \eta])$, for all $\sigma > 0$ and $\varepsilon > 0$, there exists a Dirichlet polynomial h such that $\sup_K |h - g| < \varepsilon$ and $\|h - P\|_\sigma < \varepsilon$. Then following a classical construction [2,3,8], we can can build Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$ of the form $\sum a_n n^{-s}$, with $|a_n| \leq n^{-(1-\delta_{k,\tau}(n))}$, which satisfies the universal approximation property on $[-\eta, \eta]$. Then Theorem 4.3 gives a contradiction. Thus the hypothesis (H) does not hold. Hence there exists a positive integer $N_0 = N_0(\tau) \geq q_k$ such that,

$$\forall N \geq N_0, \text{ the set } \left\{ \sum_{n=N}^{M} a_n n^{-(1-\delta_{k,\tau}(n))} n^{-it};\, M \geq N, |a_n| \leq 1 \right\} \text{ is not dense in } C([-\eta, \eta]). \tag{22}$$

It remains to check that, for any $\varepsilon > 0$, for every integer $N < N_0(\varepsilon)$ the set of Dirichlet polynomials $\left\{\sum_{n=1}^{M} a_n n^{-it};\, M \geq 1, (a_n) \in A_{k,\tau}\right\}$ is not dense in $C([-\eta, \eta])$. If not, for $\tau > 0$ fixed, for any continuous function g and $\varepsilon > 0$, one can find $M \geq 1$ and complex numbers $a_1, \ldots, a_M$, with $(a_n) \in A_{k,\tau}$, such that

$$\sup_{t \in [-\eta, \eta]} \left| \sum_{n=1}^{M} a_n n^{-it} - g(t) N_0^{1+\alpha+it} \right| < \varepsilon,$$

where $N_0 = N_0(\tau/2) \geq q_k$ ($N_0(\tau/2)$ is given by (22)) and $\alpha > 0$ satisfy the condition

$$N_0^{\alpha} \geq n^{\tau} \text{ for } n = 1, \ldots, q_k, \text{ and } \sup_{l \geq q_k} |l^{\delta_{k,\tau}(l) - \delta_{k,\tau/2}(l N_0)}| \leq N_0^{\alpha}. \tag{23}$$

We deduce

$$\sup_{t \in [-\eta, \eta]} \left| \sum_{n=1}^{M} a_n (n N_0)^{-1} N_0^{-\alpha} (n N_0)^{-it} - g(t) \right| < \varepsilon.$$

Observe that we can rewrite

$$\sum_{n=1}^{M} a_n (n N_0)^{-1} N_0^{-\alpha} (n N_0)^{-it} = \sum_{j=N_0}^{M N_0} b_j j^{-it},$$

where the coefficients b_j satisfy the following estimates (thanks to (23), $N_0 \geq q_k$ and $(a_n) \in A_{k,\tau}$):

(1) $b_j = 0$ if $j \neq n N_0$, with $n = 1, \ldots, M$,
(2) for $n = 1, \ldots, q_k - 1$,

$$|b_{n N_0}| = |a_n| N_0^{-1-\alpha} \leq \frac{1}{n^{1-\tau} N_0^{1+\alpha}} \leq (n N_0)^{-(1-\delta_{k,\tau/2}(n N_0))}$$

(3) for $n = q_k, \ldots, M$,

$$\begin{aligned} |b_{n N_0}| = |a_n| N_0^{-1-\alpha} &\leq n^{-(1-\delta_{k,\tau}(n))} N_0^{-1-\alpha} \\ &\leq (n N_0)^{-(1-\delta_{k,\tau/2}(n N_0))} n^{\delta_{k,\tau}(n) - \delta_{k,\tau/2}(n N_0)} N_0^{-\alpha} \\ &\leq (n N_0)^{-(1-\delta_{k,\tau/2}(n N_0))}. \end{aligned}$$

Thus for any continuous function g, any $\varepsilon > 0$, we have found a Dirichlet polynomial $\sum_{j=N_0}^{MN_0} b_j j^{-it}$ with $N_0 = N_0(\tau/2)$ which is defined in (22) and $|b_j| \leq j^{-(1-\delta_{k,\tau/2}(j))}$, such that

$$\sup_{t\in[-\eta,\eta]} \left| \sum_{j=N_0}^{MN_0} b_j j^{-it} - g(t) \right| < \varepsilon.$$

According to (22), we obtain a contradiction. This finishes the proof. □

5 Coefficients of universal Dirichlet series and the Riemann zeta-function

Let us consider the Dirichlet polynomial approximation again. Observe that we can rewrite the statement of [2, Theorem 3.1] as follows.

Lemma 5.1 *Let $K \subset \overline{\mathbb{C}}_-$ be a compact set with connected complement, g an entire function, $N > 0$ a positive integer, $0 < \delta < \sigma$ and $\varepsilon > 0$. Then there exist a Dirichlet polynomial $h = \sum_{n=N}^{M} a_n n^{-z}$ such that $\sup_{z\in K} |h(z) - g(z)| < \varepsilon$, $\|h\|_\sigma < \varepsilon$ and $|a_n| \leq \frac{1}{n^{1-\delta}}$, for $n = N, \ldots, M$.*

We end the paper with the Dirichlet version of a result of Gauthier [10]. Although universal Dirichlet series are generic, we do not know exhibit such elements. A similar problem holds in the case of universal power series. In a recent work [10] the Riemann-zeta function was employed to generate the coefficients of universal Taylor series in the sense of Nestoridis. The proof is constructive and is a combination of a lemma of approximation polynomial [10, Lemma 3.2] (which gives a geometric control of growth of coefficients) with Voronin's Theorem. First let us recall the latter.

Theorem 5.2 (Voronin [18]) *For each z_0 in the strip $1/2 < \sigma < 1$ and each $k = 0, 1, 2, \ldots,$ the sequence*

$$\left\{ \left(\zeta(z_0 + im), \zeta'(z_0 + im), \ldots, \zeta^{(k)}(z_0 + im) \right) : m = 1, 2, \ldots \right\}$$

is dense in $\mathbb{C}^{k+1}$.

In the case of Dirichlet series, using Lemma 5.1 and following the ideas of [10], we can build an universal Dirichlet series whose coefficients are generated by the Riemann zeta-function. To do this, we need a Dirichlet version of Lemma 3.4 of [10]. Let (g_n) be an enumeration of all Dirichlet polynomials with rational complex coefficients and (K_n) be the sequence of compact sets $K_n = [-n, 0] \times [-n, n]$.

Lemma 5.3 *Let σ be a real number with $1/2 < \sigma < 1$. Let (δ_n) be a decreasing sequence of positive real numbers with $\delta_1 < 1/2$. There exist a sequence (m_n) of integers and an increasing sequence (k_n) of integers (with $k_0 = 0$) such that*

$$|\zeta(\sigma + im_l)| \leq \frac{\log(l + 2)}{l^{1-\delta_n}} \; for \; l = 1 + k_{n-1}, \ldots, k_n,$$

and the polynomials $Q_n(z)=\sum_{l=1+k_{n-1}}^{k_n}\zeta(\sigma+im_l)l^{-z}$ *have the following approximation property*

$$\sup_{z\in K_n}\left|Q_j(z)-g_n(z)+Q_1(z)+\cdots+Q_{n-1}(z)\right|\leq\frac{2}{n}.$$

Proof First we apply Lemma 5.1 to find a Dirichlet polynomial $w_1=a_{1,1}+a_{1,2}2^{-z}+\cdots+a_{1,k_1}k_1^{-z}$ such that $|a_{1,l}|\leq l^{-1+\delta_1}$, for $l=1,\ldots,k_1$ $\sup_{z\in K_1}|w_1(z)-g_1(z)|<\frac{1}{2}$. By Theorem 5.2 there are integers $m_1,\ldots,m_{k_1}$ such that

$$|\zeta(\sigma+im_l)|\leq\log(l+2)l^{-1+\delta_1}\text{ for }l=1,\ldots,k_1,$$

and

$$\sup_{z\in K_1}|Q_1(z)-g_1(z)|<1,$$

with $Q_1(z)=\sum_{l=1}^{k_1}\zeta(\sigma+im_l)l^{-z}$. By induction, suppose for $j=1,\ldots,n-1$ we have built integers $k_1,\ldots,k_{n-1}$, and $m_1,\ldots,m_{k_{n-1}}$ such that

$$|\zeta(\sigma+im_l)|\leq\frac{\log(l+2)}{l^{1-\delta_j}}\text{ for }l=1+k_{j-1},\ldots,k_j,$$

and the polynomials $Q_j(z)=\sum_{l=1+k_{j-1}}^{k_j}\zeta(\sigma+im_l)l^{-z}$ have the following approximation property

$$\sup_{z\in K_j}\left|Q_j(z)-g_j(z)+Q_1(z)+\cdots+Q_{j-1}(z)\right|\leq\frac{2}{j}.$$

We apply Lemma 5.1 again to find a Dirichlet polynomial $w_n(z)=a_{n,k_{n-1}+1}(k_{n-1}+1)^{-z}+\cdots+a_{n,k_n}k_n^{-z}$ satisfying $|a_{n,l}|\leq l^{-1+\delta_1}$, for $l=k_{n-1}+1,\ldots,k_n$ $\sup_{z\in K_n}|w_n(z)-g_n(z)|<\frac{1}{n}$. By Theorem 5.2 there are integers $m_{k_{n-1}+1},\ldots,m_{k_n}$ such that

$$|\zeta(\sigma+im_l)|\leq\log(l+2)l^{-1+\delta_n}\text{ for }l=k_{n-1}+1,\ldots,k_n,$$

and

$$\sup_{z\in K_n}|Q_n(z)-g_n(z)+Q_1(z)+\cdots+Q_{n-1}(z)|<\frac{2}{n},$$

with $Q_n(z)=\sum_{l=k_{n-1}+1}^{k_n}\zeta(\sigma+im_l)l^{-z}$. This finishes the proof. □

Now the following statement holds.

Theorem 5.4 *Let σ be a real number with $1/2 < \sigma < 1$. There is a sequence (m_n) of integers such that the series $\sum_{n\geq 1} \zeta(\sigma + im_n)n^{-z}$ is an universal Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$.*

Proof Let (δ_n) be a decreasing sequence of positive real numbers with $\delta_1 < 1/2$. Let us consider the Dirichlet series $f(z) = \sum_{l=1}^{+\infty} \zeta(\sigma + im_l)l^{-z}$ given by Lemma 5.3. Observe that

$$f(z) = \sum_{n=1}^{+\infty} \sum_{l=k_{n-1}+1}^{k_n} \zeta(\sigma + im_l)l^{-z}.$$

Moreover for all $\tau > 0$, there exists n_τ, so that for every integer $n \geq n_\tau$, $\delta_n < \tau$. Set $\varepsilon_\tau = \tau - \delta_{n_\tau} > 0$. Thus we have

$$\begin{aligned}\|f\|_\tau &= \|\sum_{n=1}^{n_\tau} \sum_{l=k_{n-1}+1}^{k_n} \zeta(\sigma + im_l)l^{-z}\|_\tau + \sum_{l=k_{n_\tau}}^{+\infty} |\zeta(\sigma + im_l)|l^{-\tau} \\ &\leq \sum_{l=1}^{k_{n_\tau}} |\zeta(\sigma + im_l)|l^{-\tau} + \sum_{l=k_{n_\tau}}^{+\infty} \log(l+2)l^{-\varepsilon_\tau - 1} < +\infty.\end{aligned}$$

Hence f belongs to $\mathcal{D}_a(\mathbb{C}_+)$. Now let $K \subset \overline{\mathbb{C}}_-$ be a compact set with connected complement, P be a Dirichlet polynomial and $\varepsilon > 0$. By definition, there exists a positive integer N such that for all $n \geq N$, $K \subset K_n$ and one can find $n_1 \geq \max(N, 4/\varepsilon)$ such that $\sup_K |P - g_{n_1}| < \varepsilon/2$. By construction, the partial sum $\sum_{l=1}^{k_{n_1}} \zeta(\sigma + im_l)l^{-z}$ satisfies

$$\sup_{z\in K_{n_1}} \left| \sum_{l=1}^{k_{n_1}} \zeta(\sigma + im_l)l^{-z} - g_{n_1}(z) \right| < \frac{2}{n_1} < \frac{\varepsilon}{2}.$$

Therefore we have by the triangle inequality $\sup_{z\in K} \left|\sum_{l=1}^{k_{n_1}} \zeta(\sigma + im_l)l^{-z} - P(z)\right| < \varepsilon$, and f is an universal Dirichlet series. □

6 Open problems

We conclude the paper with some open problems.

(1) Let $K \in \overline{\mathbb{C}}_-$ be a compact set with connected complement and $\delta > 0$. We know that, for every $N \geq 1$, the set $\left\{\sum_{n=N}^{M} a_n n^{-(1-\delta)} n^{-s};\ M \geq N, |a_n| \leq 1\right\}$ is dense in $A(K)$. On the other hand, Corollary 4.6 ensures that this result does not hold anymore if one replaces δ by a sequence (δ_n) defined by $\delta_n = \left(\log_2(n) \ldots \log_{k-1}(n) \log_k^{1+\tau}(n)\right)^{-1}$ (for all $k \geq 2$ and $\tau > 0$). Neverthe-

less the following problem remains open: is the set of Dirichlet polynomials $\left\{\sum_{n=N}^{M} a_n n^{-\left(1-\frac{1}{\log_2(n+2)}\right)} n^{-s};\ M \geq N, |a_n| \leq 1\right\}$ dense in $A(K)$, for all $N \geq 1$?

(2) Let $\sigma < 0$, does there exist a Dirichlet series $\sum_{k\geq 1} a_k k^{-s}$ belonging to $\mathcal{D}_a(\mathbb{C}_+)$ which is universal in the strip $\{s \in \mathbb{C} : \sigma \leq \Re(s) \leq 0\}$, satisfying $\sum_{k=1}^{n} a_k k^{-s} \to \infty$, as n tends to infinity, almost everywhere in the half-plane $\{z \in \mathbb{C} : \Re(s) < \sigma\}$?

(3) An universal Dirichlet series f cannot be logarithmically summable at any point of its line of convergence. However we don't know whether the sequence of its logarithmic means $\left(\left(\frac{1}{\log(n)}\right) \sum_{k=1}^{n} k^{-1} D_k(f)(s)\right)$ still satisfies the universal approximation property.

(4) The sequence of Cesàro means of partial sums of an universal Dirichlet series remains universal, but it would be interesting to know whether the converse implication holds.

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