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On the intersections of the Besicovitch sets and the Erdös–Rényi sets

Mengjie Zhang[1] · **Li Peng**[1]

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Abstract We are interested in two properties of real numbers: the first one is the property of having given digit frequencies in the binary expansion, such as the well known Besicovitch sets, and the second one is the property of having the longest run of heads in the n independent Bernoulli trials, that is the so called Erdös–Rényi sets. In 2013, Chen and Wen (J Math Anal Appl 401:29–37, 2013) considered the intersections of these two kinds of sets by determining the Hausdorff dimension of the sets

$$\left\{x \in [0,1) : \liminf_{n\to\infty} \frac{S_n(x)}{n} \geq \alpha, \ \lim_{n\to\infty} \frac{R_n(x)}{\log_2 n} = \beta\right\}, \ \ 0 \leq \alpha \leq 1, \ 0 \leq \beta \leq +\infty,$$

where $S_n(x)$ denotes the summation of the first n digits and $R_n(x)$ is the maximal length of consecutive one digits in the first n terms of the dyadic expansion of $x \in [0,1)$. In the present paper, we complement this result by computing the Hausdorff dimension of the following sets

$$\left\{x \in [0,1) : \lim_{n\to\infty} \frac{S_n(x)}{n} = \alpha, \ \lim_{n\to\infty} \frac{R_n(x)}{\log_2 n} = \beta\right\}, \ \ 0 \leq \alpha \leq 1, \ 0 \leq \beta \leq +\infty.$$

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✉ Li Peng
lpeng@hust.edu.cn

Mengjie Zhang
mengjie_zhang@hust.edu.cn

[1] School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China

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1 Introduction

Determining the Hausdorff dimension of sets of numbers is a fundamental and important problem in number theory and the multifractal analysis of various dynamical systems in view of Borel's normal number theorem [1] in 1909. In 1934, Besicovitch [2] firstly complemented Borel's result by computing the Hausdorff dimension of sets of numbers in [0, 1] for which the limit of the frequency of digit one in the dyadic expansion is bounded from above. After that, Eggleston [3] generalized Besicovitch's work to the case of b-adic expansion, where $b \geq 2$ is an integer. In the past several decades, many people extended the results got by Besicovitch and Eggleston in diverse directions. For more details, the readers are referred to [4,7,8] and references therein. In 2013, Chen and Wen [6] generalized Besicovitch's work by considering the Hausdorff dimension of the intersections of the lower Besicovitch sets and the Erdös–Rényi sets. Based on their result, we turn to a more subtle kind of sets, which will complement the result got by Chen et al.

Now, we are going to introduce some notations and definitions that will be used later. For any real number $x \in [0, 1)$, let $x = \sum_{n=1}^{\infty} x_n/2^n = 0.\, x_1 x_2 \ldots$ be the unique non-terminating binary expansion of x, where $x_n \in \{0, 1\},\ n \geq 1$, are called the digits of the binary expansion of x. We set $S_0(x) = 0$ and write $S_n(x) = \sum_{i=1}^{n} x_i,\ n \geq 1$, the sum of the first n digits of x or the n-th partial sum of x. For the following sets

$$\underline{B}(\alpha) = \left\{ x \in [0, 1) : \ \liminf_{n\to\infty} \frac{S_n(x)}{n} \geq \alpha \right\}, \quad \frac{1}{2} \leq \alpha \leq 1,$$

which are called the lower Besicovitch sets, it can be concluded by Besicovitch's result [2] that

$$\dim_H \underline{B}(\alpha) = \frac{H(\alpha)}{\log 2}, \quad \frac{1}{2} \leq \alpha \leq 1,$$

where $\dim_H$ denotes the Hausdorff dimension and $H(\cdot)$ is the entropy function defined by

$$H(x) = -x \log x - (1 - x) \log(1 - x), \quad 0 \leq x \leq 1, \tag{1.1}$$

and we set $0 \log 0 = 0$ by convention. Furthermore, write

$$B(\alpha) = \left\{ x \in [0, 1) : \ \lim_{n\to\infty} \frac{S_n(x)}{n} = \alpha \right\}, \quad 0 \leq \alpha \leq 1, \tag{1.2}$$

which are called Besicovitch sets. It is well known that Besicovitch's research also give the conclusion that

$$\dim_H B(\alpha) = \frac{H(\alpha)}{\log 2}, \quad 0 \le \alpha \le 1.$$

Next we are ready to introduce the Erdös–Rényi sets. For any $x \in [0, 1)$, let $R_n(x)$ be the dyadic run-length function of x, namely,

$$R_n(x) = \max\{k : \ x_{i+1} = x_{i+2} = \ldots = x_{i+k} = 1, \ 0 \le i \le n-k\}.$$

In a pioneering work, Erdös and Rényi [5] proved that for almost all $x \in [0, 1)$,

$$\lim_{n\to\infty} \frac{R_n(x)}{\log_2 n} = 1,$$

which is the well known Erdös–Rényi limit theorem. So it is meaningful to study the level sets

$$E(\beta) = \left\{x \in [0, 1) : \ \lim_{n\to\infty} \frac{R_n(x)}{\log_2 n} = \beta\right\}, \quad 0 \le \beta \le \infty, \tag{1.3}$$

which are called Erdös–Rényi sets. It was proved by Ma et al. that the exceptional set of the Erdös–Rényi limit theorem has full Hausdorff dimension in [11]. In the past several years, there are abundant dimensional results concerning the run-length function. For more results, the readers are referred to [12–14]. However, few results are concerned about the relationship between the Besicovitch sets and the Erdös–Rényi sets.

By classifying the real numbers in the unit interval satisfying both the properties of Besicovitch sets and Erdös–Rényi sets, Chen et al. [6] firstly considered the fractional dimensions of intersections of the lower Besicovitch sets and the Erdös–Rényi sets in 2013. In fact, for any $0 \le \alpha \le 1, \ 0 \le \beta \le +\infty$, they showed that the Hausdorff dimension of the sets

$$\underline{S}(\alpha, \beta) = \left\{x \in [0, 1) : \ \liminf_{n\to\infty} \frac{S_n(x)}{n} \ge \alpha, \ \ \lim_{n\to\infty} \frac{R_n(x)}{\log_2 n} = \beta\right\}$$

is $\sup_{\alpha \le t \le 1} \frac{H(t)}{\log 2}$, where $H(\cdot)$ is defined as (1.1). In this paper, we turn to a more subtle kind of sets, which are the intersections of the Besicovitch sets and the Erdös–Rényi sets. For the α, β above, we define the sets

$$E(\alpha, \beta) = \left\{x \in [0, 1) : \ \lim_{n\to\infty} \frac{S_n(x)}{n} = \alpha, \ \ \lim_{n\to\infty} \frac{R_n(x)}{\log_2 n} = \beta\right\}. \tag{1.4}$$

The main result of this paper can be stated as follows.

Theorem 1.1 *For any* $0 \leq \alpha \leq 1$ *and* $0 \leq \beta \leq +\infty$, *we have that*

$$\dim_H E(\alpha, \beta) = \frac{H(\alpha)}{\log 2}.$$

For acquaintance of the definitions and properties of Hausdorff dimension, one can refer to Falconer's book [10].

2 Preliminaries

In this section, we gather some terminologies and present some lemmas which are essential to our proof.

Definition 2.1 Let $M \geq 1$ be an integer. The finite sequence $(x_1 x_2 \ldots x_M) \in \{0, 1\}^M$ is called an M-word. When $x_i = 0$ or 1 for any $1 \leq i \leq M$, such M-word is denoted by 0^M or 1^M respectively for convenience. In particular, for an integer N satisfying $0 \leq N < M$, the M-word

$$(x_1 x_2 \ldots x_{M-1}, 0) \text{ with } \sum_{i=1}^{M-1} x_i = N$$

is called an (N, M)-word. We denote the family of all the (N, M)-words by $W_M(N)$.

In order to estimate the number of the words in $W_M(N)$, we need the following result which is a direct consequence of the Stirling's formula [15].

Lemma 2.2 *For any positive integers n and k with* $0 \leq k \leq n$, *we have the following equation*

$$\log \binom{n}{k} = nH\left(\frac{k}{n}\right) + O(\log n), \text{ as } n \to \infty,$$

where $H(\cdot)$ *is the entropy function defined as* (1.1), *and the notation* $f(n) = O(g(n))$ *means that* $\frac{f(n)}{g(n)}$ *is bounded as* $n \to \infty$.

The following dimension result about homogeneous Moran sets is a classic tool to estimate the Hausdorff dimension of a fractal set from below.

Let $\{d_i\}_{i \geq 1}$ be a sequence of positive integers and $\{c_i\}_{i \geq 1}$ be a sequence of positive numbers satisfying $d_i \geq 2$, $0 < c_i < 1$, $d_1 c_1 \leq \delta$ and $d_i c_i \leq 1$ $(i \geq 2)$, where δ is some positive number. Let

$$D = \bigcup_{i \geq 0} D_i, \ D_0 = \emptyset, \ D_i = \{(\sigma_1 \ldots \sigma_i) : 1 \leq \sigma_j \leq d_j, \ 1 \leq j \leq i\}.$$

If $\sigma = (\sigma_1 \ldots \sigma_k) \in D_k$, $\tau = (\tau_1 \ldots \tau_m) \in D_m$, the concatenation of σ and τ is denoted by $\sigma * \tau = (\sigma_1 \ldots \sigma_k \tau_1 \ldots \tau_m)$.

Definition 2.3 ([9]) Suppose that $J \subset \mathbb{R}$ is a closed subinterval with diameter $\delta > 0$. Let $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ be a collection of closed subsets of J with the following properties:

(1) $J_\emptyset = J$;
(2) For any $i \geq 1$ and $\sigma \in D_{i-1}$, $J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*d_i}$ are subintervals of J_σ and $int(J_{\sigma*i}) \bigcap int(J_{\sigma*j}) = \emptyset$ $(i \neq j)$, where $int(\cdot)$ denotes the interior of a set;
(3) For any $i \geq 1$ and $\sigma \in D_{i-1}$, $1 \leq l \leq d_i$, $\frac{|J_{\sigma*l}|}{|J_\sigma|} = c_i$, where $|\cdot|$ denotes the diameter of a set.
Then $\mathbb{C}_\infty := \bigcap_{i\geq 1} \bigcup_{\sigma\in D_i} J_\sigma$ is called a homogeneous Moran set determined by $\mathcal{F}$.
For each $i \geq 1$, we call the union $C_i := \bigcup_{\sigma\in D_i} J_\sigma$ the i-th generation of $\mathbb{C}_\infty$.

Lemma 2.4 ([9]) *For the homogeneous Moran sets defined above, we have*

$$\dim_H \mathbb{C}_\infty \geq \liminf_{j\to\infty} \frac{\log d_1 d_2 \ldots d_j}{-\log c_1 c_2 \ldots c_{j+1} d_{j+1}}.$$

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. As for the upper bound, it is easy to get that $\dim_H E(\alpha, \beta) \leq \frac{H(\alpha)}{\log 2}$ since $E(\alpha, \beta) \subset B(\alpha)$ for any $0 \leq \alpha \leq 1,\ 0 \leq \beta \leq +\infty$, where $B(\alpha)$ is defined as (1.2). To obtain the lower bound of $\dim_H E(\alpha, \beta)$, we will construct a suitable homogeneous Moran subset named $\mathbb{C}_\infty$, then the conclusion can be drawn by using Lemma 2.4. For this, we need to consider the following cases.

3.1 $0 \leq \alpha \leq 1,\ 0 < \beta < +\infty$

In this subsection, we shall prove our result in the case of $0 \leq \alpha \leq 1,\ 0 < \beta < +\infty$ in detail, while the argument for other cases can be done by some minor modifications. In what follows, we will construct the desired Moran subset $\mathbb{C}_\infty$ of $E(\alpha, \beta)$. For any $x \in \mathbb{R}$, we use the notation $\lfloor x \rfloor$ to represent the maximal integer that less than or equal to x.

Choose an integer n_0 large enough such that $2^{n_0} > \lfloor (n_0 + 1)\beta \rfloor$. We define two sequences of integers $\{N_n\}_{n\geq 1}$ and $\{l_n\}_{n\geq 1}$ as follows:

$$N_n = \begin{cases} \lfloor \frac{2^{n_0} - \lfloor n_0\beta \rfloor}{\lfloor \sqrt{n_0 - 1} \rfloor} \rfloor, & n = 1; \\ \lfloor \frac{2^{n_0+n-2} - \lfloor (n_0+n-1)\beta \rfloor}{\lfloor \sqrt{n_0+n-2} \rfloor} \rfloor, & n \geq 2. \end{cases}$$

$$l_n = \begin{cases} 2^{n_0} - N_1 \lfloor \sqrt{n_0 - 1} \rfloor - \lfloor n_0 \beta \rfloor, & n = 1; \\ 2^{n_0+n-2} - N_n \lfloor \sqrt{n_0 + n - 2} \rfloor - \lfloor (n_0 + n - 1)\beta \rfloor, & n \geq 2. \end{cases}$$

Obviously, we know that $0 \le l_n < \lfloor\sqrt{n_0+n-2}\rfloor$ for any $n \ge 1$.

A family of sets $\{C_n\}_{n\ge1}$ of words which are related to $\mathbb{C}_\infty$ are defined by

$$C_n = \Big\{\omega_1\omega_2\ldots\omega_{N_n-1}\nu_{N_n} : \nu_{N_n} = \omega_{N_n}0^{l_n}1^{\lfloor(n_0+n-1)\beta\rfloor}, \\ \omega_i \in W_{\lfloor\sqrt{n_0+n-2}\rfloor}(\lfloor\alpha\sqrt{n_0+n-2}\rfloor),\ 1 \le i \le N_n\Big\},$$

for any $n \ge 1$.

Now, we are ready to give a suitable homogeneous Moran set $\mathbb{C}_\infty$ as below:

$$\mathbb{C}_\infty = \{0.u_1u_2\ldots \in [0,1] : u_i \in C_i,\ \ \forall i \ge 1\} \\ = 0.\prod_{i=1}^{\infty} C_i.$$

By the construction of $\mathbb{C}_\infty$, the two sequences $\{d_i\}_{i\ge1}$ and $\{c_i\}_{i\ge1}$ corresponding to Definition 2.3 can be written as follows:

$$d_i = \begin{cases} \binom{\lfloor\sqrt{n_0-1}\rfloor-1}{\lfloor\alpha\sqrt{n_0-1}\rfloor}, & 1 \le i \le N_1; \\ \binom{\lfloor\sqrt{n_0+p-1}\rfloor-1}{\lfloor\alpha\sqrt{n_0+p-1}\rfloor}, & \sum_{j=1}^{k+1} N_j + 1 \le i \le \sum_{j=1}^{k+2} N_j,\ k \ge 0. \end{cases}$$

$$c_i = \begin{cases} 2^{-\lfloor\sqrt{n_0-1}\rfloor}, & 1 \le i < N_1; \\ 2^{-(\lfloor\sqrt{n_0-1}\rfloor + l_1 + \lfloor n_0\beta\rfloor)}, & i = N_1; \\ 2^{-\lfloor\sqrt{n_0+k}\rfloor}, & \sum_{j=1}^{k+1} N_j + 1 \le i < \sum_{j=1}^{k+2} N_j,\ k \ge 0; \\ 2^{-(\lfloor\sqrt{n_0+k}\rfloor + l_{k+2} + \lfloor(n_0+k+1)\beta\rfloor)}, & i = \sum_{j=1}^{k+2} N_j,\ k \ge 0. \end{cases}$$

In order to get the lower bound of $\dim_H E(\alpha,\beta)$, the following two lemmas are essential.

Lemma 3.1 *Let* $0 \le \alpha \le 1,\ 0 < \beta < +\infty$, *then* $\dim_H \mathbb{C}_\infty \ge \frac{H(\alpha)}{\log 2}$.

Proof For any integer j satisfying $\sum_{i=1}^{k+1} N_i \le j < \sum_{i=1}^{k+2} N_i$ for some $k \ge 0$, we assume that $j = \sum_{i=1}^{k+1} N_i + p$, for some p with $0 \le p < N_{k+2}$. Let us consider the following two cases.

Case 1 $\ j+1 < \sum_{i=1}^{k+2} N_i$.

By the definitions of $\{d_i\}_{i\geq 1}$ and $\{c_i\}_{i\geq 1}$, an application of Lemma 2.2 implies that

$$\begin{aligned}
&\liminf_{j\to\infty} \frac{\log d_1 d_2 \ldots d_j}{-\log c_1 c_2 \ldots c_{j+1} d_{j+1}} \\
&= \liminf_{k\to\infty} \frac{\sum_{i=1}^{k+1} N_i \log \binom{\lfloor\sqrt{n_0+i-2}\rfloor - 1}{\lfloor\alpha\sqrt{n_0+i-2}\rfloor} + p\log\binom{\lfloor\sqrt{n_0+k}\rfloor-1}{\lfloor\alpha\sqrt{n_0+k}\rfloor}}{\left[2^{n_0} + \sum_{i=1}^{k} 2^{n_0+i-1} + (p+1)\lfloor\sqrt{n_0+k}\rfloor\right]\log 2 - \log\binom{\lfloor\sqrt{n_0+k}\rfloor-1}{\lfloor\alpha\sqrt{n_0+k}\rfloor}} \\
&\geq \liminf_{k\to\infty} \frac{A(k)+B(k)}{C(k)+D(k)},
\end{aligned}$$

where we denote

$$\begin{aligned}
A(k) &= \sum_{i=1}^{k+1} N_i \left[\left(\lfloor\sqrt{n_0+i-2}\rfloor - 1\right) H\left(\frac{\alpha\lfloor\sqrt{n_0+i-2}\rfloor}{\lfloor\sqrt{n_0+i-2}\rfloor-1}\right)\right. \\
&\quad \left. + O\left(\lfloor\sqrt{n_0+i-2}\rfloor - 1\right)\right] \\
&\quad + O\left(\lfloor\sqrt{n_0+k}\rfloor - 1\right), \\
B(k) &= p\left(\lfloor\sqrt{n_0+k}\rfloor - 1\right) H\left(\frac{\lfloor\sqrt{n_0+k}\rfloor-1}{\lfloor\alpha\sqrt{n_0+k}\rfloor}\right), \\
C(k) &= \left[2^{n_0} + \sum_{i=1}^{k} 2^{n_0+i-1} + \lfloor\sqrt{n_0+k}\rfloor\right]\log 2, \\
D(k) &= \left(p\lfloor\sqrt{n_0+k}\rfloor\right)\log 2.
\end{aligned}$$

Thus by the Stolz-Cesàro Theorem we know that

$$\liminf_{k\to\infty}\frac{A(k)}{C(k)} = \frac{H(\alpha)}{\log 2}, \quad \liminf_{k\to\infty}\frac{B(k)}{D(k)} = \frac{H(\alpha)}{\log 2},$$

so we have

$$\liminf_{k\to\infty}\frac{A(k)+B(k)}{C(k)+D(k)} \geq \frac{H(\alpha)}{\log 2}.$$

Case 2 $j+1 = \sum_{i=1}^{k+2} N_i$. Similar to Case 1, we have

$$\begin{aligned}
&\liminf_{j\to\infty} \frac{\log d_1 d_2 \ldots d_j}{-\log c_1 c_2 \ldots c_{j+1} d_{j+1}}\\
&= \liminf_{k\to\infty} \frac{\sum_{i=1}^{k+1} N_i \log\left(\frac{\lfloor\sqrt{n_0+i-2}\rfloor-1}{\lfloor\alpha\sqrt{n_0+i-2}\rfloor}\right) + (N_{k+2}-1)\log\left(\frac{\lfloor\sqrt{n_0+k}\rfloor-1}{\lfloor\alpha\sqrt{n_0+k}\rfloor}\right)}{\left(2^{n_0} + \sum_{i=1}^{k+1} 2^{n_0+i-1}\right)\log 2 - \log\left(\frac{\lfloor\sqrt{n_0+k}\rfloor-1}{\lfloor\alpha\sqrt{n_0+k}\rfloor}\right)}\\
&\geq \liminf_{k\to\infty} \frac{\sum_{i=1}^{k+2} N_i\left[\left(\lfloor\sqrt{n_0+i-2}\rfloor-1\right) H\left(\frac{\alpha\lfloor\sqrt{n_0+i-2}\rfloor}{\lfloor\sqrt{n_0+i-2}\rfloor-1}\right) + O\left(\log\lfloor\sqrt{n_0+i-2}\rfloor\right)\right]}{\left(2^{n_0} + \sum_{i=1}^{k+1} 2^{n_0+i-1}\right)\log 2}\\
&= \frac{H(\alpha)}{\log 2}.
\end{aligned}$$

Hence, by Lemma 2.4 we have

$$\dim_H \mathbb{C}_\infty \geq \liminf_{j\to\infty} \frac{\log d_1 d_2 \ldots d_j}{-\log c_1 c_2 \ldots c_{j+1} d_{j+1}} \geq \frac{H(\alpha)}{\log 2}.$$

□

Lemma 3.2 *Let* $0 \leq \alpha \leq 1,\ 0 < \beta < +\infty$*, then* $\mathbb{C}_\infty \subset E(\alpha, \beta)$.

Proof Let $n \in \mathbb{N}$ satisfying $2^{n_0+k} \leq n < 2^{n_0+k+1}$ for some $k \geq 1$.
Take $x = 0.x_1 x_2 \ldots \in \mathbb{C}_\infty$. In order to get that $\lim_{n\to\infty} \frac{S_n(x)}{n} = \alpha$, we have the following three cases.

Case 1 $2^{n_0+k} + p\lfloor\sqrt{n_0+k}\rfloor \leq n < 2^{n_0+k} + (p+1)\lfloor\sqrt{n_0+k}\rfloor$ for some $1 \leq p < N_{k+2}$.

$$\frac{S_n(x)}{n} = \frac{\sum_{i=1}^{k+1}\left[N_i\lfloor\alpha\sqrt{n_0+i-2}\rfloor + \lfloor(n_0+i-1)\beta\rfloor\right] + p\lfloor\alpha\sqrt{n_0+k}\rfloor + O\left(\sqrt{n_0+k}\right)}{2^{n_0+k} + p\lfloor\sqrt{n_0+k}\rfloor + O\left(\sqrt{n_0+k}\right)};$$

Case 2 $2^{n_0+k} + N_{k+2}\lfloor\sqrt{n_0+k}\rfloor \leq n < 2^{n_0+k} + N_{k+2}\lfloor\sqrt{n_0+k}\rfloor + l_{k+2}$.

$$\frac{S_n(x)}{n} = \frac{\sum_{i=1}^{k+2} N_i\lfloor\alpha\sqrt{n_0+i-2}\rfloor + \sum_{i=1}^{k+1}\lfloor(n_0+i-1)\beta\rfloor}{2^{n_0+k} + N_{k+2}\lfloor\sqrt{n_0+k}\rfloor + O\left(\sqrt{n_0+k}\right)};$$

Case 3 $2^{n_0+k} + N_{k+2}\lfloor\sqrt{n_0+k}\rfloor + l_{k+2} \leq n < 2^{n_0+k+1}$.

$$\frac{S_n(x)}{n} = \frac{\sum_{i=1}^{k+2} N_i\lfloor\alpha\sqrt{n_0+i-2}\rfloor + \sum_{i=1}^{k+1}\lfloor(n_0+i-1)\beta\rfloor + O\left(n_0+k-1\right)}{2^{n_0+k} + N_{k+2}\lfloor\sqrt{n_0+k}\rfloor + O\left(n_0+k+1\right)}.$$

Therefore, by the definition of $\{N_i\}_{i\geq 1}$, we have $\lim_{n\to\infty}\frac{S_n(x)}{n}=\alpha$ in all the cases above.

Also, by the construction of $\mathbb{C}_\infty$ one can easily check that

$$\frac{\lfloor\beta(n_0+k)\rfloor}{n_0+k+1}<\frac{R_n(x)}{\log_2 n}\leq\frac{\lfloor\beta(n_0+k+1)\rfloor}{n_0+k},$$

which means $\lim_{n\to\infty}\frac{R_n(x)}{\log_2 n}=\beta$. Hence we have $\mathbb{C}_\infty\subset E(\alpha,\beta)$. □

Using Lemmas 3.1 and 3.2, we can easily get that

$$\dim_H E(\alpha,\beta)\geq\frac{H(\alpha)}{\log 2},$$

which completes the proof of Theorem 1.1.

3.2 $0\leq\alpha\leq 1,\ \beta=0$ and $+\infty$

Bearing in mind the construction of $\mathbb{C}_\infty$, the proof for the remaining cases are similar to the proof in the case $0\leq\alpha\leq 1,\ 0<\beta<+\infty$. We only need to modify the number of 1s in the construction of $\{C_n\}_{n\geq 1}$ when β is allowed to be 0 or $+\infty$.

Case 1 $0\leq\alpha\leq 1,\ \beta=0$.

Similar to the definitions of $\{N_n\}_{n\geq 1}$ and $\{l_n\}_{n\geq 1}$ in the previous subsection, we define two sequences of integers $\{N_n^{'}\}_{n\geq 1}$ as well as $\{l_n^{'}\}_{n\geq 1}$ as follows:

$$N_n^{'}=\begin{cases}\lfloor\frac{2^{n_0}-\lfloor\sqrt{n_0}\rfloor}{\lfloor\sqrt{n_0-1}\rfloor}\rfloor, & n=1;\\ \lfloor\frac{2^{n_0+n-2}-\lfloor\sqrt{(n_0+n-1)}\rfloor}{\lfloor\sqrt{n_0+n-2}\rfloor}\rfloor, & n\geq 2.\end{cases}$$

$$l_n^{'}=\begin{cases}2^{n_0}-N_1^{'}\lfloor\sqrt{n_0-1}\rfloor-\lfloor\sqrt{n_0}\rfloor, & n=1;\\ 2^{n_0+n-2}-N_n^{'}\lfloor\sqrt{n_0+n-2}\rfloor-\lfloor\sqrt{(n_0+n-1)}\rfloor, & n\geq 2.\end{cases}$$

Obviously, we have that $0\leq l_n^{'}<\lfloor\sqrt{n_0+n-2}\rfloor$ for any $n\geq 1$.

Then we define

$$C_n=\Big\{\omega_1\omega_2\ldots\omega_{N_n^{'}-1}\nu_{N_n^{'}}:\nu_{N_n^{'}}=\omega_{N_n^{'}}0^{l_n^{'}}1^{\lfloor\sqrt{n_0+n-1}\rfloor},$$
$$\omega_i\in W_{\lfloor\sqrt{n_0+n-2}\rfloor}(\lfloor\alpha\sqrt{n_0+n-2}\rfloor),\ 1\leq i\leq N_n\Big\},$$

for any $n\geq 1$.

Case 2 $0\leq\alpha\leq 1,\ \beta=+\infty$.

In this case, take an integer n_0 large enough, and two sequences of integers $\{N_n^{''}\}_{n\geq 1}$ and $\{l_n^{''}\}_{n\geq 1}$ are defined as below:

$$N_n^{''} = \begin{cases} \lfloor \frac{2^{n_0}-n_0^2}{\lfloor\sqrt{n_0-1}\rfloor} \rfloor, & n = 1; \\ \lfloor \frac{2^{n_0+n-2}-n_0^2}{\lfloor\sqrt{n_0+n-2}\rfloor} \rfloor, & n \geq 2. \end{cases}$$

$$l_n^{''} = \begin{cases} 2^{n_0} - N_1^{''} n_0^2 - \lfloor\sqrt{n_0}\rfloor, & n = 1; \\ 2^{n_0+n-2} - N_n^{''} \lfloor\sqrt{(n_0+n-2)}\rfloor - (n_0+n-1)^2, & n \geq 2. \end{cases}$$

Also, $0 \leq l_n^{''} < \lfloor\sqrt{n_0+n-2}\rfloor$ for any $n \geq 1$.

Define

$$C_n = \Big\{\omega_1\omega_2\ldots\omega_{N_n^{''}-1}\nu_{N_n^{''}} : \nu_{N_n^{''}} = \omega_{N_n^{''}} 0^{l_n^{''}} 1^{(n_0+n-1)^2}, \\ \omega_i \in W_{\lfloor\sqrt{n_0+n-2}\rfloor}\left(\lfloor\alpha\sqrt{n_0+n-2}\rfloor\right),\ 1 \leq i \leq N_n^{''}\Big\},$$

for any $n \geq 1$.

In the two cases above, we construct the homogeneous Moran set

$$\mathbb{C}_\infty = \Big\{0.u_1u_2\ldots \in [0,1] : u_i \in C_i,\ \ \forall i \geq 1\Big\} \\ = 0.\prod_{i=1}^{\infty} C_i.$$

As we did in the Sect. 3.1, it is easy to check that Lemmas 3.1 and 3.2 still hold. So we can obtain that $\dim_H E(\alpha,\beta) \geq \frac{H(\alpha)}{\log 2}$ in all the above cases.

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