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Existence of positive solutions for a class of *p*&*q* elliptic problem with critical exponent and discontinuous nonlinearity

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Abstract In this paper we study the existence of positive solutions to a class of $p\&q$ elliptic problems given by

$$-\text{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = f(u) + |u|^{q^*-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is bounded, $2 \leq p \leq q < q^*$, $f : \mathbb{R} \to \mathbb{R}$ is a function that can have an uncountable set of discontinuity points and the function a is a continuous function. This result to extend previous ones to a larger class of $p\&q$ type problems.

Keywords Variational methods · Critical exponents · Nonlinear elliptic equations · Discontinuous nonlinearity

Mathematics Subject Classification 35A15 · 35B33 · 35B25 · 35J60

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1 Introduction

When f is a continuous function, the existence and multiplicity of solutions of $p\&q$ type problems has been extensively investigated; see for example [7,9,12,28] and [30] in bounded domain and [1,8,14,18,26] and [29] in $\mathbb{R}^N$. A check in the references of these articles will provide a complete picture of the study of this class of problems.

In this paper we are looking for positive solutions to $p\&q$ type problems when f has an uncountable set of discontinuity points. To be specific, we are looking positive solutions for the following class of quasilinear problems

$$\begin{cases} -\text{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = f(u) + |u|^{q^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $2 \leq p \leq q < q^*$. The hypotheses on the functions a and f are the following:

(a_1) The function a is continuous and there exist constants $k_0, k_1, k_2, k_3 \geq 0$ such that

$$k_0 + k_1 t^{\frac{q-p}{p}} \leq a(t) \leq k_2 + k_3 t^{\frac{q-p}{p}}, \quad \text{for all} \quad t > 0.$$

(a_2) There exists $\alpha \in (0, 1]$ such that

$$A(t) \geq \alpha a(t)t \text{ for all } t \geq 0,$$

where $A(t) = \displaystyle\int_0^t a(s)ds$.

(f_1) For all $t \in \mathbb{R}$, there are $C > 0$ and $r \in (q, q^*)$ such that

$$|f(t)| \leq C(1 + |t|^{r-1})$$

(f_2) For all $t \in \mathbb{R}$, there is $\theta \in (p\alpha, q^*)$ such that

$$0 \leq \theta F(t) = \int_0^t f(s)ds \leq t\underline{f}(t) \text{ uniformly in } \Omega, \quad \text{where}$$
$$\underline{f}(t) := \lim_{\epsilon\downarrow 0} \text{ess inf}_{|t-s|<\epsilon} f(s)$$

and

$$\overline{f}(t) := \lim_{\epsilon\downarrow 0} \text{ess sup}_{|t-s|<\epsilon} f(s), \quad \text{which are N-mensurable.}$$

(f_3) There is $\beta > 0$ that will be fixed later, such that

$$H(t - \beta) \leq f(t), \quad \text{for all} \quad t \in \mathbb{R} \quad \text{and uniformly in} \quad \Omega,$$

where H is the Heaviside function, i.e,

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

(f_4) $\limsup_{t\to 0^+} \dfrac{f(t)}{t^{q-1}} = 0$ and $f(t) = 0$ if $t \leq 0$.

A typical example of a function satisfying the conditions (f_1)–(f_4) is given by

$$f(t) = \begin{cases} 0 & \text{if } t \in]-\infty, \beta/2[\\ 1 & \text{if } t \in \mathbb{Q} \bigcap [\beta/2, \beta] \\ 0 & \text{if } t \in (\mathbb{R} \setminus \mathbb{Q}) \bigcap [0, \beta] \\ \displaystyle\sum_{k=1}^{l} \frac{|t|^{q_k-1}}{\beta^{q_k-1}} & \text{if } t > \beta, \ l \geq 1 \ \text{and} \ q_k \in (q, q^*). \end{cases}$$

Note that the function f in this example has an uncountable set of discontinuity points. By a solution for (1.1) we understand as a function $0 \leq u \in W_0^{1,q}(\Omega)$ satisfying

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla\varphi = \int_\Omega \rho\varphi\, dx + \int_\Omega |u|^{q^*-2}u\varphi\, dx,$$

for all $\varphi \in W_0^{1,q}(\Omega)$ and

$$\rho(x) \in \left[\underline{f}(u(x)), \overline{f}(u(x))\right] \text{ a.e in } \Omega.$$

Problems involving discontinuous nonlinearity appears in several physical situations. Among these, we may cite electrical phenomena, plasma physics, free boundary value problems, etc. The reader may consult Ambrosetti–CalahorranoDobarro [2], Ambrosetti–Turner [3], Arcoya–Calahorrano [4], Arcoya–Diaz–Tello [5], Badialle [6] and the references therein.

The main result of this paper is as follows.

Theorem 1.1 *Assume* (a_1)–(a_2) *and* (f_1)–(f_4)*. Then, problem* (1.1) *has a positive solution. Moreover, if* $u \in W_0^{1,q}(\Omega)$ *is a solution of problem* (1.1)*, then* $|\{x \in \Omega : u(x) > \beta\}| > 0$.

We will give some examples of functions a in order to illustrate the degree of generality of the kind of problems studied here.

Example 1.2 Considering $a(t) = t^{\frac{q-p}{p}}$, we have that the function a satisfies the hypotheses (a_1)–(a_2) with $k_0 = k_2 = 0$ and $k_1 = k_3 = 1$. Hence, Theorem 1.1 is valid for the problem

$$-\Delta_q u = f(u) + |u|^{q^*-2}u \text{ in } \Omega.$$

Example 1.3 Considering $a(t) = 1 + t^{\frac{q-p}{p}}$, we have that the function a satisfies the hypotheses (a_1)–(a_2) with $k_0 = k_1 = k_2 = k_3 = 1$. Hence, Theorem 1.1 is valid for the problem

$$-\Delta_p u - \Delta_q u = f(u) + |u|^{q^*-2}u \text{ in } \Omega.$$

Problem (pnL) comes from a general reaction–diffusion system:

$$u_t = div[D(u)\nabla u] + c(x, u), \tag{1.2}$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{N-2})$. This system has a wide range of applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction design. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.2) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x, u)$ is a polynomial of u with variable coefficients (see [15,22,23,25,31]).

Beneath we present some other examples that are also interesting from mathematical point of view.

Example 1.4 Considering $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$, we have that the function a satisfies the hypotheses (a_1)–(a_2) with $k_0 = 1$, $k_1 = 0$, $k_2 = 2$ and $k_3 = 0$. Hence, Theorem 1.1 is valid for the problem

$$-div\left(|\nabla u|^{p-2}\nabla u + \frac{|\nabla u|^{p-2}\nabla u}{(1 + |\nabla u|^p)^{\frac{p-2}{p}}}\right) = f(u) + |u|^{p^*-2}u \text{ in } \Omega.$$

Example 1.5 Considering $a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$, it follows that the function a satisfies the hypotheses (a_1)–(a_2) with $k_0 = k_1 = k_2 = 2$, and $k_3 = 1$. Hence, Theorem 1.1 is valid for the problem

$$-\Delta_p u - \Delta_N u - \text{ div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1 + |\nabla u|^p)^{\frac{p-2}{p}}}\right) = f(u) + |u|^{q^*-2}u \text{ in } \Omega.$$

Our arguments were influenced by [6,7,19] and [20], . Below we list what we believe that are the main contributions of our paper.

(1) Problem (1.1) presents combinations of discontinuous nonlinearity with critical growth and operator $p\&q$-Laplacian that at least to our knowledge, seem to be new.
(2) In [7,19,20] the nonlinearity is continuous. In this paper, the nonlinearity can have an uncountable set of discontinuity points.
(3) We adapt arguments can be found in [6] for a general class of operators.

This paper is organized as follows. In Sect. 2 we study the basic results from convex analysis and give some information on preliminary results. In Sect. 3 we study the variational framework and some Technical Lemmas. We show the existence result in Sect. 4.

2 Basic results from convex analysis

In this section, for the reader's convenience, we recall some definitions and basic results on the critical point theory of locally Lipschitz continuous functionals as developed by Chang [13], Clarke [16,17] and Grossinho and Tersian [21].

Let X be a real Banach space. A functional $I : X \to \mathbb{R}$ is locally Lipschitz continuous, $I \in Lip_{loc}(X, \mathbb{R})$ for short, if given $u \in X$ there is an open neighborhood $V := V_u \subset X$ and some constant $K = K_V > 0$ such that

$$| I(v_2) - I(v_1) | \leq K \parallel v_2 - v_1 \parallel, \; v_i \in V, \; i = 1, 2.$$

The directional derivative of I at u in the direction of $v \in X$ is defined by

$$I^0(u; v) = \limsup_{h \to 0, \; \sigma \downarrow 0} \frac{I(u + h + \sigma v) - I(u + h)}{\sigma}.$$

Hence $I^0(u; .)$ is continuous, convex and its subdifferential at $z \in X$ is given by

$$I^0(u; z) = \left\{ \mu \in X^*; I^0(u; v) \geq I^0(u; z) + \langle \mu, v - z \rangle, \; v \in X \right\},$$

where $\langle ., . \rangle$ is the duality pairing between X^* and X. The generalized gradient of I at u is the set

$$\partial I(u) = \left\{ \mu \in X^*; \langle \mu, v \rangle \leq I^0(u; v), \; v \in X \right\}.$$

Since $I^0(u; 0) = 0$, $\partial I(u)$ is the subdifferential of $I^0(u; 0)$. A few definitions and properties will be recalled below.

$$\begin{aligned} &\partial I(u) \subset X^* \text{ is convex, non-empty and weak*-compact,} \\ &\lambda(u) = \min \left\{ \parallel \mu \parallel_{X^*}; \mu \in \partial I(u) \right\}, \end{aligned}$$

and

$$\partial I(u) = \left\{ I'(u) \right\}, \text{ if } I \in C^1(X, \mathbb{R}).$$

A critical point of I is an element $u_0 \in X$ such that $0 \in \partial I(u_0)$ and a critical value of I is a real number c such that $I(u_0) = c$ for some critical point $u_0 \in X$.

A sequence $(u_n) \subset X$ is called Palais–Smale sequence at level c $(PS)_c$ if

$$I(u_n) \to c, \;\; \lambda(u_n) \to 0$$

A functional I satisfies the $(PS)_c$ condition if any Palais–Smale sequence at nivel c has a convergent subsequence.

Theorem 2.1 *Let $I \in Lip_{loc}(X, \mathbb{R})$ with $I(0) = 0$ and satisfying:*

(i) *There are $r > 0$ and $\rho > 0$, such that $I(u) \geq \rho$, for $||u|| = r$, $u \in X$;*
(ii) *There is $e \in X \backslash B_\rho(0)$ with $I(e) < 0$.*

If

$$c = \inf_{\gamma \in \Gamma} \max I(\gamma(t))_{t\in[0,1]}$$

with

$$\Gamma = \{\gamma \in C([0,1], X), \gamma(0) = 0 \ \ and \ \ I(\gamma(1)) < 0\}$$

and I satisfies the Palais–Smale condition, then $c \geq \rho$ is a critical point of I, such that there is $u \in X$ verifying

$$I(u) = c \ \ and \ \ 0 \in \partial I(u).$$

Proposition 2.2 *(Riesz representation theorem)* ([10]) *Let Φ be a bounded linear functional on $L^r(\Omega)$, $1 < r < \infty$ and $\alpha \in \mathbb{R}$. Then, there is a unique function $u \in L^{r'}(\Omega)$, $r' = \frac{r}{r-1}$, such that*

$$\langle \Phi, v \rangle = \int_\Omega uv\, dx, \quad for\ all\ v \in L^r(\Omega).$$

Moreover,

$$|u|_{r',\alpha} = \|\Phi\|_{(L^r(\Omega))^*}.$$

Proposition 2.3 ([13]) *If $\Psi(u) = \displaystyle\int_\Omega F(u)dx$, where $F(t) = \displaystyle\int_0^t f(s)ds$, then $\Psi \in Lip_{loc}(L^p(\Omega)$ and $\partial\Psi(u) \subset L^{\frac{p}{p-1}}(\Omega)$. Moreover, if $\rho \in \partial\Psi(u)$, it satisfies*

$$\rho(x) \in [\underline{f}(u(x)), \overline{f}(u(x))] \ \ a.e\ in\ \Omega.$$

3 The variational framework and some technical lemmas

We will look for solutions of problem (1.1) by finding critical points of the Euler-Lagrange functional $I : W_0^{1,q}(\Omega) \to \mathbb{R}$ given by $I(u) = Q(u) - \Psi(u)$, where

$$Q(u) = \frac{1}{p}\int_\Omega A(|\nabla u|^p)dx - \frac{1}{q^*}\int_\Omega |u|^{q*}dx,$$

and

$$\Psi(u) = \int_\Omega F(u)\, dx$$

Note that Q is $C^1(W_0^{1,q}(\Omega), \mathbb{R})$ and for all $\phi \in W_0^{1,q}(\Omega)$, we have

$$Q'(u)\phi = \int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla \phi dx - \int_\Omega |u|^{q*-2} u\phi dx$$

Note that $I \in Lip_{loc}(W_0^{1,q}(\Omega), \mathbb{R})$ and

$$\partial I(u) = \{Q'(u)\} - \partial \Psi(u),\ \forall u \in W_0^{1,q}(\Omega).$$

In the next result we prove a local Palais–Smale condition to functional I.

Lemma 3.1 *The functional I satisfies the $(PS)_c$ condition for*

$$c < \left(\frac{1}{\theta} - \frac{1}{q^*}\right)\left(Sk_1\right)^{N/q}.$$

Proof Let (u_n) be a $(PS)_c$ sequence for I. Then,

$$I(u_n) \to c \quad \text{and} \quad \lambda(u_n) \to 0.$$

Consider $(w_n) \subset \partial I(u_n)$ such that

$$\|w_n\|_* = \lambda(u_n) = o_n(1)$$

and

$$w_n = Q'(u_n) - \rho_n,$$

where $\rho_n \in \partial\Psi(u_n)$. So,

$$\begin{aligned} c + 1 + \|u_n\| \geq\ & I(u_n) - \frac{1}{\theta}\langle w_n, u_n \rangle \\ \geq\ & \frac{1}{p}\int_\Omega A(|\nabla u_n|^p)dx - \int_\Omega F(u_n)dx - \frac{1}{q^*}\int_\Omega |u_n|^{q^*} dx \\ & -\frac{1}{\theta}\int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^p dx + \frac{1}{\theta}\int_\Omega \rho_n u_n dx - \frac{1}{\theta}\int_\Omega |u_n|^{q^*}\, dx. \end{aligned}$$

From (a_2)

$$\begin{aligned} c+1+\|u_n\| &\geq I(u_n)-\frac{1}{\theta}\langle w_n,u_n\rangle \\ &\geq \left(\frac{\alpha}{p}-\frac{1}{\theta}\right)\int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^p dx+\int_\Omega\left(\frac{1}{\theta}\rho_n u_n-F(u_n)\right)dx \\ &\quad+\left(\frac{1}{\theta}-\frac{1}{q^*}\right)\int_\Omega |u_n|^{q^*}dx \end{aligned}$$

Using (f_2) we get

$$\frac{1}{\theta}\rho_n(x)u_n(x)\geq\frac{1}{\theta}\underline{f}(u_n(x))u_n(x)\geq F(u_n(x)) \quad \text{a.e in } \Omega. \tag{3.1}$$

Hence,

$$c+1+\|u_n\|\geq\left(\frac{\alpha}{p}-\frac{1}{\theta}\right)\int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^p dx+\left(\frac{1}{\theta}-\frac{1}{q^*}\right)\int_\Omega |u_n|^{q^*}dx$$

Using (a_1) and (f_2) again, we have

$$c+1+\|u_n\|\geq k_1\left(\frac{\alpha}{p}-\frac{1}{\theta}\right)\|u_n\|^q.$$

Since $\theta > p\alpha$, we conclude that (u_n) is bounded in $W_0^{1,q}(\Omega)$. Passing to a subsequence, if necessary, we obtain

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } W_0^{1,q}(\Omega), \\ &u_n \to u \quad \text{in } L^s(\Omega), \\ &u_n(x)\to u(x) \quad \text{a.e in } \Omega, \\ &|u_n(x)|\leq h(x)\in L^s(\Omega) \end{aligned}$$

where $1\leq s<q^*$.

From (f_4) and by definition of I, we can consider $u(x)\geq 0$ a.e in Ω. Moreover, using the Concentration-Compactness Principle due to Lions [27], we obtain Π an at most countable index set, sequences $(\mu_i),(\nu_i)\subset(0,\infty)$, such that

$$|\nabla u_n|^q\rightharpoonup|\nabla u|^q+\mu \quad \text{and} \quad |u_n|^{q^*}\rightharpoonup|u|^{q^*}+\nu, \tag{3.2}$$

as $n\to+\infty$, in weak*-sense of measures, where

$$\nu=\sum_{i\in\Pi}\nu_i\delta_{x_i}, \quad \mu\geq\sum_{i\in\Pi}\mu_i\delta_{x_i}, \quad S\nu_i^{q/q^*}\leq\mu_i, \tag{3.3}$$

for all $i\in\Pi$, where δ_{x_i} is the Dirac mass at $x_i\in\Omega$.

We claim that $\Pi = \emptyset$. Arguing by contradiction that $\Pi \neq \emptyset$, we fixe $i \in \Pi$. Without loss of generality we can suppose $B_2(0) \subset \Omega$. Considering $\psi \in C_0^{\infty}(\Omega)$ such that $\psi \equiv 1$ in $B_1(0)$, $\psi \equiv 0$ in $\Omega \setminus B_2(0)$ and $|\nabla \psi|_{\infty} \leq 2$, we define $\psi_{\varrho}(x) := \psi((x - x_i)/\varrho)$, where $\varrho > 0$. Hence, $(\psi_{\varrho} u_n)$ is bounded in $W_0^{1,q}(\Omega)$ and

$$o_n(1) = \langle w_n, \psi_{\varrho} u_n \rangle = \int_{\Omega} a(|\nabla u_n|^p)|\nabla u_n|^{p-2} \nabla u_n \nabla(\psi_{\varrho} u_n)\, dx \\ - \int_{\Omega} |u|^{q^*} \psi_{\varrho}\, dx - \int_{\Omega} \rho_n \psi_{\varrho} u_n\, dx.$$

So,

$$\int_{\Omega} a(|\nabla u_n|^p)|\nabla u_n|^{p-2} \nabla u_n \psi_{\varrho} = - \int_{\Omega} u_n a(|\nabla u_n|^p)|\nabla u_n|^{p-2} \nabla \psi_{\varrho}\, dx \\ + \int_{\Omega} |u|^{q^*} \psi_{\varrho}\, dx + \int_{\Omega} \rho_n \psi_{\varrho} u_n\, dx \qquad (3.4)$$

Since $supp(\psi_{\varrho})$ is compact and it is contained in $B_{2\varrho}(x_i)$ and using (a_1), we have

$$\left| \int_{\Omega} u_n a(|\nabla u_n|^p)\, |\nabla u_n|^{p-2}\, \nabla u_n \nabla \psi_{\varrho}\, dx \right| \\ \leq \int_{B_{2\varrho}(0)} a(|\nabla u_n|^p)|\nabla u_n|^{p-1} |u_n \nabla \psi_{\varrho}|\, dx \\ \leq \int_{B_{2\varrho}(0)} k_2 |\nabla u_n|^{p-1} |u_n \psi_{\varrho}|\, dx \\ + \int_{B_{2\varrho}(0)} k_3 |\nabla u_n|^{p-1} |u_n \psi_{\varrho}|\, dx$$

Using Hölder inequality and boundedness of (u_n) in $W_0^{1,q}(\Omega)$, imply

$$\left| \int_{\Omega} u_n a(|\nabla u_n|^p)\, |\nabla u_n|^{p-2}\, \nabla u_n \nabla \psi_{\varrho}\, dx \right| \\ \leq C_1 \left(\int_{B_{2\varrho}(0)} |u_n|^p |\psi_{\varrho}|^p \right) \\ + C_2 \left(\int_{B_{2\varrho}(0)} |u_n|^q |\psi_{\varrho}|^q \right)$$

Since $u_n \to u$ in $L^s(\Omega)$ and using the Dominated Convergence Theorem, we get that

$$\lim_{\varrho \to 0} \left[\lim_{n \to \infty} \left| \int_{\Omega} u_n a(|\nabla u_n|^p)|\nabla u_n|^{p-2} \nabla u_n \nabla \psi_{\varrho}\, dx \right| \right] = 0. \qquad (3.5)$$

Now, using Proposition 2.3 and (f_1), we obtain

$$0 \leq \rho_n(x) \leq C(1 + |u_n(x)|^{r-1}) \quad \text{a.e in } \Omega. \tag{3.6}$$

Then,

$$\int_{B_{2\varrho}(0)} \rho_n \psi_\varrho u_n \leq C\Big[\int_{B_{2\varrho}(0)} \psi_\varrho |u_n| dx + \int_{B_{2\varrho}(0)} \psi_\varrho |u_n|^r dx\Big]$$

so

$$\lim_{\varrho\to 0}\Big[\lim_{n\to\infty}\int_\Omega \rho_n u_n \psi_\varrho \, dx\Big] = 0. \tag{3.7}$$

Therefore

$$\int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^p \psi_\varrho dx = \int_\Omega |u|^{q^*} \psi_\varrho \, dx + o_n(1). \tag{3.8}$$

From (a_1), we have

$$k_0 \int_\Omega |\nabla u_n|^p dx + k_1 \int_\Omega |\nabla u_n|^q dx \leq \int_\Omega |u|^{q^*} \psi_\varrho \, dx$$

We can let $n \to \infty$, we obtain

$$k_1 \int_\Omega d\mu \leq \int_\Omega \psi_\varrho \, d\nu + o_\varrho(1).$$

Letting $\varrho \to 0$ we conclude that $\nu_i \geq k_1 \mu_i$. It follows from a (3.3) that $\nu_i \geq \Big(k_1 S\Big)^{N/q}$

Now we shall prove that the above expression cannot occur, and therefore the set Π is empty. Indeed, arguing by contradiction, let us suppose that $\nu_i \geq \Big(k_1 S\Big)^{N/q}$, for some $i \in \Pi$. Then, from (a_2), we get

$$\begin{aligned} c + o_n(1) &= I(u_n) - \frac{1}{\theta}\langle w_n, u_n\rangle \\ &\geq \Big(\frac{1}{\theta} - \frac{1}{q^*}\Big)\int_\Omega |u_n|^{q^*} \, dx \\ &\quad + \int_\Omega \Big[\frac{1}{\theta}\rho_n u_n - F(u_n)\Big] \, dx. \end{aligned}$$

Once that (3.1), we conclude

$$c + o_n(1) \geq \Big(\frac{1}{\theta} - \frac{1}{q^*}\Big)\int_\Omega |u_n|^{q^*} \, dx \geq \int_{B_{2\varrho}(0)} |u_n|^{q^*} \psi_\varrho \, dx$$

Letting $n \to +\infty$, we get

$$c \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \int_\Omega \psi_\varrho \, d\nu$$

and $\varrho \to 0$, we conclude

$$c \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right)\nu_i \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right)\left(Sk_1\right)^{N/q},$$

which is a contradiction. Hence Π is empty and it follows that

$$\int_\Omega |u_n|^{q*} dx \to \int_\Omega |u|^{q*} dx. \tag{3.9}$$

Now our aim is to prove that

$$u_n \to u \in W_0^{1,q}(\Omega).$$

Note that, by the (3.9) and Brezis and Lieb [11](see also [24][Lemma 4.6]

$$\int_\Omega (|u_n|^{q^*-2}u_n)(u_n - u)dx = o_n(1). \tag{3.10}$$

Moreover, using (f_1) we have

$$0 \leq \rho_n \leq C(1 + |u|^{r-1}) \;\; a.e \in \Omega.$$

Thus

$$\int_\Omega |\rho_n|^{r/r-1} dx \leq C + C\|u_n\|^r \leq C + C_1\|u_n\|^r,$$

which we conclude that (ρ_n) is bounded in $L^{r/r-1}(\Omega)$. By Holder inequality, we have

$$\int_\Omega \rho_n(u_{n-u})dx \leq |\rho_n|_{L^{r/r-1}(\Omega)}|u_n - u|_{L^r(\Omega)}$$

by the (3.9) and the boundedness of (ρ_n)

$$\int_\Omega \rho_n(u_n - u)dx = o_n(1). \tag{3.11}$$

Now by the $a(t) \geq k_1 t^{q-p/p}$ for every $t \geq 0$, which follows by the left-hand side inequality in (a_1), assumption (a_3) and arguing as [7, Lemma 2.4] we have

$$C|x - y|^q \leq < a(|x|^p)|x|^{p-2}x - a(|y|^p)|y|^{p-2}y, x - y >, \;\; \forall x, y \in \mathbb{R}^N$$

with $N \geq 1$ and $< ., . >$ the scalar product in $\mathbb{R}^N$.

Since that $(u_n - u)$ is bounded in $W_0^{1,q}(\Omega)$ and $||w_n||_* = 0_1$, we get that

$$< w_n, u_n - u >= o_n(1).$$

Now, using (3.10)and (3.11) we have

$$\begin{aligned}
C||u_n - u||^q &\leq \int_\Omega (a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n - a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)dx \\
&= \int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n \\
&\quad -\nabla u)dx - \int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u(\nabla u_n - \nabla u)dx \\
&\leq \int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)dx - \int_\Omega |u_n|^{q^*} \\
&\quad + \int_\Omega |u_n|^{q^*-2}u_n u - \int_\Omega \rho_n u_n + \int_\Omega \rho_n u \\
&= \int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)dx \\
&\quad - \int_\Omega (|u_n|^{q^*-2}u_n)(u_n - u)dx - \int_\Omega \rho_n(u_n - u)dx \\
&= < w_n, u_n - u >= o_n(1)
\end{aligned}$$

where we conclude, up to a subsequence, that

$$u_n \to u \in W_0^{1,q}(\Omega).$$

□

Lemma 3.2 (i) *There are* $v \in W_0^{1,q}(\Omega)$ *and* $T > 0$ *such that*

$$\max_{t\in[0,T]} I(tv) < c$$

(ii) *There are* $r > 0$ *and* $e \in W_0^{1,q}(\Omega) \setminus B_r(0)$ *such that* $I(e) < 0$.

(iii) *There is* $\rho > 0$ *such that* $I(u) \geq \rho$, *for* $\|u\| = r$, $u \in W_0^{1,q}(\Omega)$.

Proof Consider $v \in C_0^\infty(\Omega)$ such that $\|v\| = 1$, $|\Upsilon = \{x \in \Omega : Tv(x) > \beta\}| > 0$, T to be fixed later and the function $j : \mathbb{R} \to \mathbb{R}$ given by

$$j(t) = \frac{k_2 t^p}{p}\|v\|_{1,p}^p + \frac{k_3 t^q}{q} - \frac{t^{q^*}}{q^*}|v|_{L^{p^*}(\Omega}^{p^*}.$$

So, there is $t_* > 0$, such that

$$j(t_*) = \max_{t\geq 0} j(t)$$

Note that j is increasing in $(0, t_*)$ and decreasing in (t_*, ∞). We can choose $T > 0$ such that

(a) $T < t_*$,
(b) $j(T) < j(t_*)$
(c) $j(T) < c$

In order to prove $i)$, we use (a_1), continuous embedding and $||v|| = 1$, then

$$\begin{aligned} I(tv) &= \frac{1}{p}\int_\Omega A(|\nabla(tv)|^p)dx - \frac{t^{q^*}}{q^*}\int_\Omega |v|^{q*}dx - \int_\Omega F(tv)dx \\ &\leq \frac{k_2 t^p}{p}\int_\Omega |\nabla v|^p + \frac{k_3 t^q}{q}\int_\Omega |\nabla v|^q - \frac{t^{q^*}}{q^*}\int_\Omega |v|^{q*}dx \\ &= \frac{k_2 t^p}{p}\|v\|_{1,p}^p + \frac{k_3 t^q}{q} - \frac{t^{q^*}}{q^*}\int_\Omega |v|^{q*}dx \\ &= j(t) \leq \max_{t\in[0,T]} j(t) \leq j(T) \leq j(t_*) < c. \end{aligned}$$

Then,

$$\max_{t\in[0,T]} I(tv) < c.$$

To prove $ii)$ use (f_3) and fix $\beta = \frac{T}{2}$ we obtain $e = Tv$ with $\|e\| = T$ such that

$$\begin{aligned} I(e) = I(Tv) &= \frac{1}{p}\int_\Omega A(|\nabla(tv)|^p)dx - \frac{1}{q^*}\int_\Omega |Tv|^{q*}dx - \int_\Omega F(tv)dx \\ &\leq j(T) - \int_\Omega F(tv)dx \\ &\leq j(T) - H(tv - \beta)\int_\Upsilon |Tv|dx < 0. \end{aligned}$$

Finally we consider (f_1), (f_4) and the continuous embedding of $W_0^{1,q}(\Omega)$ in $L^r(\Omega)$ and in $L^{q^*}(\Omega)$ to obtain $C_1, C_2, C_3 > 0$ such that

$$I(u) \geq \frac{k_1}{q}C_1\|u\|^q - C_2\|u\|^r - C_3\|u\|^{q^*}.$$

Considering $0 < \gamma$ sufficient small, we obtain $\rho > 0$ such that

$$I(u) \geq \rho, \quad \text{for all } \|u\| = \gamma \text{ with } u \in W_0^{1,q}(\Omega).$$

□

4 Proof of Theorem 1.1

From Lemmas 3.2 and 3.1, follow from the Theorem 2.1 problem (1.1) has a solution $u \in W_0^{1,q}(\Omega)$.

Using u^- as a test function, we conclude

$$u = u^+ \geq 0.$$

Now we prove that the set

$$\{x \in \Omega : u(x) > \beta\}$$

has positive measure.

Suppose, by contradiction, that $u(x) \leq \beta$ a.e in Ω. Then,since u is solution, we have

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^p dx = \int_\Omega \rho u dx + \int_\Omega |u|^{q*} dx$$

Using (a_1) and (f_1) we have

$$\begin{aligned} k_1\|u\|^q = k_1 \int_\Omega |\nabla u|^q dx \leq k_0 \int_\Omega |\nabla u|^p dx + k_1 \int_\Omega |\nabla u|^q dx \\ \leq \int_\Omega a(|\nabla u|^p)|\nabla u|^p dx = \int_\Omega \rho u dx + \int_\Omega |u|^{q*} dx \\ \leq \int_\Omega C(u + |u|^r) dx + \int_\Omega |u|^{q*} dx \\ \leq C(\beta + \beta^r)|\Omega| + \beta|\Omega| \\ \leq 3\widehat{C}\beta|\Omega| \end{aligned}$$

where $\widehat{C} = \max\{C, 1\}$ and $\beta < 1$.

Since $J(u) = c > 0$, there exists $M > 0$ such that $\|u\| \geq M$. Then,

$$k_1 M^q \leq 3\widehat{C}\beta|\Omega|, \tag{4.1}$$

But this inequality is impossible if we choose

$$\beta = \min\left\{1/2, \frac{T}{2}, \frac{k_1 M^r}{3\widehat{C}|\Omega|}\right\}.$$

□

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