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Liouville-type theorem for a nonlocal operator on the half plane

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Abstract In this article we consider the following integral equation associated to the BO–ZK operator in the half plane. By combining the lifting regularity and the moving planes method for integral forms, we demonstrate that there is no positive solution for this integral equation.

Keywords Integral equation · BO–ZK operator · Half plane · Nonexistence

Mathematics Subject Classification 45E10 · 35Q35 · 35B65 · 35B07

1 Introduction

In [11] Dancer studied the nonexistence of positive solutions for the following nonlinear elliptic equation

$$\begin{cases} -\Delta u(x) = u^r(x), & x \in \mathbb{R}^2_+, \\ u(x) = 0, & x \in \partial\mathbb{R}^2_+ \end{cases} \tag{1}$$

proved a Liouville-type result by showing that problem (1) has no bounded positive solution. During the last years there has been an increasing interest in the study of linear and nonlinear integral operators, especially nonlocal and integral operators.

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Chen et al. [5] recently investigated the following integral equation on the upper half space

$$u(x) = c_n \int_{\mathbb{R}^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|\bar{x}-y|^{n-\alpha}} \right) u^r(y) \mathrm{d}y,$$

where $r > 1$ and $\alpha < n$, and established that the above integral equation is equivalent to the poly-harmonic semi-linear equation

$$(-\Delta)^\alpha u = u^r, \quad u > 0, \tag{2}$$

with Navier boundary conditions on the half-space. They applied the method of moving planes in integral forms and showed that there is no nonnegative solution $u \in L^{\frac{n(r-1)}{\alpha}}_{\mathrm{loc}}(\mathbb{R}^n_+)$, if $\frac{n}{n-\alpha} < r < \frac{n+\alpha}{n-\alpha}$. One can see, for instance, [6,21] for some good surveys on some Liouville-type theorems for (2).

In the present paper we study the the following integral equation on the upper half space

$$u(x) = \int_{\mathbb{R}^2_+} G_\alpha(x,y) u^r(y)\, \mathrm{d}y, \quad x = (x_1, x_2) \in \mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+, \tag{3}$$

where $r > 0$, $\alpha \in (0,1)$,

$$G_\alpha(x,y) = K_\alpha(x-y) - K_\alpha(\overline{x}-y), \tag{4}$$

and K_α is the kernel of the Benjamin–Ono–Zakharov–Kuznetsov (BO–ZK) operator $\mathcal{L}_\alpha = I + D^{2\alpha}_{x_1} - \partial^2_{x_2}$, i.e.

$$K_\alpha(x) = C_\alpha \int_0^\infty \mathrm{e}^{-t} \mathrm{e}^{-\frac{x_2^2}{4t}} t^{-\frac{1}{2}} H_\alpha(x_1, t)\, \mathrm{d}t, \tag{5}$$

with

$$H_\alpha(x_1,t) = \int_{\mathbb{R}} \mathrm{e}^{-t|\xi|^{2\alpha}} \mathrm{e}^{\mathrm{i}x_1\xi} \mathrm{d}\xi,$$

where C_α is a positive constant, depending on α. Here D_{x_1} is defined by $(-\Delta_{x_1})^{1/2}$ and $\bar{x}$ is the reflection of x about $x_2 = 0$. It can be easily seen that under suitable decay assumptions on the solutions, (3) is equivalent to the equation

$$\begin{cases} \mathcal{L}_\alpha u(x) = u^r(x), & x \in \mathbb{R}^2_+, \\ u(x) = 0, & x \in \partial\mathbb{R}^2_+. \end{cases} \tag{6}$$

The operator $\mathcal{L}_\alpha$ appears in the study of toy models [1,2,10], parabolic equations for which local diffusions occur only in certain directions and nonlocal diffusions. See

[16] for some results on regularity and rigidity properties of the operator $\mathcal{L}_\alpha$. Equation (6) appears in the study of solitary waves of the generalized BO–ZK equation

$$u_t + \partial_{x_1}\left(-D_{x_1}^{2\alpha}u + \partial_{x_2}^2 u + u^r\right) = 0. \tag{7}$$

See also [22] for some local and global well-posedness results for (7). In the case $\alpha = 1/2$, Eq. (7) turns into

$$u_t + \partial_{x_1}\left(-\mathscr{H}\partial_{x_1}u + \partial_{x_2}^2 u + u^r\right) = 0, \tag{8}$$

which was proposed as a model to describe the electromigration in thin nanoconductors on a dielectric substrate (see [13,14]). Here $\mathscr{H}$ stands for the Hilbert transform in the x_1-variable such that $\mathscr{H}\partial_{x_1} = D_{x_1}$. In this case the kernel $K_{1/2}$ can be represented [14] by

$$K_{1/2}(x) = C_{1/2}\int_0^\infty \frac{\sqrt{t}}{t^2 + x_1^2} e^{-\frac{x_2^2}{4t}}\, dt.$$

It was proved in [14] that the regular solitary waves of (8) do exist in the fractional Sobolev–Liouville spaces (see [12]), if $2 \le r < 5$.

Before stating our main result, we recall that if $f \in L_{x_2}^q L_{x_1}^p(\mathbb{R}_+^2)$, its norm is denoted by $\|f\|_{L_{x_2}^q L_{x_1}^p(\mathbb{R}_+^2)} = \left\| \|f\|_{L_{x_1}^p(\mathbb{R})} \right\|_{L_{x_2}^q(\mathbb{R}^+)}$.

Theorem 1 *Suppose that $p_0, q_0 \ge r$ satisfies*

$$\frac{\alpha}{p_0} + \frac{1}{q_0} < \frac{2\alpha}{r-1}. \tag{9}$$

Then there is no positive solution $u \in L_{x_2}^{p_0} L_{x_1}^{q_0}(\mathbb{R}_+^2)$ of (3).

Corollary 1 *Assume that $p_0, q_0 \ge r$ satisfies (9). If $u \in L_{x_2}^{p_0} L_{x_1}^{q_0}(\mathbb{R}_+^2)$ is a nonnegative solution of (6), then $u \equiv 0$.*

To prove the non-existence of positive solutions for (3), we use regularity lifting by contracting operators appearing in the integral equations [7,19] to boost the positive solutions for integral equation (3) to $L^1(\mathbb{R}_+^2) \cap L^\infty(\mathbb{R}_+^2)$.

Theorem 2 *Let u be a positive solution of (3). Suppose that $u \in L_{x_2}^{p_0} L_{x_1}^{q_0}(\mathbb{R}_+^2)$, where $p_0, q_0 \ge r$ satisfies (9). Then $u \in L^1(\mathbb{R}_+^2) \cap L^\infty(\mathbb{R}_+^2)$.*

Next step to prove Theorem 1 is to employ the method of moving planes in integral forms. We move the plane along x_2 direction to show that the solutions must be monotone increasing in x_2 and thus derive a contradiction.

Theorem 3 *Under the assumption of Theorem 2, we have that u must be symmetric about the line $x_2 = c$, for some constant c. Moreover u is strictly monotone increasing with respect to x_2.*

For more related results regarding the method of moving planes and integral equations, refer the reader to [6,8,9,15,19,20] and the references therein.

This paper is organized as follows. Section 2 is devoted to the preliminaries of the kernel K_α and also the proof of Theorem 2. The symmetry and the nonexistence result of the solutions are proved in Sect. 3.

For the simplicity and without loss of generality, we assume henceforth $C_\alpha = 1$.

Throughout the paper, the notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \le CB$. The notation $A \gtrsim B$ is similarly defined. We will also write $A \approx B$ to mean $A \lesssim B$ and $A \gtrsim B$.

2 Properties of K_α

In this section we will give some key properties of the kernel K_α.

Lemma 1 *Let $\alpha > 0$. The following properties hold:*

(i) $K_\alpha \in L^p_{x_2} L^q_{x_1}(\mathbb{R}^2) \cap L^q_{x_1} L^p_{x_2}(\mathbb{R}^2)$ *for $p, q \ge 1$ with $\alpha(1 + 1/p) > 1 - 1/q$.*

(ii) $K_\alpha(x) > 0$, *for $x \in \mathbb{R}^2$, and is an even function which is strictly decreasing in $|x_1|$ and $|x_2|$ and smooth for $x_1 \neq 0$.*

(iii) *For $0 < \alpha < 1$, we have the following bound*

$$K_\alpha(x) \lesssim |x_1|^{\alpha-1} e^{-|x_2|}, \quad x \in \mathbb{R}^2,$$

where C depends only on α.

(iv) *There is a constant C, depending only on α, such that*

$$K_\alpha(x) \lesssim |x_1|^{-1-2\alpha} e^{-\frac{|x_2|}{4}}, \quad \textit{if } |x_1| \ge 1. \tag{10}$$

and

$$K_\alpha(x) \gtrsim |x_1|^{-1-2\alpha} e^{-\frac{x_2^2}{4}}, \quad \textit{if } |x_1| \ge 1 \ge |x_2|. \tag{11}$$

In particular, $0 < \lim_{|x_1|,|x_2| \to +\infty} |x_1|^{1+2\alpha} e^{x_2^2/4} K_\alpha(x) < +\infty$.

Proof The decay properties of H_α are obtained in [4] (see also [3]). In particular, it is proved that

$$\lim_{|x_1| \to \infty} |x_1|^{1+2\alpha} H_\alpha(x_1, 1) < \infty.$$

Using this estimate and the scaling property

$$H_\alpha(x_1, t) = t^{-\frac{1}{2\alpha}} H_\alpha\left(t^{-\frac{1}{2\alpha}} x_1, 1\right),$$

it is easy to see that

$$H_\alpha(x_1, t) \approx \min\left\{t^{-1/2\alpha}, t|x_1|^{-1-2\alpha}\right\}. \tag{12}$$

So that

$$K_\alpha(x) \lesssim |x_1|^{-1-2\alpha} \int_0^{|x_1|^{2\alpha}} \mathrm{e}^{-t-\frac{x_2^2}{4t}} t^{\frac{1}{2}} \mathrm{d}t + \int_{|x_1|^{2\alpha}}^{\infty} \mathrm{e}^{-t-\frac{x_2^2}{4t}} t^{-\frac{1}{2}\left(1+\frac{1}{\alpha}\right)} \mathrm{d}t. \tag{13}$$

It follows then from a change of variable and the elementary inequality $s+y^2/s \geq 2|y|$, for all $s \geq 0$, that

$$K_\alpha(x) \lesssim |x_1|^{\alpha-1} \mathrm{e}^{-|x_2|} \left(\int_0^1 t^{1/2} \mathrm{d}t + \int_1^\infty t^{-1/2(1+1/\alpha)} \mathrm{d}t \right) \lesssim |x_1|^{\alpha-1} \mathrm{e}^{-|x_2|}. \tag{14}$$

To prove (10), we have for $|x_1| \geq 1$ from (13) and the inequality

$$t + \frac{x_2^2}{4t} \geq \frac{t}{2} + \frac{1}{2t} + \frac{|x_2|}{4}$$

that

$$K_\alpha(x) \lesssim |x_1|^{-2\alpha-1} \mathrm{e}^{-\frac{|x_2|}{4}} \int_0^\infty \mathrm{e}^{-t} \mathrm{e}^{-\frac{1}{t}} t^{\frac{1}{2}} \mathrm{d}t \lesssim |x_1|^{-2\alpha-1} \mathrm{e}^{-\frac{|x_2|}{4}}. \tag{15}$$

In order to prove (11), we have for $|x_1| \geq 1 \geq |x_2|$ from (12) and the inequality $x_2^2/t \leq x_2^2 + 1/t$ that

$$K_\alpha(x) \geq |x_1|^{-1-2\alpha} \int_0^1 \mathrm{e}^{-t} \mathrm{e}^{-x_2^2/4t} t^{1/2} \gtrsim |x_1|^{-1-2\alpha} \mathrm{e}^{-x_2^2/4}.$$

The properties of K_α in (ii) follow from the positivity and the monotonicity of H_α in [4, 17]. Finally the property (i) is deduced from (13) and the Minkowski inequality. □

The following lemma gives a Hardy–Littlewood–Sobolev-type inequality; and is a direct consequence of Lemma 1 (see also [18]) and the Young inequality

$$\|f * g\|_{L^p_{x_2} L^q_{x_1}(\mathbb{R}^2_+)} \leq \|f\|_{L^{p_1}_{x_2} L^{q_1}_{x_1}(\mathbb{R}^2_+)} \|g\|_{L^{p_2}_{x_2} L^{q_2}_{x_1}(\mathbb{R}^2_+)}, \tag{16}$$

with $1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $1 + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Lemma 2 *Let $\alpha > 0$ and $c > 0$ and $f \in L^p_{x_2} L^q_{x_1}(\mathbb{R}^2)$. Then $M_\alpha(f) = K_\alpha * f \in L^{p_1}_{x_2} L^{q_1}_{x_1}(\mathbb{R}^2)$ and*

$$\|M_\alpha(f)\|_{L^{p_1}_{x_2} L^{q_1}_{x_1}(\mathbb{R}^2)} \leq C \|f\|_{L^p_{x_2} L^q_{x_1}(\mathbb{R}^2)},$$

provided $\alpha(2 + 1/p_1) + 1/q_1 > 1/q + \alpha/p$, where $*$ is the convolution operator. The same result holds for $L^q_{x_1} L^p_{x_2}(\mathbb{R}^2)$ and $L^{q_1}_{x_1} L^{p_1}_{x_2}(\mathbb{R}^2)$.

To prove Theorem 2, we apply the regularity lifting by contracting operators.

Definition 1 Let V be a topological vector space with two extended norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, where $X=\{v\in V;\ \|v\|_X<\infty\}$ and $Y=\{v\in V;\ \|v\|_Y<\infty\}$. The operator $T:X\to Y$ is said to be a contraction if

$$\|Tx-Ty\|_Y\leq\theta\|x-y\|_X,$$

for all $x,y\in X$ and some $0<\theta<1$.

We now recall the following regularity lifting theorem (see [7,19]).

Theorem 4 [19, Lemma 2.2] *Let T be a contracting operator from X to itself and from Y to itself, and assume that X, Y are both complete. If $f\in X$, and there exists $g\in Z=X\cap Y$ such that $f=Tf+g$ in X, then $g\in Z$.*

The proof of Theorem 2 is a direct corollary of the following result and the Young inequality (16).

Theorem 5 *Under the same conditions of Theorem 2, we have $u\in L^p_{x_2}L^q_{x_1}(\mathbb{R}^2_+)$ for all $1<p,q<\infty$.*

Proof Define the linear operator

$$T_vw=\int_{\mathbb{R}^2_+}G_\alpha(x,y)(|v(y)|^{r-1}w(y))\,\mathrm{d}y.$$

For a fixed real number $a>0$, define

$$u_a(x)=\begin{cases}u(x), & |u(x)|>a \text{ or } |x|>a,\\ 0, & \text{otherwise.}\end{cases}$$

Write $u_b=u-u_a$, which is uniformly bounded by a in $B_a(0)$. It is evident that $u^q=(u_a+u_b)^q=u_a^q+u_b^q$ for all $q>0$. Since $u=u_a+u_b$ satisfies (3), we have $u=T_{u_a}u_a+T_{u_b}u_b$. Let $g=T_{u_b}u_b-u_b$. Then we can see that $g\in L^\infty(\mathbb{R}^2_+)\cap L^1(\mathbb{R}^2_+)$, so that $g\in L^p_{x_2}L^q_{x_1}(\mathbb{R}^2_+)$ for all $1<p,q<\infty$. Thus $u_a=T_{u_a}u_a+g$. Now by using Lemma 2 and the Hölder inequality we have for $w\in L^p_{x_2}L^q_{x_1}(\mathbb{R}^2_+)$ that

$$\|T_{u_a}w\|_{L^p_{x_2}L^q_{x_1}(\mathbb{R}^2_+)}\leq C\|u_a\|^{r-1}_{L^{p_0}_{x_2}L^{q_0}_{x_1}(\mathbb{R}^2_+)}\|w\|_{L^p_{x_2}L^q_{x_1}(\mathbb{R}^2_+)}.$$

By virtue of $u\in L^{p_0}_{x_2}L^{q_0}_{x_1}(\mathbb{R}^2_+)$, choose a large enough such that

$$C\|u_a\|^{r-1}_{L^{p_0}_{x_2}L^{q_0}_{x_1}(\mathbb{R}^2_+)}<1/2.$$

This combining with T_{u_a} being a linear operator, implies that T_{u_a} is a contraction map from $L^p_{x_2}L^q_{x_1}(\mathbb{R}^2_+)$ into itself, for all $p,q>r$. Applying Theorem 4, the solution $w=T_{u_a}w+g$ belongs to $L^{p_0}_{x_2}L^{q_0}_{x_1}(\mathbb{R}^2_+)\cap L^p_{x_2}L^q_{x_1}(\mathbb{R}^2_+)$; and by uniqueness of solution, $u_a\in L^{p_0}_{x_2}L^{q_0}_{x_1}(\mathbb{R}^2_+)\cap L^p_{x_2}L^q_{x_1}(\mathbb{R}^2_+)$, since u_a is a solution of $w=T_{u_a}w+g$. Therefore $u\in L^p_{x_2}L^q_{x_1}(\mathbb{R}^2_+)$ for all $r<p,q<\infty$.

Finally an application of Lemma 1 shows that $u \in L^p_{x_2} L^q_{x_1}(\mathbb{R}^2_+)$ for all $1 < p, q < \infty$. □

Remark 1 Lemmas 1 and 2 show that Theorem 5 is also true if we consider $L^q_{x_1} L^p_{x_2}(\mathbb{R}^2_+)$ instead of $L^p_{x_2} L^q_{x_1}(\mathbb{R}^2_+)$.

We show that the positive solution u of (3) are continuous. We believe that u is also Lipschitz continuous, but we are not able to show it.

Proposition 1 *Under the same conditions of Theorem 2, the positive solution u of (3) is continuous.*

Proof It follows from (3) that

$$\begin{aligned} u(x) - u(y) &= \int_{\mathbb{R}^2_+} (G_\alpha(x,z) - G_\alpha(y,z))\, u^r(z)\mathrm{d}z \\ &= \int_{B_\delta(x)} (G_\alpha(x,z) - G_\alpha(y,z))\, u^r(z)\mathrm{d}z \\ &\quad + \int_{\mathbb{R}^2_+ \setminus B_\delta(x)} (G_\alpha(x,z) - G_\alpha(y,z))\, u^r(z)\mathrm{d}z. \end{aligned} \tag{17}$$

By Theorem 2, $\int_{\mathbb{R}^2_+} G_\alpha(x,z) u^r(z)\mathrm{d}z < +\infty$, and thus the second term of the right hand side of (17) is small enough if we choose δ sufficiently large. On the other hand, we have from Lemma 1 that $K_\alpha(x-z) - K_\alpha(y-z) \to 0$, as $|x-y| \to 0$, for any $z \in B_\delta(x)$. Hence

$$\lim_{x \to y} \int_{B_\delta(x)} (K_\alpha(x-z) - K_\alpha(y-z))\, u^r(z)\mathrm{d}z = 0.$$

Since $|\bar{x} - \bar{y}| \to 0$ as $|x-y| \to 0$, we have

$$\lim_{x \to y} \int_{B_\delta(x)} (K_\alpha(\bar{x}-z) - K_\alpha(\bar{y}-z))\, u^r(z)\mathrm{d}z = 0.$$

Therefore we deduce

$$\lim_{x \to y} \int_{B_\delta(x)} (G_\alpha(x,z) - G_\alpha(y,z))\, u^r(z)\mathrm{d}z = 0.$$

Thus the first term of the right hand side of (17) is finite, and consequently the solution u of (3) is continuous. □

3 Symmetry

For a given real number $\lambda > 0$, we may define a family of moving planes

$$\Omega_\lambda = \{x \in \mathbb{R}^2_+,\ x_2 = \lambda\}.$$

and moving regions $\Sigma_\lambda = \{x \in \mathbb{R}^2_+,\ 0 < x_2 < \lambda\}$. Let us list some properties of G_α. For any $x \in \mathbb{R}^2_+$, we denote its reflection through the plane Ω_λ by $x_\lambda = (x_1, 2\lambda - x_2)$.

Lemma 3 (i) *For any $x, y \in \Sigma_\lambda$ with $x \neq y$, we have*

$$\max\{G_\alpha(x_\lambda, y), G_\alpha(x, y_\lambda)\} < G_\alpha(x_\lambda, y_\lambda) \tag{18}$$

and

$$|G_\alpha(x_\lambda, y) - G_\alpha(x, y_\lambda)| < G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y). \tag{19}$$

(ii) *For any $x \in \Sigma_\lambda$ and $y \in \Sigma_\lambda^c = \mathbb{R}^2_+ \setminus \Sigma_\lambda$, it holds that*

$$G_\alpha(x, y) < G_\alpha(x_\lambda, y). \tag{20}$$

Proof For any x, y, let $d(x, y) = |x_2 - y_2|^2$. Recalling (4), one has

$$\begin{aligned} G_\alpha(x, y) &= K_\alpha(x - y) - K_\alpha(\bar{x} - y) \\ &= \int_0^\infty \mathrm{e}^{-t} t^{-\frac{1}{2}} H_\alpha(x_1 - y_1, t) \left(\mathrm{e}^{-\frac{(x_2 - y_2)^2}{4t}} - \mathrm{e}^{-\frac{(-x_2 - y_2)^2}{4t}}\right) \mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-t} t^{-\frac{1}{2}} H_\alpha(x_1 - y_1, t) \left(\mathrm{e}^{-\frac{d(x,y)}{4t}} - \mathrm{e}^{-\frac{d(x,y)+\psi(x,y)}{4t}}\right) \mathrm{d}t, \end{aligned}$$

where $\psi(x, y) = 4x_2 y_2$. It is clear that $G_\alpha(x, y) > 0$. By direct computations, one obtains that

$$\frac{\partial G_\alpha}{\partial d} < 0, \quad \frac{\partial G_\alpha}{\partial d} > 0, \quad \frac{\partial^2 G_\alpha}{\partial \psi \partial d} < 0. \tag{21}$$

On the other hand, it is obvious to see for any $x, y \in \Sigma_\lambda$ that

$$d(x_\lambda, y_\lambda) = d(x, y) < d(x_\lambda, y) = d(x, y_\lambda), \tag{22}$$

and

$$\psi(x, y) \leq \max\{\psi(x_\lambda, y), \psi(x, y_\lambda)\} \leq \psi(x_\lambda, y_\lambda). \tag{23}$$

The proof of lemma follows from (21)–(23). □

Lemma 4 *For any positive solution u of (3), we have for any $x \in \Sigma_\lambda$ that*

$$u(x) - u_\lambda(x) \leq \int_{\Sigma_\lambda} (G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda)) \left(u^r(y) - u^r_\lambda(y)\right) \mathrm{d}y,$$

where $u_\lambda(x) = u(x_\lambda)$.

Proof Let $\tilde{\Sigma}_\lambda = \{x_\lambda,\ x \in \Sigma_\lambda\}$. It is easy to see that

$$\begin{aligned} u(x) &= \int_{\Sigma_\lambda} G_\alpha(x, y)u^r(y)\mathrm{d}y + \int_{\tilde{\Sigma}_\lambda} G_\alpha(x, y)u^r(y)\mathrm{d}y \\ &\quad + \int_{\Sigma_\lambda^c \setminus \tilde{\Sigma}_\lambda} G_\alpha(x, y)u^r(y)\mathrm{d}y \\ &= \int_{\Sigma_\lambda} G_\alpha(x, y)u^r(y)\mathrm{d}y + \int_{\Sigma_\lambda} G_\alpha(x, y_\lambda)u_\lambda^r(y)\mathrm{d}y \\ &\quad + \int_{\Sigma_\lambda^c \setminus \tilde{\Sigma}_\lambda} G_\alpha(x, y)u^r(y)\mathrm{d}y. \end{aligned} \tag{24}$$

Substituting x by x_λ, we get

$$\begin{aligned} u(x) - u(x_\lambda) &= \int_{\Sigma_\lambda} \left(G_\alpha(x, y) - G_\alpha(x_\lambda, y)\right) u^r(y)\mathrm{d}y \\ &\quad + \int_{\Sigma_\lambda} \left(G_\alpha(x, y_\lambda) - G_\alpha(x_\lambda, y_\lambda)\right) u_\lambda^r(y)\mathrm{d}y \\ &\quad + \int_{\Sigma_\lambda^c \setminus \tilde{\Sigma}_\lambda} \left(G_\alpha(x, y) - G_\alpha(x_\lambda, y)\right) u^r(y)\mathrm{d}y. \end{aligned} \tag{25}$$

It is deduced from Lemma 3 that

$$\begin{aligned} u(x) - u(x_\lambda) &\le \int_{\Sigma_\lambda} \left(G_\alpha(x, y) - G_\alpha(x_\lambda, y)\right) u^r(y)\mathrm{d}y \\ &\quad - \int_{\Sigma_\lambda} \left(G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda)\right) u_\lambda^r(y)\mathrm{d}y \\ &\le \int_{\Sigma_\lambda} \left(G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda)\right) u^r(y)\mathrm{d}y \\ &\quad - \int_{\Sigma_\lambda} \left(G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda)\right) u_\lambda^r(y)\mathrm{d}y \\ &= \int_{\Sigma_\lambda} \left(G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda)\right) \left(u^r(y) - u_\lambda^r(y)\right) \mathrm{d}y; \end{aligned}$$

and the proof is completed. □

Lemma 5 *For* $0 < \lambda \ll 1$, $\Sigma_\lambda^- = \{x \in \Sigma_\lambda,\ u(x, y) > u_\lambda(x, y)\}$ *has measure zero.*

Proof It is easy to see from Lemma 3, for any $x \in \Sigma_\lambda^-$, that

$$\begin{aligned}0 < u(x) - u_\lambda(x) &\leq \int_{\Sigma_\lambda} (G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda)) \left(u^r(y) - u_\lambda^r(y)\right) \mathrm{d}y \\ &= \int_{\Sigma_\lambda^-} (G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda)) \left(u^r(y) - u_\lambda^r(y)\right) \mathrm{d}y \\ &\quad + \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} (G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda)) \left(u^r(y) - u_\lambda^r(y)\right) \mathrm{d}y.\end{aligned}$$

By Lemma 3, $u^r(x) \leq u_\lambda^r(x)$ on $\Sigma_\lambda \setminus \Sigma_\lambda^-$ and $G_\alpha(x_\lambda, y_\lambda) \geq G_\alpha(x, y_\lambda)$ for $y \in \Sigma_\lambda \setminus \Sigma_\lambda^-$, then the last integral in the above inequality is negative. Hence we get

$$\begin{aligned}u(x) - u_\lambda(x) &\leq \int_{\Sigma_\lambda^-} (G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda)) \left(u^r(y) - u_\lambda^r(y)\right) \mathrm{d}y \\ &\leq \int_{\Sigma_\lambda^-} G_\alpha(x_\lambda, y_\lambda) \left(u^r(y) - u_\lambda^r(y)\right) \mathrm{d}y \\ &\leq \int_{\Sigma_\lambda^-} K_\alpha(x_\lambda - y_\lambda) \left(u^r(y) - u_\lambda^r(y)\right) \mathrm{d}y \\ &\lesssim \int_{\Sigma_\lambda^-} G_\alpha(x_\lambda, y_\lambda)\varphi^{r-1}(y) \left(u(y) - u_\lambda(y)\right) \mathrm{d}y \\ &\lesssim \int_{\Sigma_\lambda^-} G_\alpha(x_\lambda, y_\lambda)u^{r-1}(y) \left(u(y) - u_\lambda(y)\right) \mathrm{d}y,\end{aligned} \tag{26}$$

where we have used the mean value theorem with $\varphi(y)$ valued between $u(y)$ and $u_\lambda(y)$, and the fact that $0 \leq u_\lambda(y) \leq \varphi(y) \leq u(y)$ on Σ_λ^-.

It follows first from Lemma 2 and then the Hölder inequality that

$$\|u - u_\lambda\|_{L^{p_0}_{x_2} L^{q_0}_{x_1}(\Sigma_\lambda^-)} \lesssim \|u\|^{r-1}_{L^{p_0}_{x_2} L^{q_0}_{x_1}(\Sigma_\lambda)} \|u - u_\lambda\|_{L^{p_0}_{x_2} L^{q_0}_{x_1}(\Sigma_\lambda^-)}.$$

Since $u \in L^{p_0}_{x_2} L^{q_0}_{x_1}(\mathbb{R}^2)$, by choosing $\lambda \ll 1$ we deduced that $\|u - u_\lambda\|_{L^{p_0}_{x_2} L^{q_0}_{x_1}(\Sigma_\lambda^-)} = 0$, and therefore Σ_λ^- has measure zero. □

Proof of Theorem 3 Define

$$\lambda_0 = \sup\{\lambda,\ u_\mu(x) \geq u(x), \forall \mu \leq \lambda,\ x \in \Sigma_\mu\}.$$

We assume that $\lambda_0 < +\infty$, because the case $\lambda_0 = +\infty$ gives the proof by the definition of λ_0 and the assumption $u \in L^{p_0}_{x_2} L^{q_0}_{x_1}(\mathbb{R}^2_+)$. We show that the solution $u(x)$ is monotone increasing with respect to the x_2-variable and symmetric about Ω_{λ_0}, that is, $u(x) = u_{\lambda_0}(x)$ on Σ_{λ_0}. Suppose by the contradiction argument that $u \leq u_{\lambda_0}$ and $u \not\equiv u_\lambda$ on Σ_{λ_0}. We prove that there exists an $\epsilon > 0$ such that, for any $\lambda_0 \leq \lambda < \lambda_0 + \epsilon$, it holds on Σ_λ that

$$u(x) \leq u_\lambda(x).$$

By using an argument analogous to the proof of Lemma 5, we can obtain that

$$\|u-u_\lambda\|_{L^{p_0}_{x_2}L^{q_0}_{x_1}(\Sigma_\lambda)} \le C\|u\|^{r-1}_{L^{p_0}_{x_2}L^{q_0}_{x_1}(\Sigma^-_\lambda)}\|u-u_\lambda\|_{L^{p_0}_{x_2}L^{q_0}_{x_1}(\Sigma_\lambda)}. \tag{27}$$

Now if we establish for $\epsilon \ll 1$ that

$$C\|u\|^{r-1}_{L^{p_0}_{x_2}L^{q_0}_{x_1}(\Sigma^-_\lambda)} \le \frac{1}{4}, \tag{28}$$

then it follows from (27) that Σ^-_λ is a set of zero measure; and consequently $u_\lambda(x) \ge u(x)$ for any $x \in \Sigma_\lambda$ and $\lambda \in [\lambda_0, \lambda_0+\epsilon)$. This contradicts with the definition of λ_0 and the result follows.

Now we prove inequality (28). Choose, for any small $\varsigma > 0$, a large enough number $\delta > 0$ such that

$$\|u\|^{r-1}_{L^{p_0}_{x_2}L^{q_0}_{x_1}(\mathbb{R}^2_+\setminus B_\delta(0))} < \varsigma, \tag{29}$$

where $B_\delta(0)$ is the ball of radius $\delta > 0$ centered at zero in $\mathbb{R}^2_+$. It is straightforward to see that $u < u_\lambda$ in Σ_{λ_0}. Indeed by contrary suppose that $u_\lambda(x_0) = u(x_0)$, for some $x_0 \in \Sigma_{\lambda_0}$. It follows then from Lemma 3 and the proof of Lemma 4 that

$$\begin{aligned} 0 = u_\lambda(x_0) - u(x_0) &\ge \int_{\Sigma_{\lambda_0}} \left(K_\alpha(x_{\lambda_0}, y_{\lambda_0}) - K_\alpha(x, y_{\lambda_0})\right)\left(u^r_{\lambda_0}(y) - u^r(y)\right)\mathrm{d}y \\ &\quad + \int_{\Sigma^c_{\lambda_0}\setminus\tilde{\Sigma}_{\lambda_0}} \left(K_\alpha(x_{\lambda_0}, y) - K_\alpha(x, y)\right)u^r_{\lambda_0}(y)\mathrm{d}y \\ &\ge \int_{\Sigma^c_{\lambda_0}\setminus\tilde{\Sigma}_{\lambda_0}} \left(K_\alpha(x_{\lambda_0}, y) - K_\alpha(x, y)\right)u^r_{\lambda_0}(y)\mathrm{d}y. \end{aligned}$$

By applying again Lemma 3 it yields that $u(y) = 0$ for all $y \in \Sigma^c_{\lambda_0} \setminus \tilde{\Sigma}_{\lambda_0}$ which contradicts with the positivity of u. Now for any $\kappa > 0$ define

$$B_\kappa = \{x \in \Sigma_{\lambda_0} \cap B_\delta(0),\ u_{\lambda_0}(x) - u(x) > \kappa\}, \quad B^c_\kappa = \Sigma_{\lambda_0} \cap B_\delta(0) \setminus B_\kappa;$$

and denote, for $\lambda > \lambda_0$,

$$\tilde{B}_\lambda = (\Sigma_\lambda \setminus \Sigma_{\lambda_0}) \cap B_\delta(0).$$

Note that the measure of B^c_κ tends to zero as $\kappa \to 0$. Moreover

$$\Sigma^-_\lambda \cap B_\delta(0) \subset (\Sigma^-_\lambda \cap B_\kappa) \cup B^c_\kappa \cup \tilde{B}_\lambda \tag{30}$$

and the measure of $\tilde{B}_\lambda$ is small as λ is close to λ_0. We show that the measure of $\Sigma_\lambda^- \cap B_\kappa$ is sufficiently small as λ is close to λ_0. Actually, since for any $x \in \Sigma_\lambda^- \cap B_\kappa$ we have

$$u_\lambda(x) - u(x) = u_\lambda(x) - u_{\lambda_0}(x) + u_{\lambda_0}(x) - u(x) < 0,$$

then $u_{\lambda_0}(x) - u_\lambda(x) > \kappa$. And hence

$$\Theta_\kappa := \{x \in B_\delta(0),\ u_{\lambda_0}(x) - u_\lambda(x) > \kappa\} \supset \Sigma_\lambda^- \cap B_\kappa. \tag{31}$$

Therefore it is deduced from the Chebyshev inequality that

$$|\Theta_\kappa| \le \frac{1}{\kappa^{r+1}} \int_{\Theta_\kappa} |u_{\lambda_0}(x) - u_\lambda(x)|^{r+1} \mathrm{d}x \le \frac{1}{\kappa^{r+1}} \int_{B_\delta(0)} |u_{\lambda_0}(x) - u_\lambda(x)|^{r+1} \mathrm{d}x.$$

The above integral and consequently $\Sigma_\lambda^- \cap B_\kappa$ is small enough as λ is close to λ_0. Finally by combining (30) and (31) we obtain that the measure of $\Sigma_\lambda^- \cap B_\delta(0)$ is sufficiently small for λ close to λ_0. This completes the proof. □

Proof of Theorem 1 Suppose that u is a nontrivial nonnegative solution of (3). Then there exists $y_0 \in \mathbb{R}_+^2$ such that $u(y_0) > 0$. By the continuity of u from Proposition 1, there exists a neighborhood N_{y_0} of y_0 in $\mathbb{R}_+^2$ such that $u(y) > 0$ for any $y \in N$. Since $G_\alpha > 0$ in $\mathbb{R}_+^2$, then

$$u(x) = \int_{\mathbb{R}_+^2} G_\alpha(x, y) u^r(y) \mathrm{d}y \ge \int_{N_{y_0}} G_\alpha(x, y) u^r(y) \mathrm{d}y > 0. \tag{32}$$

Due to Theorem 3, we know that the plane Ω_λ can be moved to the limiting position Ω_{λ_0}. We show that $\lambda_0 = +\infty$, which gives a simple contradiction argument. Assume $\lambda_0 < +\infty$, then the symmetric image of the boundary of $\mathbb{R}_+^2$ through the line Ω_{λ_0} is the plane $x_2 = 2\lambda_0$. And therefore $u(x) = 0$ for any $x \in \Omega_{2\lambda_0}$, which is a contradiction to (32). Now making use again of Theorem 3, it can be derived that $u(x)$ is monotone increasing with respect to x_2. This leads to a contradiction with the assumption $u \in L_{x_2}^{p_0} L_{x_1}^{q_0}(\mathbb{R}_+^2)$. As a result, the positive solution of (3) does not exist. □

Remark 2 Under the assumptions of Theorem 1, one can easily repeat, with some modifications, the proof of Theorem 1 and demonstrate the non-existence of positive solution for the integral equation (3) in $\mathbb{R}^+ \times \mathbb{R}$. The key point is that by Bernstein's theorem [17], one can write H_α in terms of subordination formula

$$H_\alpha(x_1, t) = \int_0^\infty \frac{1}{\sqrt{2s}} \mathrm{e}^{-\frac{t^2 x_1^2}{4s}} \mathrm{d}\mu_\alpha(s),$$

with some nonnegative finite measure $\mu_\alpha \ge 0$ with $\mu_\alpha \not\equiv 0$.

Theorem 1 can be extended to a general nonlinearity.

Theorem 6 *Let the assumptions of Theorem 1 hold. Suppose that* $u \in L^{p_0}_{x_2} L^{q_0}_{x_1}(\mathbb{R}^2_+)$ *is the nonnegative solution of*

$$u(x) = \int_{\mathbb{R}^2_+} G_\alpha(x, y) f(y, u(y))\, \mathrm{d}y, \quad x = (x_1, x_2) \in \mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+, \tag{33}$$

Suppose also that $f(x, u)$ *is nondecreasing in the variable* x_2 *and nondecreasing with respect to* u*, and* $\frac{\partial f}{\partial u} \in L^{p_1}_{x_2} L^{q_1}_{x_1}(\mathbb{R}^2_+)$ *is non-decreasing with respect to* u*, where* $1 \le p_1, q_1 \le \infty$ *with*

$$\frac{1}{p_1} + \frac{1}{p_0} \le 1 \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_0} \le 1.$$

Then u *is identically equal to zero.*

The proof is basically the same as that of Theorem 1. In order to avoid repetition, we explain the key modifications of the proof of Theorem 6.

Applying the same arguments as in Lemma 4, we can observe for any $x \in \Sigma_\lambda$ from the monotonicity of $f(x, u)$ with respect to x_2 that

$$\begin{aligned} u(x) - u_\lambda(x) &\le \int_{\Sigma_\lambda} (G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda))\,(f(y, u(y)) - f(y, u_\lambda(y)))\, \mathrm{d}y \\ &\quad + \int_{\Sigma_\lambda} (G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda))\,(f(y, u_\lambda(y)) - f(y_\lambda, u_\lambda(y)))\, \mathrm{d}y \\ &\le \int_{\Sigma_\lambda} (G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda))\,(f(y, u(y)) - f(y, u_\lambda(y)))\, \mathrm{d}y. \end{aligned} \tag{34}$$

Recall the definition of Σ^-_λ in Lemma 5. Now since f is nondecreasing in u, we have

$$\begin{aligned} u(x) - u_\lambda(x) &\le \int_{\Sigma_\lambda} (G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda))\,(f(y, u(y)) - f(y, u_\lambda(y)))\, \mathrm{d}y \\ &\le \int_{\Sigma^-_\lambda} (G_\alpha(x_\lambda, y_\lambda) - G_\alpha(x, y_\lambda))\,(f(y, u(y)) - f(y, u_\lambda(y)))\, \mathrm{d}y \\ &\le \int_{\Sigma^-_\lambda} G_\alpha(x, y)\,(f(y, u(y)) - f(y, u_\lambda(y)))\, \mathrm{d}y \\ &\le \int_{\Sigma^-_\lambda} K_\alpha(x - y)\,(f(y, u(y)) - f(y, u_\lambda(y)))\, \mathrm{d}y \\ &\le \int_{\Sigma^-_\lambda} G_\alpha(x, y) \frac{\partial f}{\partial u}(y, u(y))(u(y) - u_\lambda(y))\mathrm{d}y, \end{aligned} \tag{35}$$

where in the last inequality we have used the mean value theorem and the monotonicity of $\frac{\partial f}{\partial u}$ in u. The rest of proof is the same as that of Theorem 1 by using Lemma 2.

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