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Unique strong and strict solutions for inhomogeneous hyperbolic differential equations in Banach space (revised version)

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Abstract This paper is a revised version of a recent paper by the author with the same title. The purpose of the present paper is the improvement of results for solutions of the inhomogeneous differential equation:

$$u'(t) + A(t)u(t) + f(t) = 0, \quad t \in (t_1, t_2)$$
$$u(t_1) = \varphi$$

in reflexive Banach space X. For $f(s) \in C^1\big([t_1, t_2], X\big)$, Kato obtained a unique strict solution under some conditions on the operator family $\{A(t)\}_{t_1 \le t \le t_2}$ to ensure the hyperbolicity of the problem. In a previous paper, the author obtained in abstract Hilbert space H a unique strong solution $u(t) \in C^{0,1}\big([t_1, t_2], H\big) \cap D$ if $f \in BV\big([t_1, t_2], H\big)$ and a strict solution if additionally $f \in C^0\big([t_1, t_2], H\big)$. Here is $D = D\big((A(t)\big)$ independent of t. In the present paper we extend these results to reflexive Banach spaces X.

Keywords Evolution equations · Regularity of integrals · Abstract hyperbolic differential equations · Strong solutions · Strict solutions

Mathematics Subject Classification Primary 47D06 · 47D03 · 35L90; Secondary 64G05 · 58D25 · 34K30 · 34G20

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1 Introduction

This paper is a revised version of a recent paper by the author with the same title. New in this version is the crucial Lemma 2.1 and its correct application in Theorem 3.1. This Lemma was first mentioned in the paper of Kato [6] from 1953 as Lemma 5, but the proof there is not quite right, because from the boundedness of the sequence $\{Ax_n\}$ its weak convergence is inferred, while only the existence of a weak convergent subsequence is certain. In the present paper we give proof for this Lemma. For the treatment of the inhomogeneous differential equation:

$$\begin{cases} u'(t) + A(t)u(t) + f(t) = 0\,, & t_1 < t < t_2 \\ u(t_1) = \varphi \end{cases} \tag{1.1}$$

in reflexive Banach space X, we use as usual the associated integral equation:

$$u(t) = U(t, t_1)\varphi - \int_{t_1}^{t} U(t, s) f(s)\mathrm{d}s. \tag{1.2}$$

Here $U(t, s)$ is the evolution operator generated by the family of closed operators $\big\{A(t)\big\}_{t_1 \le t \le t_2}$ according to Kato [7, page 246] (see precisely the assumptions on $A(t)$ in Sect. 2).

The classic of Kato [6] from 1953 represents the beginning. After his introduction of the stability concept in his revision [7] from 1970 intense activities arose with respect to the homogeneous part of (1.1) while the inhomogeneous problem was remaining disregarded. So, this paper is devoted to this topic. For details on these activities around the homogeneous equation, we refer to the paper of Bárta [1].

The stability concept due to Kato means essentially the admission that the operators $A(t)$ generate C_0-semigroups instead of the merely C_0-semigroups of contractions at Kato [6].

As is to be observed, the integral term in (1.2) plays the leading role. In this context, we show in Theorem 3.1(I) for $f \in BV\big([t_1, t_2], X\big)$ first of all the relation:

$$v(t) := \int_{t_1}^{t} U(t, s) f(s)\mathrm{d}s \in D, \quad t \in [t_1, t_2]. \tag{1.3}$$

This relation means nearness to the a.e. differentiability of $v(t)$. For comparison see Kato [7, page 255] and Pazy [8, page 148].

Further we also obtain in Theorem 3.1 a similar estimate to the estimate (3.2) for $\|A(t)v(t)\|$ in $[t_1, t_2]$ from Jawad [5].

The chapter after deals with the investigation of the differentiability of $v(t)$ by showing of the Lipschitz continuity of $v(t)$ so that $u(t) := U(t, t_1)\varphi - v(t)$ fullfils (1.1) strongly and uniquely for $f \in BV\big([t_1, t_2], X\big)$ and strictly for additionally $f \in C^0\big([t_1, t_2], X\big)$. Regarding the strict solution, that means the renunciation of the usually requirement of the continuously differentiability of $f(t)$ (see Kato [6, Theorem 5] and

Kato [7, Theorem 7.2]). Further we show the uniqueness of $u(t)$ even for absolutely instead of Lipschitz continuous $u(t)$.

Regarding the classification of the solution, we note:

1. *Mild solution* The solution $u(t)$ satisfies the integral equation (1.2). For that it is sufficient that $f \in L^1\big((t_1, t_2), X\big)$.
2. *Strong solution* $u(t)$ fullfils the differential equation (1.1) a.e. in $[t_1, t_2]$ with $u'(t) \in L^1\big((t_1, t_2), X\big)$ and $u(t) \in D$, $t \in [t_1, t_2]$. Pazy [8, page 109].
3. *Classical solution* $u(t) \in C^0\big([t_1, t_2], X\big) \bigcap C^1\big((t_1, t_2], X\big), u(t) \in D, t \in [t_1, t_2]$. The differential equation (1.1) is satisfied on $(t_1, t_2]$, see Pazy [8, page 139].
4. *Strict solution* $u(t) \in C^0\big([t_1, t_2], D\big) \bigcap C^1\big([t_1, t_2], X\big)$ and $u(t)$ satisfies the differential equation on the whole $[t_1, t_2]$.

2 Preliminaries

In this chapter we state the conditions on the operator family $\big\{A(t)\big\}_{t_1 \le t \le t_2}$ as well as the resulting conclusions and tools:

(1) The closed operators $A(t)$ are infinitesimal generators of C_0 -semigroups on the reflexive Banach space X.

(2) The operator family $\big\{A(t)\big\}_{t_1 \le t \le t_2}$ is stable, i.e. there exist constants $M \geq 1$ and ω such that:

$$\left\| \prod_{j=1}^{k} \big(\lambda + A(\tau_j)\big)^{-1} \right\| \leq \frac{M}{(\lambda - \omega)^k} \tag{2.1}$$

for $\lambda > \omega$ and every finite sequence $t_1 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_k \leq t_2,\ k = 1, 2, \ldots$. The constants M, ω are called the stability constants.

(3) The domains of definition $D\big(A(t)\big)$ are independent of t, i.e. $D\big(A(t)\big) = D,\ t \in [t_1, t_2]$.

(4) For every $x \in D$, $A(t)x$ is continuously differentiable in $[t_1, t_2]$.

These conditions are the basis for Theorem 7.2 of Kato [7].They imply the following propositions:

(a) For the operator family $\big\{A(t)\big\}_{t_1 \le t \le t_2}$ there exists the unique evolution operator $U(t, s) \in B(X)$, $t_1 \leq s \leq t \leq t_2$, with the following properties according to Kato [7] (Theorem 6.1, page 252):

(b) For every $x \in X$, $U(t, s)x$ is jointly continuous in $t,\ s \in [t_1, t_2]$ with $U(s, s) = I$ and $\|U(t, s)\| \leq Me^{\omega(t-s)}$.

(c) $U(t, s) = U(t, r)U(r, s),\ \ s \leq r \leq t.$

(d) $U(t, s)D \subset D.$

(e) For $x \in D$ it is in $[t_1, t_2]$:

$$\frac{\partial}{\partial t} U(t, s)x = -A(t)U(t, s)x \tag{2.2}$$

$$\frac{\partial}{\partial s} U(t, s)x = U(t, s)A(s)x. \tag{2.3}$$

(f) For every $y \in D$, $A(t)U(t,s)y$ and $U(t,s)A(s)y$ are continuous in t, s.

As it is usual in the sources (see Kato [7, page 254, 255]) we may also assume for simplicity and without loss of generality the existence of the inverses $A^{-1}(t),\ t \in [t_1, t_2]$, on X.

By setting $S(t) := A(t)$ on page 253 from Kato [7], we have further:

(g) For every $x \in X$, the expression $A(t)U(t,s)A^{-1}(s)x$ is continuous in t, s (separately) and there is a constant k_1 such that:

$$\|A(t)U(t,s)A^{-1}(s)\| \leq k_1\,, \qquad t, s \in [t_1, t_2]. \tag{2.4}$$

(h) For every $x \in X,\ A'(s)A^{-1}(s)x$ is continuous in s with:

$$\|A'(s)A^{-1}(s)\| \leq k_2 \quad on \ \ [t_1, t_2]. \tag{2.5}$$

For the proof of (h) see the author [5], Hilfssatz 2.3.

Lemma 2.1 *Let B be a reflexive Banach space and A be a closed linear operator with non-empty resolvent set. Let $\{x_n\}$ be a sequence such that $x_n \in D(A),\ x_n \rightharpoonup x \quad (n \to \infty)$ and $\{\|Ax_n\|\}$ is bounded. Then $x \in D(A)$ and $Ax_n \rightharpoonup Ax \quad (n \to \infty)$.*

Proof We may assume without loss of generality that $\lambda = 0$ belongs to the resolvent set of A. Thus A^{-1} exists and is bounded so that $(A^{-1})^*$ also exists. Since $\{\|Ax_n\|\}$ is bounded, there exists a weakly convergent subsequence $\{Ax_{n_k}\}$. Set $Ax_{n_k} \rightharpoonup y \in B \ \ (k \to \infty)$. It is for $z \in B^*$:

$$\langle Ax_{n_k}, (A^{-1})^* z\rangle \to \langle y, (A^{-1})^* z\rangle = \langle A^{-1}y, z\rangle \quad (k \to \infty).$$

On the other hand we have for the same convergent sequence of complex numbers:

$$\langle Ax_{n_k}, (A^{-1})^* z\rangle = \langle A^{-1}Ax_{n_k}, z\rangle = \langle x_{n_k}, z\rangle \to \langle x, z\rangle \quad (k \to \infty).$$

That implies: $\langle A^{-1}y, z\rangle = \langle x, z\rangle$, precisely for every $z \in B^*$, i.e. $A^{-1}y = x$. Hence $x \in D(A),\ \ y = Ax$.

Let $\{Ax_{n_j}\}$ be any another weakly convergent subsequence with $Ax_{n_j} \rightharpoonup y'$ $(j \to \infty)$. The same procedure shows $y' = Ax = y$. Therefore $\{Ax_n\}$ is weakly convergent and $Ax_n \rightharpoonup Ax \ \ (n \to \infty)$. □

3 The regularity of v

Theorem 3.1 *Let the Banach space X be reflexive and the conditions (1)–(4) from the Preliminaries be satisfied. Then it holds:*

(I) For $f \in BV\big([t_1, t_2], X\big)$ and every $t \in [t_1, t_2]$ it is:

$$v(t) = \int_{t_1}^{t} U(t,s)f(s)ds \in D \tag{3.1}$$

with

$$\|A(t)v(t)\| \leq \|\{I - A(t)U(t,t_1)A^{-1}(t_1)\}f(t_1)\| \\ + k_1k_2\left(\sup_{s\in[t_1,t_2]}\|f(s)\|\right)(t-t_1) + (2k_1+1)\int_{t_1}^{t}\|df(s)\| \tag{3.2}$$

in $[t_1, t_2]$*, where* $k_1,\ k_2$ *are from* (2.4) *and* (2.5).

(II) $v(t) \in C^{0,1}\big([t_1, t_2], X\big)$.

(III) If additionally $f \in C^0\big([t_1, t_2], X\big)$ *then:*

$$A(t)v(t) \in C^0\big([t_1, t_2], X\big).$$

Proof (I) For a fixed $t \in [t_1, t_2]$, our proceeding consists of the approximation of the integral $v(t)$ by a sequence of elements $v_n \in D,\ n = 1, 2, \ldots$. Afterwards we show the boundedness of $A(t)v_n$ so that finally $v(t) \in D$ because X is reflexive. The estimate (3.2) is then a consequence of the weak convergence:

$$A(t)v_n \rightharpoonup A(t)v(t), \quad (n \to \infty).$$

So, let $t \in [t_1, t_2]$ be fixed. Since $f \in BV\big([t_1, t_2]X\big)$, there exists a sequence of step functions $\big\{f_n(s)\big\}_{n=0}^{\infty}$ which converges uniformly to $f(s)$ *on* $[t_1, t_2]$ (see Dieudonné [2, page 139]). For every $f_n(s),\ n = 0, 1, 2, \ldots$, there exists a partition of the interval $[t_1, t]$:

$$Z : t_1 = s_0 < s_1 < s_2 < \cdots < s_m < s_{m+1} = t$$

so that it holds with $f_n = c_i$ on (s_{i-1}, s_i) for $i = 1, \ldots, m$:

$$\begin{aligned} v_n(t) &:= \int_{t_1}^{t} U(t,s)f_n(s)ds \\ &= \int_{t_1=s_0}^{s_1} U(t,s)c_0ds + \int_{s_1}^{s_2} U(t,s)c_1ds + \int_{s_2}^{s_3} U(t,s)c_2ds \\ &\quad + \int_{s_3}^{s_4} U(t,s)c_3ds + \int_{s_4}^{s_5} U(t,s)c_4ds + \int_{s_5}^{s_6} U(t,s)c_5ds \\ &\quad + \cdots + \int_{s_{m-1}}^{s_m} U(t,s)c_{m-1}ds + \int_{s_m}^{s_{m+1}=t} U(t,s)c_mds, \end{aligned} \tag{3.3}$$

where $c_i = f(s_i^*),\ s_i < s_i^* < s_{i+1},\ i = 0, 1, \ldots, m$ and $c_i \in X$ are constants.

We show now that every one of these integrals belongs to D and that $A(t)$ applied on these integrals (i.e. on v_n) implies the boundedness of the sequence $A(t)v_n(t)$.

Since $f(s)$ is of bounded variation on $[t_1, t_2]$, $f(s)$ is bounded on $[t_1, t_2]$. The uniform boundedness of the operator $U(t, s) : X \to X$ implies the uniform convergence:

$$U(t,s)f_n(s) \longrightarrow U(t,s)f(s), \quad (n \to \infty)$$

on $[t_1, t]$ so that it holds via Lebesgue's Theorem:

$$\begin{aligned} v(t) &= \int_{t_1}^{t} U(t,s) f(s) ds = \int_{t_1}^{t} \lim_{n\to\infty} U(t,s) f_n(s) ds \\ &= \lim_{n\to\infty} \int_{t_1}^{t} U(t,s) f_n(s) ds = \lim_{n\to\infty} v_n(t). \end{aligned}$$

For $x \in D$, $A(s)x$ is continuously differentiable in s. Moreover the inverses $A^{-1}(s)$ exist and are bounded. Using the identity: $\big(A^{-1}(s)x\big)' = -A^{-1}(s)A'(s)A^{-1}(s)x$, $x \in X$, and applying (e) from Preliminaries yields via integration by parts on an arbitrary integral from (3.3):

$$\begin{aligned} I_i :&= \int_{s_i}^{s_{i+1}} U(t,s)c_i ds \\ &= \int_{s_i}^{s_{i+1}} \Big\{\frac{\partial}{\partial s} U(t,s)\Big\} A^{-1}(s) c_i ds \\ &= \{U(t,s_{i+1})A^{-1}(s_{i+1}) - U(t,s_i)A^{-1}(s_i)\} c_i \\ &\quad + \int_{s_i}^{s_{i+1}} U(t,s)A^{-1}(s)A'(s)A^{-1}(s) c_i ds. \end{aligned}$$

That means via (g) and (h) from Prliminaries:

$$\begin{aligned} v_n &= \sum_{i=0}^{m} I_i \\ &= \sum_{i=0}^{m} \big\{U(t,s_{i+1})A^{-1}(s_{i+1}) - U(t,s_i)A^{-1}(s_i)\big\} c_i \\ &\quad + \sum_{i=0}^{m} \int_{s_i}^{s_{i+1}} U(t,s)A^{-1}(s)A'(s)A^{-1}(s) c_i ds \\ &= A^{-1}(t)\Bigg(\sum_{i=0}^{m} \big\{A(t)U(t,s_{i+1})A^{-1}(s_{i+1}) - A(t)U(t,s_i)A^{-1}(s_i)\big\} c_i \\ &\quad + \sum_{i=0}^{m} \int_{s_i}^{s_{i+1}} \big\{A(t)U(t,s)A^{-1}(s)\big\}\big\{A'(s)A^{-1}(s)\big\} c_i ds \Bigg) \quad \in D. \end{aligned}$$

So, we have by reordering of the terms of the finite sum:

$$\begin{aligned} A(t)v_n(t) = c_m &- A(t)U(t,t_1)A^{-1}(t_1)c_0 \\ &- A(t)U(t,s_1)A^{-1}(s_1)\big(c_1 - c_0\big) - A(t)U(t,s_2)A^{-1}(s_2)\big(c_2 - c_1\big) \\ &- A(t)U(t,s_3)A^{-1}(s_3)\big(c_3 - c_2\big) - \dots \end{aligned}$$

$$
\begin{aligned}
&- A(t)U(t, s_m)A^{-1}(s_m)\big(c_m - c_{m-1}\big)\\
&+ \sum_{i=0}^{m} \int_{s_i}^{s_{i+1}} \big\{A(t)U(t, s)A^{-1}(s)\big\}\big\{A'(s)A^{-1}(s)\big\}c_i ds,
\end{aligned}
$$

i.e. with $\|c_i\| \leq \sup_{s\in[t_1,t_2]} \|f(s)\|$ and $s_i \leq s_i^* \leq s_{i+1}$:

$$
\begin{aligned}
\|A(t)v_n(t)\| &\leq \|f(s_m^*) - A(t)U(t, t_1)A^{-1}(t_1)f(s_0^*)\|\\
&\quad + k_1 \sum_{i=0}^{m-1} \|c_{i+1} - c_i\| + k_1k_2\big(\sup_{s\in[t_1,t_2]} \|f(s)\|\big)(t - t_1)\\
&= \|f(s_m^*) - f(t_1) + f(t_1) - A(t)U(t, t_1)A^{-1}(t_1)\{f(s_0^*) - f(t_1) + f(t_1)\}\|\\
&\quad + \ldots\\
&\leq \|f(s_m^*) - f(t_1)\| + k_1\|f(s_0^*) - f(t_1)\| + \|\{I - A(t)U(t, t_1)A^{-1}(t_1)\}f(t_1)\|\\
&\quad + k_1 \int_{t_1}^{t} \|df(s)\| + k_1k_2\big(sup_{s\in[t_1,t_2]}\|f(s)\|\big)(t - t_1)\\
&\leq \|\{I - A(t)U(t, t_1)A^{-1}(t_1)\}f(t_1)\| + (1 + 2k_1) \int_{t_1}^{t} \|df(s)\|\\
&\quad + k_1k_2\big(\sup_{s\in[t_1,t_2]} \|f(s)\|\big)(t - t_1).
\end{aligned}
$$

Lemma 2.1 yields finally that $v(t) \in D$ and $A(t)v_n(t) \rightharpoonup A(t)v(t)$ $(n \to \infty)$ and the reflexivity of X implies:

$$
\begin{aligned}
\|A(t)v(t)\| &\leq \liminf_{n\to\infty} \|A(t)v_n(t)\|\\
&\leq \|\{I - A(t)U(t, t_1)A^{-1}(t_1)\}f(t_1)\| + (1 + 2k_1) \int_{t_1}^{t} \|df(s)\|\\
&\quad + k_1k_2\big(sup_{s\in[t_1,t_2]}\|f(s)\|\big)(t - t_1),
\end{aligned}
$$

i.e. the desired estimate (3.2).

(II) We prove now the Lipschitz continuity of $v(t)$. By the properties of the evolution operator (see Sect. 2):

$$
\begin{aligned}
v(t + \varepsilon) - v(t) &= \int_{t_1}^{t+\varepsilon} U(t + \varepsilon, s)f(s)ds - \int_{t_1}^{t} U(t, s)f(s)ds\\
&= U(t + \varepsilon, t) \int_{t_1}^{t} U(t, s)f(s)ds + \int_{t}^{t+\varepsilon} U(t + \varepsilon, s)f(s)ds\\
&\quad - \int_{t_1}^{t} U(t, s)f(s)ds\\
&= \big\{U(t + \varepsilon, t) - I\big\}v(t) + \int_{t}^{t+\varepsilon} U(t + \varepsilon, s)f(s)ds.
\end{aligned}
$$

Here Lipschitzcontinuity of the integral term is obvious. By the just proved boundedness of $\|A(t)v(t)\|$ we obtain:

$$\begin{aligned}\Big\|\big\{U(t+\varepsilon,t)-I\big\}v(t)\Big\| &= \Big\|\int_0^1 \frac{\partial}{\partial\tau}U(t+\tau\varepsilon,t)v(t)d\tau\Big\| \\ &= \varepsilon\Big\|\int_0^1 A(t+\tau\varepsilon)U(t+\tau\varepsilon,t)v(t)d\tau\Big\| \\ &\le \varepsilon\int_0^1 \Big\|A(t+\tau\varepsilon)U(t+\tau\varepsilon,t)A^{-1}(t)\Big\|\|A(t)v(t)\|d\tau \\ &\le K_1\varepsilon.\end{aligned}$$

So, we have $v(t)\in C^{0,1}\big([t_1,t_2],X\big)$.

(III) Let $f\in C^0\big([t_1,t_2],X\big)$. We show in $[t_1,t_2]$:

$$A(t+\varepsilon)v(t+\varepsilon)-A(t)v(t)\longrightarrow 0,\qquad (\varepsilon\to 0).$$

For $\varepsilon>0$ (for $\varepsilon<0$ it is similar), it is:

$$\begin{aligned}&A(t+\varepsilon)\int_{t_1}^{t+\varepsilon}U(t+\varepsilon,s)f(s)ds-A(t)\int_{t_1}^{t}U(t,s)f(s)ds \\ &= A(t+\varepsilon)\int_{t_1}^{t}U(t+\varepsilon,s)f(s)ds-A(t)\int_{t_1}^{t}U(t,s)f(s)ds \\ &\quad +A(t+\varepsilon)\int_{t}^{t+\varepsilon}U(t+\varepsilon,s)f(s)ds \\ &= \big\{A(t+\varepsilon)U(t+\varepsilon,t)-A(t)\big\}\int_{t_1}^{t}U(t,s)f(s)ds \\ &\quad +A(t+\varepsilon)\int_{t}^{t+\varepsilon}U(t+\varepsilon,s)f(s)ds \\ &=: T_1(\varepsilon)+T_2(\varepsilon).\end{aligned}$$

Part (I) above together with (g) from Preliminaries yields:

$T_1(\varepsilon)=\big\{A(t+\varepsilon)U(t+\varepsilon,t)A^{-1}(t)-I\big\}A(t)\int_{t_1}^{t}U(t,s)f(s)ds\longrightarrow 0,\ (\varepsilon\to 0).$

Regarding the second term above, it is first according to estimate (3.2) from part (I):

$$\begin{aligned}\Big\|T_2(\varepsilon)\Big\| &= \Big\|A(t+\varepsilon)\int_{t}^{t+\varepsilon}U(t+\varepsilon,s)f(s)ds\Big\| \\ &\le \Big\|\big\{I-A(t+\varepsilon)U(t+\varepsilon,t)A^{-1}(t)\big\}f(t)\Big\| \\ &\quad + k_1k_2\big(sup_{s\in[t_1,t_2]}\|f(s)\|\big)\varepsilon+(2k_1+1)\int_{t}^{t+\varepsilon}\|df(s)\|.\end{aligned}$$

With ($\varepsilon \to 0$), the first term tends to 0 due to (g) from Preliminaries, the same holds for the last integral term because of the continuity of $f(t)$ (see Jawad [4], Lemma 2.2, holds in Banach space as well).

Summarized, the assertion of (III) follows. □

4 The differentiability of v

In this closing chapter, we prove the differentiability of the integral:

$$v(t) = \int_{t_1}^{t} U(t,s) f(s) ds.$$

We show then that $u(t) := U(t, t_1)\varphi - v(t)$ represents the unique strong solution of our differential equation (1.1).

Theorem 4.1 *(I) (Uniqueness) Let the function* $u : [t_1, t_2] \to X$ *be absolutely continuous and satisfy* (1.1) *a.e. with* $f \in L^1\big([t_1, t_2], X\big)$*. Then* $u(t)$ *is uniquely determind by* (1.2)*.*

(II) Let $f \in BV\big([t_1, t_2], X\big)$*, then:*

$$\frac{d^{\pm} v(t)}{dt} = f(t \pm 0) + A(t) \int_{t_1}^{t} U(t,s) f(s) ds \tag{4.1}$$

$$= f(t \pm 0) - A(t) v(t), \qquad t \in [t_1, t_2) \quad resp. \quad (t_1, t_2]. \tag{4.2}$$

According to (I), $u(t) := U(t, t_1)\varphi - v(t)$ *is the unique strong solution of* (1.1)*.*

(III) If additionally $f \in C^0\big([t_1, t_2], X\big)$*, then it is moreover:*

$$v(t) \in C^0\big([t_1, t_2], D\big) \cap C^1\big([t_1, t_2], X\big). \tag{4.3}$$

This means that $u(t)$ *is a strict solution.*

Proof (I) Multiplying both sides of (1.1) by $U(t, s)$ yields:

$$U(t,s)\big(u'(s) + A(s)u(s) + f(s)\big) = 0,$$

i.e.

$$\frac{\partial}{\partial s} U(t,s) u(s) + U(t,s) f(s) = 0.$$

Now integration from $s = t_1$ to t leads to the integral equation (1.2).

(II) Since the function f is of bounded variation, $f(t \pm 0)$ exist, $t \in [t_1, t_2)$ resp. $(t_1, t_2]$. It is for $\varepsilon > 0$:

$$
\begin{aligned}
v(t+\varepsilon)-v(t) &= \int_{t_1}^{t+\varepsilon} U(t+\varepsilon,s)f(s)ds - \int_{t_1}^{t} U(t,s)f(s)ds \\
&= \left\{U(t+\varepsilon,t)-I\right\}\int_{t_1}^{t} U(t,s)f(s)ds + \int_{t}^{t+\varepsilon} U(t+\varepsilon,s)f(s)ds,
\end{aligned}
$$

where according to Theorem 3.1(I) and (e) from Preliminaries:

$$
\frac{1}{\varepsilon}\left\{U(t+\varepsilon,s)-I\right\}\int_{t_1}^{t} U(t,s)f(s)ds \longrightarrow A(t)\int_{t_1}^{t} U(t,s)f(s)ds, \quad (\varepsilon\to 0).
$$

So, it remains only to show:

$$
\frac{1}{\varepsilon}\int_{t}^{t+\varepsilon} U(t+\varepsilon,s)f(s)ds \longrightarrow f(t+0), \qquad (\varepsilon\to 0+). \tag{4.4}
$$

It is first:

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_{t}^{t+\varepsilon} U(t+\varepsilon,s)f(s)ds &\overset{(s:=t+\tau)}{=} \frac{1}{\varepsilon}\int_{0}^{\varepsilon} U(t+\varepsilon,t+\tau)f(t+\tau)d\tau \\
&= \frac{1}{\varepsilon}\int_{0}^{\varepsilon} U(t+\varepsilon,t+\tau)\{f(t+\tau)-f(t+0)\}d\tau \\
&\quad + \frac{1}{\varepsilon}\int_{0}^{\varepsilon} U(t+\varepsilon,t+\tau)f(t+0)d\tau \\
&=: I_1(\varepsilon)+I_2(\varepsilon)
\end{aligned}
$$

with ((b) from Preliminaries):

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\|I_1(\varepsilon)\| &\le \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{0}^{\varepsilon} Me^{\omega(\varepsilon-\tau)}\|f(t+\tau)-f(t+0)\|d\tau \\
&\le M\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{0}^{\varepsilon}\|f(t+\tau)-f(t+0)\|d\tau = 0.
\end{aligned}
$$

So, we just have to show:

$$
\lim_{\varepsilon\to 0} I_2(\varepsilon) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{0}^{\varepsilon} U(t+\varepsilon,t+\tau)f(t+0)d\tau = f(t+0), \tag{4.5}
$$

which is equivalent to:

$$
\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{0}^{\varepsilon}\left\{U(t+\varepsilon,t+\tau)-I\right\}f(t+0)d\tau = 0.
$$

It is in this context:

$$
\begin{aligned}
\frac{1}{\varepsilon}\left\| \int_0^{\varepsilon} \left\{U(t+\varepsilon, t+\tau) - I\right\} f(t+0) d\tau \right\| \\
&\leq \frac{1}{\varepsilon} \int_0^{\varepsilon} \|\{U(t+\varepsilon, t+\tau) - I\} f(t+0)\| d\tau \\
&\leq \sup_{\tau \in [0,\varepsilon]} \|\{U(t+\varepsilon, t+\tau) - I\} f(t+0)\| \cdot \frac{1}{\varepsilon} \int_0^{\varepsilon} d\tau \\
&= \|\{U(t+\varepsilon, t+\varepsilon_0) - I\} f(t+0)\|, \quad \varepsilon_0 \in [0, \varepsilon].
\end{aligned}
$$

For every $\varepsilon > 0$ there exists at least one $\varepsilon_0 \in [0, \varepsilon]$ so that this relation holds. On the other hand, we have according to (b) from Preliminaries that for every $x \in X$, $U(t, s)x$ is jointly continuous in s and t. It follows:

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \|\{U(t+\varepsilon, t+\varepsilon_0) - I\} f(t+0)\| &= \lim_{\varepsilon_0 \leq \varepsilon \to 0} \|\{U(t+\varepsilon, t+\varepsilon_0) - I\} f(t+0)\| \\
&= 0,
\end{aligned}
$$

and (4.5) holds. Summarized, we have:

$$
\begin{aligned}
\frac{d^+ v(t)}{dt} &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{v(t+\varepsilon) - v(t)\} \\
&= f(t+0) - A(t) \int_{t_1}^{t} U(t, s) f(s) ds \\
&= f(t+0) - A(t) v(t), \qquad t \in [t_1, t_2).
\end{aligned}
$$

Arguing similarly for $\varepsilon < 0$ we obtain:

$$
\begin{aligned}
\frac{d^- v(t)}{dt} &= f(t-0) - A(t) \int_{t_1}^{t} U(t, s) f(s) ds \\
&= f(t-0) - A(t) v(t), \qquad t \in (t_1, t_2].
\end{aligned}
$$

Consequently, $u(t) := U(t, t_1)\varphi - v(t)$ is the unique strong solution of (1.1), for obviously $u'(t) \in L^1\big([t_1, t_2], X\big)$ (see also (3.2)).

(III) Follows immediately from part (II) and Theorem 3.1(III). □

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