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The radii of fully starlikeness and fully convexity of a harmonic operator

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Abstract Let $f = h + \overline{g}$ be a normalized harmonic mapping in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. In this paper, we study the radius of fully starlikeness and the radius of fully convexity of the following harmonic operator

$$\bigwedge_{0,1}[f] = \int_0^z \frac{h(\xi)}{\xi}\, d\xi + \overline{\int_0^z \frac{g(\xi)}{\xi}\, d\xi},$$

where the coefficients of the analytic functions h and g satisfy the conditions of the harmonic Bieberbach coefficient conjecture. We also study the radius of uniform starlikeness, and uniform convexity of harmonic mappings.

Keywords Analytic · Univalent · Starlike · Convex · Close-to-convex · Harmonic functions · Operator

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1 Introduction and definitions

A continuous complex-valued function $f = u+iv$ is harmonic in a domain $D \subset \mathbb{C}$, if u and v are real harmonic in D. If D is a simply connected domain, f can be decomposed as $f = h + \overline{g}$, where h and g are analytic functions in D. Here h and g are called the analytic and co-analytic parts of f respectively. Let $\mathcal{H}$ be the class of complex-valued harmonic mappings $f = h + \overline{g}$ defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, normalized by $h(0) = 0 = h'(0) - 1$, and having the following series representation

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Also let $\mathcal{H}_0 := \{f \in \mathcal{H} : f_{\overline{z}} = 0\}$. The Jacobian of a function f is defined as $J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2$. A result of Lewy [12], together with the inverse function theorem, shows that a harmonic function $f \in \mathcal{H}$ is locally univalent and sense preserving in $\mathbb{D}$ if, and only if, the Jacobian $J_f(z) > 0$ in $\mathbb{D}$. Let $\mathcal{S}_{\mathcal{H}}$ be the subclass of $\mathcal{H}$ consisting of univalent, i.e., one-to-one, and sense-preserving harmonic mappings on $\mathbb{D}$, and also $\mathcal{S}_{\mathcal{H}}^0 = \{f \in \mathcal{S}_{\mathcal{H}} : f_{\overline{z}}(0) = 0\}$. For any function $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^0$, its analytic and co-analytic parts can be represented as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1.1}$$

We note that if the co-analytic part of functions in the class $\mathcal{S}_{\mathcal{H}}$ is zero, then $\mathcal{S}_{\mathcal{H}}$ reduces to $\mathcal{S}$, the class of normalized analytic and univalent functions in $\mathbb{D}$. The geometric subclasses of $\mathcal{S}_{\mathcal{H}}$ consisting of the convex, starlike and close-to-convex functions in $\mathbb{D}$ are denoted by $\mathcal{K}_{\mathcal{H}}$, $\mathcal{S}_{\mathcal{H}}^*$ and $\mathcal{C}_{\mathcal{H}}$ respectively. Let $\mathcal{K}_{\mathcal{H}}^0$, $\mathcal{S}^{*0}_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{H}}^0$ denote the subclasses of $\mathcal{K}_{\mathcal{H}}$, $\mathcal{S}_{\mathcal{H}}^*$ and $\mathcal{C}_{\mathcal{H}}$ with the condition $f_{\overline{z}}(0) = 0$ respectively. The class $\mathcal{S}_{\mathcal{H}}$ and its subclasses on $\mathbb{D}$ have been extensively discussed by Clunie and Sheil-Small in [6] (see also [4,5,9]).

In 2012, Aleman and Constantin [1] studied nice a connection between harmonic mappings and ideal fluid flows. Indeed, they have developed ingenious technique to solve the incompressible two dimensional Euler equations in terms of univalent harmonic mappings. More precisely, the problem of finding all solutions which in Lagrangian variables describing the particle paths of the flow present a labelling by harmonic mappings is reduced to solve on explicit nonlinear differential system in $\mathbb{C}^n$ with $n = 3$ or $n = 4$ (see also [8]).

Let $\mathcal{K}$ and $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ consisting of univalent and analytic functions $\phi : \mathbb{D} \to \mathbb{C}$ with convex and starlike ranges respectively, and satisfying $\phi(0) = 0 = \phi'(0) - 1$. By a classical theorem of Alexander [2], $\mathcal{K}$ and $\mathcal{S}^*$ are related in that $\phi \in \mathcal{S}^*$ if, and only if, $\int_0^z \phi(\xi)/\xi \, d\xi \in \mathcal{K}$. In 1990, Sheil-Small [15] extended this theorem to harmonic mappings and proved the following.

Theorem A [15] *If $f = h + \overline{g} : \mathbb{D} \to \mathbb{C}$ fixes zero, is univalent, and has a starlike range, and H and G are the analytic functions in $\mathbb{D}$ defined by*

$$zH'(z) = h(z), \quad zG'(z) = -g(z), \quad H(0) = G(0) = 0,$$

then $F = H + \overline{G}$ is univalent, and has a convex range.

In 1969, Bernardi [3] generalized Alexander's theorem by introducing the function $\phi_\lambda : \mathbb{D} \to \mathbb{C}$ defined by

$$\phi_\lambda(z) = \frac{\lambda + 1}{z^\lambda} \int_0^z \xi^{\lambda - 1} \phi(\xi)\, d\xi,$$

where ϕ is analytic in $\mathbb{D}$, with $\phi(0) = 0 = \phi'(0) - 1$.

In [6], it was conjectured that if $f \in \mathcal{S}^0_{\mathcal{H}}$, with $h(z)$ and $g(z)$ given by (1.1), then

$$|a_n| \le \frac{1}{6}(2n + 1)(n + 1) \quad \text{and} \quad |b_n| \le \frac{1}{6}(2n - 1)(n - 1) \tag{1.2}$$

for all $n \ge 2$. This coefficient conjecture remains an open problem for the full class $\mathcal{S}^0_{\mathcal{H}}$. However, it has been verified for some subclasses of $\mathcal{S}^0_{\mathcal{H}}$, such as typically real functions [6], starlike functions [15] and close-to-convex functions [16]. The extremal function in these cases is the harmonic Koebe function K, given by

$$\begin{aligned} K(z) &= \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3} + \overline{\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3}} \\ &= z + \sum_{n=2}^{\infty} \frac{1}{6}(2n + 1)(n + 1)z^n + \sum_{n=2}^{\infty} \frac{1}{6}(2n - 1)(n - 1)\overline{z^n}. \end{aligned} \tag{1.3}$$

If $f \in \mathcal{K}^0_{\mathcal{H}}$, then Clunie and Sheil-Small [6] proved that

$$|a_n| \le \frac{1}{2}(n + 1) \quad \text{and} \quad |b_n| \le \frac{1}{2}(n - 1) \tag{1.4}$$

for all $n \ge 2$, with equality for all harmonic left-plane mappings

$$\begin{aligned} L(z) &= \frac{z - z^2/2}{(1 - z)^2} + \overline{\frac{-z^2/2}{(1 - z)^2}} \\ &= z + \sum_{n=2}^{\infty} \frac{1}{2}(n + 1)z^n + \sum_{n=2}^{\infty} \frac{1}{2}(n - 1)\overline{z^n}. \end{aligned} \tag{1.5}$$

Convexity and starlikeness are hereditary properties for conformal mappings. That is, if an analytic function maps $\mathbb{D}$ on to a convex, or a starlike domain, then it also maps each concentric subdisk onto a convex, or starlike domain respectively. However such hereditary properties cannot be generalized to harmonic mappings. The failure of hereditary properties such as starlike and convex harmonic mappings leads to the notion of fully starlike and fully convex functions, discussed in [7], which we define next.

Definition 1.1 A harmonic map f on $\mathbb{D}$ is said to be fully convex of order α, $0 \leq \alpha < 1$, if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a convex curve satisfying

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}f(re^{i\theta})\right)\right) > \alpha, \qquad 0 \leq \theta < 2\pi, \quad 0 < r < 1. \tag{1.6}$$

If $\alpha = 0$, then f is said to be fully convex.

Definition 1.2 A harmonic map f on $\mathbb{D}$ with $f(0) = 0$ is said to be fully starlike of order α, $0 \leq \alpha < 1$, if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin satisfying

$$\frac{\partial}{\partial\theta}\left(arg\left(f(re^{i\theta})\right)\right) > \alpha, \qquad 0 \leq \theta < 2\pi, \quad 0 < r < 1. \tag{1.7}$$

If $\alpha = 0$, then f is said to be fully starlike.

Let $\mathcal{FK}_{\mathcal{H}}(\alpha)$ and $\mathcal{FS}^*_{\mathcal{H}}(\alpha)$ denote the subclass of $\mathcal{K}_{\mathcal{H}}$ consisting of fully convex functions of order α, and the subclass of $\mathcal{S}^*_{\mathcal{H}}(\alpha)$ consisting of fully starlike functions of order α respectively.

The following two lemmas give sufficient conditions for functions f in $\mathcal{H}$ to belong to $\mathcal{FK}_{\mathcal{H}}(\alpha)$ and $\mathcal{FS}^*_{\mathcal{H}}(\alpha)$ respectively.

Lemma 1.1 [10] *Let $f = h + \overline{g}$, where h and g are given by* (1.1). *Further, let*

$$\sum_{n=2}^{\infty}\frac{n(n-\alpha)}{(1-\alpha)}|a_n| + \sum_{n=2}^{\infty}\frac{n(n+\alpha)}{(1-\alpha)}|b_n| \leq 1$$

and $0 \leq \alpha < 1$. *Then f is harmonic univalent in $\mathbb{D}$, and $f \in \mathcal{FK}_{\mathcal{H}}(\alpha)$.*

Lemma 1.2 [11] *Let $f = h + \overline{g}$, where h and g are given by* (1.1). *Further, let*

$$\sum_{n=2}^{\infty}\frac{(n-\alpha)}{(1-\alpha)}|a_n| + \sum_{n=2}^{\infty}\frac{(n+\alpha)}{(1-\alpha)}|b_n| \leq 1$$

and $0 \leq \alpha < 1$. *Then f is harmonic univalent in $\mathbb{D}$, and $f \in \mathcal{FS}^*_{\mathcal{H}}(\alpha)$.*

According to the Radó–Kaneser–Choquet theorem (see [6]), fully convex harmonic functions of order α are necessarily univalent in $\mathbb{D}$. However, fully harmonic starlike functions need not be univalent in $\mathbb{D}$.

For $\operatorname{Re}\lambda \geq 0, \alpha \in \mathbb{D}$ and $f \in \mathcal{S}_{\mathcal{H}}$, Muir [13] generalized the Bernardi integral operator to harmonic functions, by defining $\bigwedge_{\lambda,\alpha}[f] : \mathbb{D} \to \mathbb{C}$ as follows.

$$\bigwedge_{\lambda,\alpha}[f](z) = \frac{\lambda+1}{z^{\lambda}} \int_0^z \xi^{\lambda-1} h(\xi)\, d\xi + \alpha \overline{\frac{\lambda+1}{z^{\lambda}} \int_0^z \xi^{\lambda-1} g(\xi)\, d\xi}. \tag{1.8}$$

Clearly $\bigwedge_{0,-1}$ is the harmonic analogue of Alexander's operator, given in Theorem A. In view of Theorem A, the classes $\mathcal{K}_{\mathcal{H}}$ and $\mathcal{S}^*_{\mathcal{H}}$ are preserved under the operator $\bigwedge_{0,-1}$. However $\mathcal{K}_{\mathcal{H}}$ and $\mathcal{S}^*_{\mathcal{H}}$ are not necessarily preserved under the operator $\bigwedge_{0,1}$. Indeed $\bigwedge_{0,1}[K] \notin {\mathcal{S}^*_{\mathcal{H}}}^0$ and $\bigwedge_{0,1}[L] \notin \mathcal{K}^0_{\mathcal{H}}$, where K is the harmonic Koebe function defined by (1.3), and L is the harmonic right half-plane mapping defined by (1.5).

Definition 1.3 A locally univalent function $f = h + \overline{g}$ is said to be uniformly starlike in the unit disk $\mathbb{D}$, if f is fully starlike in $\mathbb{D}$, and maps every circular arc γ_ζ contained in $\mathbb{D}$ with center ζ also in $\mathbb{D}$, to the arc $f(\gamma_\zeta)$ which is starlike with respect to $f(\zeta)$.

Definition 1.4 A locally univalent function $f = h + \overline{g}$ is said to be uniformly convex in the unit disk $\mathbb{D}$, if f is fully convex in $\mathbb{D}$, and maps every circular arc γ_ζ contained in $\mathbb{D}$ with center ζ also in $\mathbb{D}$, onto a convex arc $f(\gamma_\zeta)$.

Let $\mathcal{US}^*_{\mathcal{H}}$ (respectively ${\mathcal{US}^*_{\mathcal{H}}}^0$) denote the classes of functions in $f \in \mathcal{S}_{\mathcal{H}}$ (respectively $f \in \mathcal{S}^0_{\mathcal{H}}$) which are uniformly starlike in $\mathbb{D}$. Clearly $\mathcal{US}^*_{\mathcal{H}} \subset \mathcal{FS}^*_{\mathcal{H}}$. Let $\mathcal{UK}_{\mathcal{H}}$ (respectively $\mathcal{UK}^0_{\mathcal{H}}$) denote the classes of all functions $f \in \mathcal{S}_{\mathcal{H}}$ (respectively $f \in \mathcal{S}^0_{\mathcal{H}}$) which are uniformly convex in $\mathbb{D}$.

The following two lemmas give sufficient conditions for functions in $\mathcal{H}$ to belong to $\mathcal{US}^*_{\mathcal{H}}$ and $\mathcal{UK}_{\mathcal{H}}$, respectively.

Lemma 1.3 [14] *Let $f = h + \overline{g}$, where h and g are given by* (1.1)*, and satisfy the condition*

$$\sum_{n=2}^{\infty} n\left(|a_n| + |b_n|\right) \leq \frac{1}{2}. \tag{1.9}$$

Then $f \in {\mathcal{US}^*_{\mathcal{H}}}^0$.

Lemma 1.4 [14] *Let $f = h + \overline{g}$, where h and g are given by* (1.1)*, and satisfy the condition*

$$\sum_{n=2}^{\infty} n(2n-1)\left(|a_n| + |b_n|\right) \leq 1. \tag{1.10}$$

Then $f \in \mathcal{UK}^0_{\mathcal{H}}$.

The remainder of this paper is organized as follows. In Sect. 2, we study the radii of fully starlikeness and fully convexity of order α of the harmonic operator $\bigwedge_{0,1}[f]$. As a consequence (by taking $\alpha = 0$), we obtain the radii of fully starlikeness and fully convexity of the harmonic operator $\bigwedge_{0,1}[f]$. In Sect. 3, we obtain the radii of uniformly starlikeness and uniformly convexity of the harmonic mappings.

2 Radii of fully starlikeness and fully convexity of harmonic operator of order α

In this section, we obtain the radii of fully starlikeness and fully convexity of order α of the harmonic operator $\bigwedge_{0,1}[f]$. The following elementary identities are used to prove our results.

Lemma 2.1 *We have*

(i) $\sum_{n=2}^{\infty} nr^{n-1} = \dfrac{r(2-r)}{(1-r)^2}$,

(ii) $\sum_{n=2}^{\infty} n^2 r^{n-1} = \dfrac{r(4-3r+r^2)}{(1-r)^3}$,

(iii) $\sum_{n=2}^{\infty} n^3 r^{n-1} = \dfrac{r(8-5r+4r^2-r^3)}{(1-r)^4}$.

Theorem 2.1 *Let $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^0$ be given by (1.1), and the coefficients satisfy the conditions*

$$|a_n| \leq \frac{1}{2}(n+1) \quad \text{and} \quad |b_n| \leq \frac{1}{2}(n-1) \tag{2.1}$$

for $n \geq 2$. Then $F = \bigwedge_{0,1}[f]$ is fully convex of order alpha in $|z| < r_c(\alpha)$, where $r_c(\alpha)$ is the unique real root of $p_c(r,\alpha) = 0$ in $(0,1)$, and where

$$p_c(r,\alpha) = (1-\alpha) - (7-4\alpha)r + (6-5\alpha)r^2 - 2(1-\alpha)r^3. \tag{2.2}$$

Proof Let $f = h + \overline{g}$ be in $\mathcal{S}_{\mathcal{H}}^0$ where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n$. Then

$$\begin{aligned} F(z) = \bigwedge_{0,1}[f](z) &= \int_0^z \frac{h(\xi)}{\xi}\, d\xi + \overline{\int_0^z \frac{g(\xi)}{\xi}\, d\xi} \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n + \overline{\sum_{n=2}^{\infty} \frac{b_n}{n} z^n}. \end{aligned} \tag{2.3}$$

For $r \in (0,1)$, it is sufficient to show that $F_r \in \mathcal{FK}_{\mathcal{H}}(\alpha)$ where

$$F_r(z) = \frac{F(rz)}{r} = z + \sum_{n=2}^{\infty} \frac{a_n}{n} r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} \frac{b_n}{n} r^{n-1} z^n}. \tag{2.4}$$

Consider this time, the sum

$$\begin{aligned} S(\alpha) &= \sum_{n=2}^{\infty} \left(\frac{n(n-\alpha)}{(1-\alpha)} \frac{|a_n|}{n} + \frac{n(n+\alpha)}{(1-\alpha)} \frac{|b_n|}{n} \right) r^{n-1} \\ &= \sum_{n=2}^{\infty} \left(\frac{(n-\alpha)}{(1-\alpha)} |a_n| + \frac{(n+\alpha)}{(1-\alpha)} |b_n| \right) r^{n-1}. \end{aligned} \tag{2.5}$$

In view of (2.1) and Lemma 2.1, (2.5) gives

$$S(\alpha) \leq \sum_{n=2}^{\infty} \left(\frac{(n-\alpha)(n+1)}{2(1-\alpha)} + \frac{(n+\alpha)(n-1)}{2(1-\alpha)} \right) r^{n-1}$$
$$= \frac{r(4-3r+r^2) - \alpha r(1-r)^2}{(1-\alpha)(1-r)^3} =: X_1.$$

Thus Lemma 1.1 implies that in order to show that $F_r \in \mathcal{FK}_{\mathcal{H}}(\alpha)$, it is sufficient to show that $X_1 \leq 1$. It is easy to show that $X_1 \leq 1$, when $p_c(r, \alpha) \geq 0$. Since $p_c(r, 0) = 1 - \alpha \geq 0$ and $p_c(r, 1) = -2 < 0$, $p_c(r, \alpha)$ has at least one zero in (0, 1). Now we shall show that (2.2) has exactly one zero in the interval (0, 1) for every $\alpha \in [0, 1)$. A straight forward computation shows that

$$p_c'(r, \alpha) = -6(1-\alpha)r^2 + 2(6-5\alpha)r - (7-4\alpha).$$

To prove $p_c(r, \alpha)$ has exactly one zero in (0, 1), it is sufficient to prove that $p_c(r, \alpha)$ has a monotonic property on (0, 1).

A direct computation shows that

$$p_c'(0, \alpha) = -(7-4\alpha) < 0,$$
$$p_c'(1, \alpha) = -1 < 0,$$
$$p_c''(r, \alpha) = -12(1-\alpha)r + 2(6-5\alpha).$$

Again, by straightforward computation we have

$$p_c''(0, \alpha) = 2(6-5\alpha) > 0,$$
$$p_c''(1, \alpha) = 2\alpha \geq 0,$$
$$p_c'''(r, \alpha) = -12(1-\alpha) < 0 \quad \text{for} \quad \alpha \in [0, 1).$$

Hence $p_c''(r, \alpha)$ is strictly monotonically decreasing on (0, 1) for every $\alpha \in [0, 1)$. By combining the monotonic property of $p_c''(r, \alpha)$ with $p_c''(1, \alpha) \geq 0$, we conclude that $p_c''(r, \alpha)$ is strictly positive on (0, 1) for every $\alpha \in [0, 1)$. This shows that $p_c'(r, \alpha)$ is strictly monotonically increasing on (0, 1). Again, by combining the monotonic increasing property of $p_c'(r, \alpha)$ with $p_c'(1, \alpha) < 0$, we conclude that $p_c'(r, \alpha) < 0$ on (0, 1) for every $\alpha \in [0, 1)$, and so $p_c(r, \alpha)$ is strictly monotonically decreasing on (0, 1) for every $\alpha \in [0, 1)$. Thus $p_c(r, \alpha) = 0$ has exactly one real root (say $r_c(\alpha)$) in (0, 1). Therefore $p_c(r, \alpha) \geq 0$ for $0 < r < r_c(\alpha)$. Hence F_r belongs to $\mathcal{FK}_{\mathcal{H}}(\alpha)$ for $|z| < r_c(\alpha)$. □

Theorem 2.2 *Let* $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^0$ *be given by* (1.1), *and the coefficients satisfy the conditions*

$$|a_n| \leq \frac{1}{6}(2n+1)(n+1) \quad \textit{and} \quad |b_n| \leq \frac{1}{6}(2n-1)(n-1) \tag{2.6}$$

for $n \geq 2$. Then $F = \bigwedge_{0,1}[f]$ is fully convex of order α in $|z| < r_c(\alpha)$, where $r_c(\alpha)$ is the unique real root of $q_c(r,\alpha) = 0$ in $(0, 1)$, and where

$$q_c(r,\alpha) = 3(1-\alpha)-(30-18\alpha)r+33(1-\alpha)r^2-24(1-\alpha)r^3+6(1-\alpha)r^4. \quad (2.7)$$

Proof Let $f = h + \overline{g} \in \mathcal{S}^0_{\mathcal{H}}$ where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n$. Then

$$\begin{aligned} F(z) = \bigwedge_{0,1}[f](z) &= \int_0^z \frac{h(\xi)}{\xi}\, d\xi + \overline{\int_0^z \frac{g(\xi)}{\xi}\, d\xi} \quad (2.8)\\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n + \overline{\sum_{n=2}^{\infty} \frac{b_n}{n} z^n}. \end{aligned}$$

For $r \in (0, 1)$, it is sufficient to show that $F_r \in \mathcal{FK}_{\mathcal{H}}(\alpha)$, where

$$F_r(z) = \frac{F(rz)}{r} = z + \sum_{n=2}^{\infty} \frac{a_n}{n} r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} \frac{b_n}{n} r^{n-1} z^n}. \quad (2.9)$$

Consider again the sum

$$\begin{aligned} S(\alpha) &= \sum_{n=2}^{\infty} \left(\frac{n(n-\alpha)}{(1-\alpha)} \frac{|a_n|}{n} + \frac{n(n+\alpha)}{(1-\alpha)} \frac{|b_n|}{n} \right) r^{n-1} \quad (2.10)\\ &= \sum_{n=2}^{\infty} \left(\frac{(n-\alpha)}{(1-\alpha)} |a_n| + \frac{(n+\alpha)}{(1-\alpha)} |b_n| \right) r^{n-1}. \end{aligned}$$

Using the coefficient conditions (2.6) and applying Lemma 2.1 in (2.10), we obtain

$$\begin{aligned} S(\alpha) &\leq \sum_{n=2}^{\infty} \left(\frac{(n-\alpha)(2n+1)(n+1)}{6(1-\alpha)} + \frac{(n+\alpha)(2n-1)(n-1)}{6(1-\alpha)} \right) r^{n-1}\\ &= \frac{(18-6\alpha)r - 15(1-\alpha)r^2 + 12(1-\alpha)r^3 - 3(1-\alpha)r^4}{3(1-\alpha)(1-r)^4} =: X_3. \end{aligned}$$

As before, in view of Lemma 1.1, in order to show that $F_r \in \mathcal{FK}_{\mathcal{H}}(\alpha)$, it is sufficient to show that $X_3 \leq 1$. A straightforward calculation shows that $X_3 \leq 1$ when $q_c(r,\alpha) \geq 0$, where $q_c(r,\alpha)$ is defined by (2.7). Since $q_c(0,\alpha) = 3(1-\alpha) \geq 0$ and $q_c(1,\alpha) = -12 < 0$, the polynomial $q_c(r,\alpha)$ has at least one zero in $(0, 1)$. We now need to show that $q_c(r,\alpha) = 0$ defined by (2.7), has exactly one zero in $(0, 1)$ for every $\alpha \in [0, 1)$. Thus in order to show that the polynomial $q_c(r,\alpha)$ has exactly one zero in $(0, 1)$, it sufficies to show that $q_c(r,\alpha)$ has a monotonic property on $(0, 1)$.

Using the fact that $0 < r^3 < r < 1$, a simple computation gives

$$\begin{aligned} q_c'(r,\alpha) &= -(30-18\alpha)+66(1-\alpha)r-72(1-\alpha)r^2+24(1-\alpha)r^3 \\ &< -(30-18\alpha)+66(1-\alpha)r-72(1-\alpha)r^2+24(1-\alpha)r \\ &= -(30-18\alpha)+90(1-\alpha)r-72(1-\alpha)r^2 =: l(r,\alpha). \end{aligned}$$

To prove that $q_c(r,\alpha)$ is monotonically decreasing, we need to show that $l(r,\alpha)$ is negative in $r \in (0,1)$ for every $\alpha \in [0,1)$. Further,

$$\begin{aligned} l'(r,\alpha) &= 90(1-\alpha)-144(1-\alpha)r \\ &= 18(1-\alpha)(5-8r) \\ &= 0 \end{aligned}$$

if $r = 5/8$. As, $l''(r,\alpha) = -144(1-\alpha) < 0$ for $\alpha \in [0,1)$, it is evident that $l(r,\alpha)$ attains its maximum value at $r = 5/8$. Clearly the value of $l(r,\alpha)$ at $r = 5/8$ is

$$l\left(\frac{5}{8},\alpha\right) = \frac{-(15+81\alpha)}{8} < 0.$$

This shows that $l(r,\alpha) < 0$ on $r \in (0,1)$ for every $\alpha \in [0,1)$. Hence $q_c'(r,\alpha) < 0$ on $r \in (0,1)$ for every $\alpha \in [0,1)$. Therefore $q_c(r,\alpha)$ is strictly monotonically decreasing on $(0,1)$ for every $\alpha \in [0,1)$, and so $q_c(r,\alpha)$ has a unique real root (say $r_c(\alpha)$) in $(0,1)$. A simple computation shows that $q_c(r,\alpha)$ is positive for $0 < r < r_c(\alpha)$. Hence F_r belongs to $\mathcal{FK}_{\mathcal{H}}(\alpha)$ for $|z| < r_c(\alpha)$. □

Theorem 2.3 *Under the hypothesis of Theorem 2.1, $F = \bigwedge_{0,1}[f]$ is fully starlike of order α in $|z| < r_s(\alpha)$, where $r_s(\alpha)$ is the unique real root of $p_s(r,\alpha) = 0$ in $(0,1)$, where*

$$p_s(r,\alpha) = 2(1-\alpha)r^3 - 4(1-\alpha)r^2 + (1-2\alpha)r - \alpha(1-r)^2\ln(1-r). \tag{2.11}$$

Proof Let $f = h + \overline{g} \in \mathcal{S}^0_{\mathcal{H}}$, where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n$. Then

$$\begin{aligned} F(z) = \bigwedge_{0,1}[f](z) &= \int_0^z \frac{h(\xi)}{\xi}\,d\xi + \overline{\int_0^z \frac{g(\xi)}{\xi}\,d\xi} \\ &= z + \sum_{n=2}^{\infty}\frac{a_n}{n}z^n + \overline{\sum_{n=2}^{\infty}\frac{b_n}{n}z^n}. \end{aligned} \tag{2.12}$$

For $r \in (0,1)$, it is sufficient to show that $F_r \in \mathcal{FS}^*_{\mathcal{H}}(\alpha)$, where

$$F_r(z) = \frac{F(rz)}{r} = z + \sum_{n=2}^{\infty}\frac{a_n}{n}r^{n-1}z^n + \overline{\sum_{n=2}^{\infty}\frac{b_n}{n}r^{n-1}z^n}. \tag{2.13}$$

As above, consider the sum

$$S(\alpha)=\sum_{n=2}^{\infty}\left(\frac{(n-\alpha)}{(1-\alpha)}\frac{|a_n|}{n}+\frac{(n+\alpha)}{(1-\alpha)}\frac{|b_n|}{n}\right)r^{n-1}. \tag{2.14}$$

In view of Lemma 2.1, and the coefficient inequalities in (2.1), we can see that (2.14) gives

$$\begin{aligned}S(\alpha)&\le\sum_{n=2}^{\infty}\left(\frac{(n-\alpha)(n+1)}{2n(1-\alpha)}+\frac{(n+\alpha)(n-1)}{2n(1-\alpha)}\right)r^{n-1}\\&=\frac{1}{(1-\alpha)}\frac{r^2(2-r)+\alpha r(1-r)^2+\alpha(1-r)^2\ln(1-r)}{r(1-r)^2}=:X_2.\end{aligned}$$

An application of Lemma 1.2 shows that $F_r\in\mathcal{FS}^*_{\mathcal{H}}(\alpha)$ if $X_2\le 1$. A simple computation show that $X_2\le 1$ if, and only if, $p_s(r,\alpha)\ge 0$, where $p_s(r,\alpha)$ is defined by (2.11). It is not difficult to shows that $p_s(r,\alpha)\ge 0$ for $|z|<r_s(\alpha)$, where $r_s(\alpha)$ is the unique real root of $p_s(r,\alpha)=0$. Thus $F_r\in\mathcal{FS}^*_{\mathcal{H}}(\alpha)$ for $|z|<r_s(\alpha)$. □

By taking $\alpha=0$ in Theorems 2.1, 2.2 and 2.3, we obtain the followings results on radii of fully starlikeness and fully convexity of harmonic operator $\bigwedge_{0,1}[f]$.

Corollary 2.1 *Under the hypothesis of Theorem 2.1, $F=\bigwedge_{0,1}[f]$ is fully convex in $|z|<r_c$, where $r_c\approx 0.16487$ is the unique real root of $p_c(r)=0$ in $(0,1)$, where*

$$p_c(r)=1-7r+6r^2-2r^3.$$

Corollary 2.2 *Under the hypothesis of Theorem 2.2, $F=\bigwedge_{0,1}[f]$ is fully convex in $|z|<r_c$, where $r_c\approx 0.112903$ is the unique real root of $q_c(r)=0$ in the interval $(0,1)$, where*

$$q_c(r)=1-10r+11r^2-8r^3+2r^4.$$

Corollary 2.3 *Under the hypothesis of Theorem 2.1, $F=\bigwedge_{0,1}[f]$ is fully starlike in $|z|<r_s$, where $r_s\approx 0.2928$ is the unique real root of $p_s(r)$ in $(0,1)$, where*

$$p_s(r)=1-4r+2r^2.$$

Since the proofs of Corollaries 2.1, 2.2 and 2.3 are similar to that of Theorems 2.1, 2.2 and 2.3 respectively, we omit the details.

3 Radii of uniformly starlikeness and uniformly convexity of harmonic mappings

In this section we study the radii of uniformly starlikeness and uniformly convexity for functions in $\mathcal{S}^0_{\mathcal{H}}$.

Theorem 3.1 *Let $f = h + \overline{g} \in \mathcal{S}^0_{\mathcal{H}}$ be given by (1.1), and the coefficients satisfy the conditions*

$$|a_n| \leq \frac{1}{2}(n+1) \quad and \quad |b_n| \leq \frac{1}{2}(n-1) \tag{3.1}$$

for $n \geq 2$. Then f is uniformly starlike on the disk $|z| < r_{us}$, where $r_{us} \approx 0.0986023$ is the unique positive root of $p_{us}(r) = 0$ in $(0, 1)$, where

$$p_{us}(r) = 1 - 11r + 9r^2 - 3r^3. \tag{3.2}$$

Proof Let $f = h+\overline{g} \in \mathcal{S}^0_{\mathcal{H}}$ be given by (1.1), and the coefficients satisfy the conditions (3.1) for $n \geq 2$. For $0 < r < 1$, let

$$f_r(z) = r^{-1} f(rz) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} b_n r^{n-1} z^n}, \quad z \in \mathbb{D}. \tag{3.3}$$

Consider the sum

$$S = \sum_{n=2}^{\infty} n|a_n|r^{n-1} + \sum_{n=2}^{\infty} n|b_n|r^{n-1}. \tag{3.4}$$

In view of lemma 3.1 and the conditions in (3.1), we see that (3.4) satisfies

$$S \leq \sum_{n=2}^{\infty} n^2 r^{n-1} = \frac{r(4-3r+r^2)}{(1-r)^3} =: S_4. \tag{3.5}$$

From Lemma 1.3, we note that $f_r(z)$ is uniformly starlike on $\mathbb{D}$ if $S_4 \leq 1/2$, which implies that $1 - 11r + 9r^2 - 3r^3 \geq 0$.

Now let $p_{us} := 1 - 11r + 9r^2 - 3r^3$, so that $S_4 \leq 1/2$ whenever $p_{us}(r) \geq 0$. It is easy to see that $p_{us}(0) = 1 > 0$ and $p_{us}(1) = -4 < 0$, and hence p_{us} has a real root in $(0, 1)$.

To show that $p_{us}(r)$ has exactly one zero in $(0, 1)$, it is sufficient to prove that $p_{us}(r)$ is monotonic on $(0, 1)$. A simple computation shows that

$$\begin{aligned} &p'_{us}(0) = -11 < 0 \\ &p'_{us}(1) = -2 < 0 \\ &p''_{us}(r) = 18(1-r) > 0 \quad \text{for} \quad r \in (0, 1). \end{aligned}$$

Hence $p'_{us}(r)$ is a strictly monotonic increasing function in $(0, 1)$. Combining the monotonicity property of $p'_{us}(r)$ with $p'_{us}(1) < 0$, we conclude that $p'_{us}(r) < 0$ on $(0, 1)$. This shows that $p_{us}(r)$ is strictly monotonically decreasing in $(0, 1)$. Thus $p_{us}(r)$ has exactly one zero (say $r_{us} \approx 0.0986023$) in $(0, 1)$. Since $p_{us}(r)$ is strictly monotonically decreasing in $(0, 1)$ with $p_{us}(0) > 0$ and $p_{us}(r_{us}) = 0$, it is easy to see that $p_{us}(r) \geq 0$ for $0 < r \leq r_{us}$. Hence f is uniformly starlike in $|z| < r_{us}$. ◻

Theorem 3.2 *Under the hypothesis of Theorem 3.1, f is uniformly convex on the disk $|z| < r_{uc}$, where $r_{uc} \approx 0.064723$ is the unique positive real root of $p_{uc}(r) = 0$ in $(0, 1)$, and where*

$$p_{uc}(r) = 2r^4 - 8r^3 + 9r^2 - 16r + 1. \tag{3.6}$$

Proof Let $f = h+\overline{g} \in \mathcal{S}^0_{\mathcal{H}}$ be given by (1.1), and the coefficients satisfy the conditions (3.1) for $n \geq 2$. For $0 < r < 1$, let

$$f_r(z) = r^{-1} f(rz) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} b_n r^{n-1} z^n}, \quad z \in \mathbb{D}. \tag{3.7}$$

Consider again the sum

$$S = \sum_{n=2}^{\infty} n(2n-1)\left(|a_n| + |b_n|\right) r^{n-1}. \tag{3.8}$$

Using (3.1) and Lemma 2.1 in (3.8), we obtain

$$S \leq \sum_{n=2}^{\infty} (2n-1) n^2 r^{n-1} = \frac{r(12 - 3r + 4r^2 - r^3)}{(1-r)^4} =: S_5. \tag{3.9}$$

In order to show that f_r is uniformly convex on $\mathbb{D}$, in view of Lemma 1.4, it is enough to show that $S_5 \leq 1$. It is easy to see that $S_5 \leq 1$ implies

$$12r - 3r^2 + 4r^3 - r^4 \leq 1 - 4r + 6r^2 - 4r^3 + r^4,$$

which is equivalent to

$$2r^4 - 8r^3 + 9r^2 - 16r + 1 \geq 0.$$

Therefore $S_5 \leq 1$, when $p_{uc}(r) \geq 0$. Since $p_{uc}(0) = 1 > 0$ and $p_{uc}(1) = -12 < 0$, $p_{uc}(r)$ has a real root in $(0, 1)$. We now show that $p_{uc}(r)$ has exactly one real root in $(0, 1)$. As before, it is sufficient to show that $p_{uc}(r)$ is either monotonically increasing, or monotonically decreasing on $(0, 1)$. A direct computation shows that

$$\begin{aligned} p'_{uc}(r) &= -16 + 18r - 24r^2 + 8r^3 \\ &\leq -16 + 18r - 24r^2 + 8r \\ &= -16 + 26r - 24r^2 =: \xi(r). \end{aligned}$$

We note that $\xi'(r) = 0$ at $r = 13/24$. Since $\xi''(r) < 0$, $\xi(r)$ attains its maximum value at $r = 13/24$ and $\xi(13/24) < 0$. Hence $\xi(r) < 0$ for all $r \in (0, 1)$. Thus $p'_{uc}(r) < 0$ in $(0, 1)$. This shows that $p_{uc}(r)$ is strictly monotonically decreasing in $(0, 1)$, and hence $p_{uc}(r)$ has exactly one real root (say $r_{uc} \approx 0.064723$) in $(0, 1)$. Since $p_{uc}(r)$ is monotonically decreasing with $p_{uc}(0) > 0$ and $p_{uc}(r_{uc}) = 0$, we conclude that $p_{uc}(r) \geq 0$ for $0 < r \leq r_{uc}$. Thus f is uniformly convex in $|z| < r_{uc}$. □

Theorem 3.3 *Let* $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^0$ *be given by* (1.1), *and satisfy the coefficient conditions*

$$|a_n| \leq \frac{1}{6}(2n+1)(n+1) \quad and \quad |b_n| \leq \frac{1}{6}(2n-1)(n-1) \tag{3.10}$$

for $n \geq 2$. *Then* f *is uniformly starlike on the disk* $|z| < r_{us}$, *where* $r_{us} \approx 0.0667343$ *is the unique positive real root of* $q_{us}(r) = 0$ *in* $(0, 1)$, *and where*

$$q_{us}(r) = 3r^4 - 12r^3 + 16r^2 - 16r + 1. \tag{3.11}$$

Proof Let $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^0$ be given by (1.1), and the coefficients a_n and b_n satisfy the conditions (3.11) for $n \geq 2$. For $0 < r < 1$, let

$$f_r(z) = r^{-1} f(rz) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} b_n r^{n-1} z^n}, \quad z \in \mathbb{D}. \tag{3.12}$$

Again consider the sum

$$S = \sum_{n=2}^{\infty} n|a_n| r^{n-1} + \sum_{n=2}^{\infty} n|b_n| r^{n-1}. \tag{3.13}$$

In view of (3.11) and Lemma 2.1, (3.13) reduces to

$$S \leq \sum_{n=2}^{\infty} n \frac{2n^2+1}{3} r^{n-1} = \frac{r(6 - 5r + 4r^2 - r^3)}{(1-r)^4} =: S_6. \tag{3.14}$$

To show that $f_r(z)$ is uniformly starlike on $\mathbb{D}$, in view of Lemma 1.3, it is sufficient to show that $S_6 \leq 1/2$. A simple calculation shows that $S_6 \leq 1/2$ implies

$$12r - 10r^2 + 8r^3 - 2r^4 \leq 1 - 4r + 6r^2 - 4r^3 + r^4,$$

or

$$3r^4 - 12r^3 + 16r^2 - 16r + 1 \geq 0.$$

Let $q_{us}(r) := 3r^4 - 12r^3 + 16r^2 - 16r + 1$. Therefore $S_6 \leq 1/2$ if, and only if, $q_{us}(r) \geq 0$. Since $q_{us}(0) = 1 > 0$ and $q_{us}(1) = -8 < 0$, $q_{us}(r)$ has a real root in $(0, 1)$. As before we now show that $q_{us}(r)$ has exactly one real root in $(0, 1)$, so it is sufficient to show that $q_{us}(r)$ has a monotonicity property on $(0, 1)$. It is easy to see that

$$\begin{aligned} q'_{us}(r) &= -16 + 32r - 36r^2 + 12r^3 \\ &\leq -16 + 32r - 36r^2 + 12r \end{aligned}$$

$$= -16 + 44r - 36r^2 =: \phi(r).$$

Therefore $\phi'(r) = 44 - 72r = 0$ at $r = 11/18$. Since $\phi''(r) < 0$, $\phi(r)$ attains its maximum value at $r = 11/18$ and $\phi(11/18) < 0$. Hence $\phi(r) < 0$ for all $r \in (0, 1)$. Thus $p'_{uc}(r) < 0$ in $(0, 1)$. Therefore $q_{us}(r)$ is strictly monotonically decreasing in $(0, 1)$, and $q_{us}(r)$ has exactly one real root (say $r_{us} \approx 0.0667343$) in $(0, 1)$. Since $q_{us}(r)$ is monotonically decreasing with $q_{us}(0) > 0$ and $q_{us}(r_{us}) = 0$, we conclude that $q_{us}(r) \geq 0$ for $0 < r \leq r_{us}$. Hence f is uniformly convex in $|z| < r_{us}$. □

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