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Existence and uniqueness of a renormalized solution of parabolic problems in Orlicz spaces

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Abstract

In this work, we shall be concerned with the existence and uniqueness result to the nonlinear parabolic equations whose prototype is

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \Delta_M u - \operatorname{div}\left(\overline{c}(x,t)\overline{M}^{-1} M\left(\dfrac{\alpha_0}{\lambda}|b(u)|\right)\right) = f & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ b(u)(t=0) = b(u_0) & \text{in } \Omega, \end{cases}$$

where $-\Delta_M u = -\operatorname{div}((1+|u|)^2 Du \frac{\log(e+Du)}{|Du|})$, $\overline{c} \in (L^\infty(Q_T))^N$ and $M(t) = t\log(e+t)$ is an N-function. The data f and $b(u_0)$ in $L^1(Q_T)$ and $L^1(\Omega)$.

Keywords Nonlinear parabolic equations · Orlicz spaces · Renormalized solutions · Uniqueness

Mathematics Subject Classification Primary 47A15; Secondary 46A32 · 47D20

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1 Introduction

Let Ω be a bounded open set of $I\!R^N$ $(N \geq 2)$, T is a positive real number, and $Q_T = \Omega \times (0, T)$. Consider the following nonlinear Dirichlet equation:

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + \operatorname{div}(\Phi(x,t,u)) = f \text{ in } Q_T, \\ u(x,t) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega \times (0,T), \\ b(u)(t=0) = b(u_0) \qquad\qquad\qquad\qquad\qquad\quad\ \text{in } \Omega, \end{cases} \tag{1}$$

where $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is a Leary-Lions operator defined on the inhomogeneous Orlicz–Sobolev space $W_0^{1,x}L_M(Q_T)$, M is an N-function related to the growth of $A(u)$ (see Assumptions (9)–(11)), and to the growth of the lower order Carathéodory function $\Phi(x,t,u)$ (see Assumption (12)). $b : I\!R \longrightarrow I\!R$ is a strictly increasing C^1-function, the second term f in $L^1(Q_T)$.

In the classical Lebesgue spaces $L^p(0, T, W_0^{1,p}(\Omega))$, the notion of renormalized solution of (1) was early introduced by Di-Perna and Lions [14] for the study of Boltzmann equation and Blanchard, Murat and Redwane were adapted it to parabolic equations with L^1-data in [9,11] where they treated the existence and uniqueness with $b(u)$ a linear function $(b(u) = u)$ and $a(x,t,u,\nabla u) + \Phi(u)$ with $\Phi \in \mathcal{C}^\infty(I\!R)$, $u \in L^\infty(0, T, L^1(\Omega))$ and the source data is a measure $\mu = f - \operatorname{div}(G)$.

Recently Blanchard et al. [12] have studied Stefan problem the function in the evolution term b is maximal graph on $I\!R$ and Aberqi et al. [1] where b is a general strictly increasing $\mathcal{C}^1(I\!R)$-function.

Another approach to define a suitable generalized solution is that of entropy solution which was introduced in [8] in the elliptic case and by Prignet [26] in the parabolic case.

Aharouch and Bennouna [3] have proved the existence and uniqueness of entropy solutions in the framework of Orlicz-Sobolev spaces $W_0^1L_M(\Omega)$ assuming the $\triangle_2$-condition on the N-function M. Recently, Mukminov [24,25] proved the uniqueness of renormalized solutions to the Cauchy problem for parabolic equation using Kruzhkovis method of doubling the variable.

In the generalized-Orlicz spaces, the work [5] is a continuation of [3] where Al-Hawmi, Benkirane, Hjiaj and Touzani proved the existence and uniqueness of entropy solution for

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = f \text{ in } \Omega, \\ u(x) = 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Phi = 0$ and $\overline{M}$ satisfy the $\triangle_2$-condition. Antontsev and Shmarev [6] proved theorems of existence and uniqueness of weak solutions of Dirichlet problem for a class of nonlinear parabolic equations with nonstandard anisotropic growth conditions in the variable exponent Lebesgue spaces. Equations of this class generalize the evolution $p(x,t)$-Laplacian of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sum_i \dfrac{\partial}{\partial x_i}[a_i(x,t,u)|D_i u|^{p_i(x,t)-2} D_i u + b_i(x,t,u)] = 0 & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3}$$

The study of the problem in the framework of renormalized solutions is motivated by the lack of regularity of the distributional formulation (it's not enough to assure the uniqueness, for more detail see [26] and the counterexample in [29]).

Our novelty in the present paper is to give an existence and uniqueness result of renormalized solution of (1) in the general framework of inohomogeneous Orlicz spaces with a lower order term Φ which depends on x, t and u, that is with $a(x,t,u,\nabla u)$ replaced by $a(x,t,u,\nabla u) + \Phi(x,t,u)$. The difficulty encountered during the proof of the existence of the solution is that the term Φ does not satisfy the coercivity condition. Nonlinearities are characterized by N-functions M, for which $\triangle_2$-conditions are not imposed, losing the reflexivity of the spaces $L_M(Q_T)$ and $W_0^1 L_M(Q_T)$.

In the literature up to our knowledge there is no result on the uniqueness of the operator $a(x,t,u,\nabla u) + \Phi(x,t,u)$ in the framework of Orlicz spaces. So the crucial question that we will focus in this paper is to impose appropriate conditions on each term of problem (1) in order to obtain a uniqueness result (see Theorem 3).

This paper is organized as follows. In the Sect. 2, we recall some well-known preliminaries properties and results of Orlicz-Sobolev spaces. Section 3 is devoted to specify the essential assumptions on b, a, Φ and f and we introduce the Definition 1 of a renormalized solution of (1) and the existence result given in Theorem 2. In Sect. 4 we prove Theorem 2 and in Sect. 5 we establish the uniqueness result. The proof of Lemma 8 is given in the "Appendix".

2 *N*-function and Orlicz spaces

Let $M : I\!R^+ \to I\!R^+$ be an N-function, that is, M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$, and $M(t)/t \to +\infty$ as $t \to +\infty$. Equivalently, M admits the representation $M(t) = \int_0^t a(s)ds$, where $a : I\!R^+ \to I\!R^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$, and $a(t) \to +\infty$ as $t \to +\infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{a}(s)ds$, where $\overline{a} : I\!R^+ \to I\!R^+$, is given by $\overline{a}(t) = \sup\{s : a(s) \le t\}$.

We extend these N-functions to even functions on all $I\!R$.

Example 1 For $M(t) = \dfrac{|t|^p}{p}$, $\overline{M}(t) = \dfrac{|t|^q}{q}$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $p, q \in (1; +\infty)$.

For $M(t) = \exp(t^2) - 1 - |t|$, $\overline{M}(t) = (1 + |t|)\ln(1 + |t|) - |t|$.

The N-function M is said to satisfy the $\triangle_2$-condition if, for some k,

$$M(2t) \le kM(t) \quad \text{for all} \quad t \in I\!R.$$

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, that is, for each $\epsilon > 0$, $\lim_{t\to+\infty} \dfrac{P(t)}{Q(\epsilon t)} = 0$.

Proposition 1 *$P \ll M$ if and only if, for all $\epsilon > 0$ there exists a constant c_ϵ such that,*

$$P(t) \le M(\epsilon t) + c_\epsilon, \quad \forall t \ge 0. \tag{4}$$

Proof Let $\epsilon > 0$, then by the definition of $P \ll M$, there exists $t_\epsilon > 0$ such that $\forall t > t_\epsilon$,

$$P(t) \le M(\epsilon t). \tag{5}$$

On the other hand, for $t \in [0, t_\epsilon]$, we use the continuity of P and then there exists a constant C_ϵ such that

$$P(t) \le C_\epsilon, \tag{6}$$

where $C_\epsilon = \sup_{t\in[0,t_\epsilon]} P(t)$. We combine (5) and (6) we conclude (4).

The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))dx < +\infty \quad \left(\text{resp. } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \quad \text{for some} \quad \lambda > 0\right).$$

The set $L_M(\Omega)$ is Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \le 1\},$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The dual $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the duality pairing $\int_\Omega uv dx$ and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|u\|_{\overline{M},\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{M,\Omega}.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$

[resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [2]). □

We recall the following lemmas:

Theorem 1 (Banach–Alaoglu–Bourbaki [13]) *Let E be a Banach space and E' be the dual space.*
The closed unit ball $B_{E'} = \{f \in E'; \|f\| \leq 1\}$ is compact in the weak- topology $\sigma(E', E)$.*

Lemma 1 (Dominated convergence) *Let f_k, f in $L_M(\Omega)$.*

If $f_k \to f$ a.e. and $|f_k| \leq |g|$ a.e. and $\int_\Omega M(\lambda|g|)dx < \infty$ for every $\lambda > 0$, then $f_k \to f$ in $L_M(\Omega)$.

Lemma 2 (See [19], [22, p. 132]) *If a sequence $g_n \in L_M(\Omega)$ converges a.e. to g and g_n remains bounded in $L_M(\Omega)$, then $g \in L_M(\Omega)$ and $g_n \to g$ in $\sigma(L_M, E_{\overline{M}})$.*

Lemma 3 *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W^1L_M(\Omega)$. Then $F(u) \in W^1L_M(\Omega)$.*

Moreover if the set D of discontinuity points of F' is finite, then

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \text{ in } \{x \in \Omega; u(x) \notin D\} \\ 0 & a.e. \text{ in } \{x \in \Omega; u(x) \in D\}. \end{cases}$$

Proof It is easily adapted from that given in [21] in the case $W^1L_M(\Omega)$, by Theorem 1 of [20] instead of Theorem 4 of [20] (see also Remark 5 of [21]). □

Lemma 4 (See [17]) *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function. Then the mapping $F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak-* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

Inhomogeneous Orlicz-Sobolev spaces :

Let M be an N-function, for each $\alpha \in \mathbb{N}^N$, denote by ∇_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces are defined as follows

$$W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) : \nabla_x^\alpha u \in L_M(Q_T),\ \forall\alpha \in \mathbb{N}^N,\ |\alpha| \leq 1\},$$
$$W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) : \nabla_x^\alpha u \in E_M(Q_T),\ \forall\alpha \in \mathbb{N}^N,\ |\alpha| \leq 1\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leq 1} \|\nabla_x^\alpha u\|_{M,Q_T}.$$

The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure $W^{1,x}E_M(Q_T)$ of $\mathcal{D}(Q_T)$. We can easily show that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit in $W_0^{1,x}E_M(Q_T)$, of some subsequence in $\mathcal{D}(Q_T)$ for the modular convergence.

This implies that $\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}$. This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore, $W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_M$, and the dual space of $W_0^{1,x}E_M(Q_T)$ will be denoted by

$$W^{-1,x}L_{\overline{M}}(Q_T) = \left\{ f = \sum_{|\alpha| \le 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in L_{\overline{M}}(Q_T) \right\}.$$

This space will be equipped with the usual quotient norm $\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\overline{M}, Q_T}$.

Remark 1 We can easily check, using Lemma 3, that each uniformly Lipschitzian mapping F, with $F(0) = 0$, acts in inhomogeneous Orlicz-Sobolev spaces of order 1: $W^{1,x}L_M(Q_T)$ and $W_0^{1,x}L_M(Q_T)$.

Lemma 5 (See [15]) *For all $u \in W_0^1 L_M(Q_T)$ with $meas(Q_T) < +\infty$ one has*

$$\int_{Q_T} M\left(\frac{|u|}{\lambda}\right) dxdt \le \int_{Q_T} M(|\nabla u|) dxdt \tag{7}$$

where $\lambda = diam(Q_T)$, is the diameter of Q_T.

3 Essential assumptions and the existence result

Throughout this paper, we assume that the following assumptions hold true.

Let M and P be two N-functions such that $P \ll M$.

$b : I\!R \to I\!R$ is a strictly increasing $\mathcal{C}^1(I\!R)$-function, $b(0) = 0$,

$$b_0 < b'(s) < b_1, \quad \forall s \in I\!R \quad \text{such that} \quad b_1 < \frac{1}{\alpha_0}. \tag{8}$$

where α_0 is the constant appearing in (12).

$a : Q_T \times I\!R \times I\!R^N \to I\!R^N$ is Carathéodory function and there exists a constant $\nu > 0$ such that for a.e. $x \in \Omega$ and for all $s \in I\!R$, $\xi, \xi^* \in I\!R^N$, $\xi \ne \xi^*$:

$$|a(x,t,s,\xi)| \le \nu(a_0(x,t) + \overline{M}^{-1} P(|s|) + \overline{M}^{-1} M(|\xi|)) \tag{9}$$

with $a_0(.,.) \in E_{\overline{M}}(Q_T)$,

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi - \xi^*) > 0, \tag{10}$$

$$a(x,t,s,\xi).\xi \ge \alpha M(|\xi|). \tag{11}$$

$\Phi : Q_T \times I\!R \to I\!R^N$ is a Carathéodory function such that

$$|\Phi(x,t,s)| \le c(x,t)\overline{M}^{-1} M\left(\frac{\alpha_0}{\lambda}|b(s)|\right), \tag{12}$$

where $\lambda = \operatorname{diam}(Q_T)$, $\|c(.,.)\|_{L^\infty(Q_T)} \leq \min(\frac{\alpha}{\alpha_0+1}; \frac{\alpha}{2(\alpha_0 b_1+1)})$ and $0 < \alpha_0 < 1$.

$$f \in L^1(Q_T), \tag{13}$$

$$u_0 \in L^1(\Omega) \text{ such that } b(u_0) \in L^1(\Omega). \tag{14}$$

Note that $\langle , \rangle$ means for either the pairing between $W_0^{1,x}L_M(Q_T) \cap L^\infty(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ or between $W_0^{1,x}L_M(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T)$.

Let T_k, $k > 0$ denotes the truncation function at level k defined on $I\!R$ by $T_k(r) = \max(-k; \min(k; r))$.

The definition of a renormalized solution of problem (1) can be stated as follows.

Definition 1 A measurable function u defined on Q_T is a renormalized solution of problem (1), if it satisfies the following conditions:

$$b(u) \in L^\infty(0, T; L^1(\Omega)), \tag{15}$$

$$T_k(b(u)) \in W_0^{1,x}L_M(Q_T), \quad \forall k > 0, \tag{16}$$

$$\lim_{m\to+\infty} \int_{\{(x,t)\in Q_T : m \leq |u| \leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt = 0, \tag{17}$$

and if, for every function $S \in W^{2,\infty}(I\!R)$ which is piecewise C^1 and such that S' has a compact support, we have in the sense of distributions

$$\frac{\partial S(b(u))}{\partial t} - \operatorname{div}\Big(S'(b(u))(a(x,t,u,\nabla u) - \Phi(x,t,u))\Big)$$
$$+S''(b(u))\Big(a(x,t,u,\nabla u) + \Phi(x,t,u)\Big) = f S'(b(u)) \quad \text{in} \quad \mathcal{D}(Q_T), \tag{18}$$

$$S(b(u))(t=0) = S(b(u_0)) \qquad \text{in} \quad \Omega. \tag{19}$$

Theorem 2 *Assume that (8)–(14) hold true. Then there exists at least one renormalized solution u of the problem (1) in the sense of the definition 1.*

4 The stages of the Proof of Theorem 2

Truncated problem.

For each $n > 0$, we define the following approximations:

$$b_n(s) = b(T_n(s)), \ \forall s \in I\!R, \tag{20}$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \quad \text{a.e. } (x,t) \in Q_T, \ \forall s \in I\!R, \ \forall \xi \in I\!R^N, \tag{21}$$

$$\Phi_n(x,t,s) = \Phi(x,t,T_n(s)) \quad \text{a.e. } (x,t) \in Q_T, \ \forall s \in I\!R, \tag{22}$$

$$\text{Let } f_n \text{ be a smooth function such that } f_n \to f \text{ strongly in } L^1(Q_T) \tag{23}$$

and

$$u_{0n} \in \mathcal{C}_0^\infty(\Omega) \text{ such that } b_n(u_{0n}) \to b(u_0) \text{ strongly in } L^1(\Omega). \tag{24}$$

Consider the approximate problem:

$$\begin{cases} \dfrac{\partial b_n(u_n)}{\partial t} - div(a_n(x,t,u_n,\nabla u_n) + \Phi_n(x,t,u_n)) = f_n & \text{in } Q_T, \\ u_n(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ b_n(u_n)(t=0) = b_n(x,u_{0n}) & \text{in } \Omega. \end{cases} \tag{25}$$

Let $u_n \in W_0^{1,x}L_M(Q_T)$, then for any fixed $n > 0$, there exists at least one solution $u_n \in W_0^{1,x}L_M(Q_T)$ of (25), (see [23]).

Note that by Lemma 3 and Remark 1, we have $T_k(u_n) \in W_0^{1,x}L_M(Q_T)$, and by (8), (11), (12) and Young inequality, the quantity $\int_{Q_T} M(|\nabla T_k(u_n)|)dxdt$ is finite for all $k > 0$.

Remark 2 The explicit dependence in x and t of the functions a and Φ will be omitted so that $a(x,t,u,\nabla u) = a(u,\nabla u)$ and $\Phi(x,t,u) = \Phi(u)$.

Step 1: A priori estimates.

Lemma 6 *Let u_n be a solution of the approximate problem (25), then for all $k > 0$, there exists a constant C and for a subsequence, still indexed by n we have*

$$\int_{Q_T} M(|\nabla T_k(u_n)|)dxdt \leq kC, \tag{26}$$

$$u_n \to u \text{ a.e in } Q_T, \text{ where } u \text{ is a measurable function on } Q_T, \tag{27}$$

$$b_n(u_n) \to b(u) \quad \text{a.e in} \quad Q_T, \quad b(u) \in L^\infty(0,T,L^1(\Omega)), \tag{28}$$

$$a_n(T_k(u_n), \nabla T_k(u_n)) \quad \text{is bounded in} \quad (L_{\overline{M}}(Q_T))^N, \tag{29}$$

Proof Fix $k > 0$ and $\tau \in (0,T)$. Let $T_k(u_n)\chi_{(0,\tau)}$ a test function in problem (25). Using the Young inequality we get

$$\int_\Omega B_n(u_n(\tau))dx + \int_{Q_\tau} a_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt + \int_{Q_\tau} \Phi_n(u_n)\nabla T_k(u_n)dxdt \\ \leq k[\|f_n\|_{L^1(Q_T)} + \|b(u_0)\|_{L^1(Q_T)}], \tag{30}$$

where $B_n(r) = \int_0^r \frac{\partial b_n(s)}{\partial s} T_k(s)ds$.

By definition, we have $\int_\Omega B_n(u_n(\tau))dx \geq 0$ and $\int_\Omega B_n(u_n(0))dx \leq kb_1\|b(u_0)\|_{L^1(Q_T)}$.

By (12), (8) and Young inequality we have

$$\int_{Q_\tau} \Phi_n(u_n)\nabla T_k(u_n)dxdt \leq \|c(.,.)\|_{L^\infty(Q_T)}[\alpha_0 b_1 \int_{Q_\tau} M(\frac{|T_k(u_n)|}{\lambda})dxdt \\ + \int_{Q_\tau} M(|\nabla T_k(u_n)|)dxdt],$$

thanks to Lemma 5, we obtain

$$\int_{Q_\tau} \Phi_n(u_n)\nabla T_k(u_n)dxdt \leq \|c(.,.)\|_{L^\infty(Q_T)}(\alpha_0 b_1 + 1)\int_{Q_\tau} M(|\nabla T_k(u_n)|)dxdt,$$

return to (30) and using (11) we get

$$\int_{Q_\tau} a_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq \frac{(\alpha_0 b_1+1)}{\alpha}\int_{Q_\tau} a_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt$$
$$+k[\|f_n\|_{L^1(Q_T)} + \|b(u_0)\|_{L^1(Q_T)}],$$

thus

$$\left[1-\frac{(\alpha_0 b_1+1)}{\alpha}\|c(.,.)\|_{L^\infty(Q_T)}\right]\int_{Q_T} a_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kc_1.$$

We take $\frac{1}{c_2} = [1-\frac{(\alpha_0 b_1+1)}{\alpha}\|c(.,.)\|_{L^\infty(Q_T)}]$.

By (12) we have $c_2 > 0$ and we obtain

$$\int_{Q_\tau} a(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kC,$$

where $C = c_1c_2$. And by (11) we have (26).

We conclude that $(T_k(u_n))$ is bounded in $W_0^{1,x}L_M(Q_T)$ independently of n. Since $(E_{\overline{M}}(Q_T))' = L_M(Q_T)$ then by Theorem 1, the set $\{(T_k(u_n))\}$ is compact for the weak topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ so there exists a subsequence still denoted by u_n and there exists a measurable function ξ_k such that $T_k(u_n)\rightharpoonup\xi_k$ for the weak topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.

On the other hand, using Lemma 5, we have

$$M\left(\frac{k}{\lambda}\right)\text{meas}\{|u_n|>k\} = \int_{\{|u_n|>k\}} M\left(\frac{|T_k(u_n)|}{\lambda}\right)dxdt$$
$$\leq \int_{Q_T} M(|\nabla T_k(u_n)|)dxdt \leq kC,$$

then $\text{meas}\{|u_n|>k\} \leq \frac{kC}{M(\frac{k}{\lambda})}$ for all n and for all k.

Thus, we get $\lim_{k\to\infty} \text{meas}\{|u_n|>k\} = 0$. □

Proof of (27) *and* (28): For $k < n$, let $g_k \in W^{2,\infty}(I\!R)$, such that g_k', has a compact support $\text{supp}(g_k') \subset [-k,k]$. We multiply the Eq. (25) by $g_k'(u_n)$, to obtain in $\mathcal{D}'(Q_T)$,

$$\frac{\partial B^n_{g_k}(u_n)}{\partial t} = div(g_k'(u_n)(a_n(u_n,\nabla u_n)+\Phi_n(u_n))$$
$$-g_k''(u_n)(a_n(u_n,\nabla u_n)+\Phi_n(u_n))\nabla u_n + f_n g_k'(u_n) \qquad (31)$$

where $B^n_{g_k}(r) = \displaystyle\int_0^r g'_k(s)\frac{\partial b_n(s)}{\partial s}ds.$

Then, we show that

$$\left(B^n_{g_k}(u_n)\right) \quad \text{is bounded in} \quad W_0^{1,x}L_M(Q_T), \tag{32}$$

and

$$\left(\frac{\partial B^n_{g_k}(u_n)}{\partial t}\right) \quad \text{is bounded in} \quad L^1(Q_T) + W^{-1,x}L_{\overline{M}}(Q_T) \tag{33}$$

independently of n.

Indeed, first we have $|\nabla B^n_{g_k}(u_n)| \le b_1|\nabla T_k(u_n)|\|g'_k\|_{L^\infty(I\!R)}$ a.e. in Q_T, and using (26) we obtain (32). To show that (33) holds true, since supp(g'_k) and supp(g''_k) are both included in $[-k, k]$, u_n may be replaced by $T_k(u_n)$ in each of these terms. As a consequence, each term in the right hand side of (31) is bounded either in $W^{-1,x}L_{\overline{M}}(Q_T)$ or in $L^1(Q_T)$ which shows that (33) holds true.

Arguing again as in [10] estimates (32), (33) and the following remark, imply that, for a subsequence, still indexed by n,

$$u_n \to u \text{ a.e in } Q_T \text{ and } b(u) \in L^\infty(0, T, L^1(\Omega)),$$

where u is a measurable function defined on Q_T.

Remark 3 For every $g \in W^{2,\infty}(I\!R)$, nondecreasing function such that supp$(g') \subset [-k, k]$ and (8), we have

$$b_0|g(r) - g(r')| \le |B_g(r) - B_g(r')| \le b_1|g(r) - g(r')| \quad \text{for every} \quad \text{in} \quad I\!R. \tag{34}$$

Proof of (29) : As in [4], we may deduce that $a_n(T_k(u_n), \nabla T_k(u_n))$ is a bounded sequence in $(L_{\overline{M}}(Q_T))^N$, and we obtain (29).

Step 2: Almost everywhere convergence of the gradients. In order to show that the gradient converges almost everywhere, we need to prove the next proposition.

Proposition 2 *Let u_n be a solution of the approximate problem* (25)*, then*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m\le|u_n|\le m+1\}} a(u_n, \nabla u_n)\nabla u_n dxdt = 0, \tag{35}$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m\le|u_n|\le m+1\}} \Phi(u_n)\nabla u_n dxdt = 0. \tag{36}$$

For any $r > 0$ and $0 < \delta < 1$, we have

$$\lim_{n\to+\infty} \int_{Q_T} [(a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)\chi_{\{|\nabla T_k(u)|\le k\}}) \times(\nabla T_k(u_n) - \nabla T_k(u)\chi_{\{|\nabla T_k(u)|\le k\}})]^\delta \, dxdt = 0, \tag{37}$$

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad Q_T. \tag{38}$$

Proof Taking $Z_m(u_n) = T_1(u_n - T_m(u_n))$ as a test function in the approximate Eq. (25) we get

$$\int_{\{m\le|u_n|\le m+1\}} a_n(u_n,\nabla u_n)\nabla u_n dxdt \le C[\int_{Q_T} f_n Z_m(u_n)\,dxdt + \int_{\{|u_{0n}|>m\}} |b_n(u_{0n})|dxdt,$$

where $\frac{1}{C} = [1 - \frac{(\alpha_0 b_1 + 1)}{\alpha}\|c(.,.)\|_{L^\infty(Q_T)}] > 0$.

Passing to the limit as $n \to +\infty$, using the pointwise convergence of u_n and strongly convergence in $L^1(Q_T)$ of f_n we get

$$\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a_n(u_n,\nabla u_n)\nabla u_n dxdt \le C[\int_{Q_T} f Z_m(u)\,dxdt + \int_{\{|u_0|>m\}} |b(u_0)|dxdt.$$

Owning to Lebesgue's theorem and passing to the limit as $m \to +\infty$, in the all terms of the right-hand side, we get (35).

From (11), we deduce also

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} M(|\nabla Z_m(u_n)|)dxdt = 0. \tag{39}$$

On the other hand, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} \Phi_n(u_n)\nabla Z_m(u_n)dxdt \le \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} M(|\nabla Z_m(u_n)|)dxdt, + \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} \overline{M}(|\Phi_n(u_n)|)dxdt.$$

Using the pointwise convergence of u_n and by Lebegue's theorem, in the second term of the right side of this last expression, we get

$$\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} \overline{M}(|\Phi_n(u_n)|)dxdt = \int_{\{m\le|u|\le m+1\}} \overline{M}(|\Phi(u)|)dxdt,$$

and also, by Lebesgue's theorem

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} \overline{M}(|\Phi(u)|)dxdt = 0. \tag{40}$$

Thus with (39) and (40), we get the (36).

The proof of (37) is the same as the corresponding result in [18,27].

Finally, for the almost everywhere convergence of the gradients we use the following lemma and same techniques as in [4] and [18].

Lemma 7 (See [7]) *Under the Assumptions* (8)–(14)*, let* (z_n) *be a sequence in* $W_0^{1,x}L_M(Q_T)$ *such that:*

$$z_n \rightharpoonup z \quad for \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{41}$$

$$(a(x,t,z_n,\nabla z_n)) \quad is\ bounded\ in\ (L_{\overline{M}}(Q_T))^N, \tag{42}$$

$$\int_{Q_T} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]dxdt \to 0 \tag{43}$$

as n and s tend to $+\infty$*, and where* χ_s *is the characteristic function of* $Q^s = \{x \in Q_T; |\nabla z| \leq s\}$.

Then,

$$\nabla z_n \to \nabla z \quad a.e.\ in\ Q_T, \tag{44}$$

$$\lim_{n\to+\infty} \int_{Q_T} a(x,t,z_n,\nabla z_n)\nabla z_n dxdt = \int_{Q_T} a(x,t,z,\nabla z)\nabla z dxdt, \tag{45}$$

$$M(|\nabla z_n|) \to M(|\nabla z|) \quad in\ L^1(Q_T). \tag{46}$$

□

Step 3: We show that u satisfies the conditions of Definition 1 For this, let show that (17) holds.

We have for any $m > 0$,

$$\begin{aligned}\int_{\{m\leq|u_n|\leq m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt &= \int_{Q_T} a(u_n,\nabla u_n)[\nabla T_{m+1}(u_n) - \nabla T_m(u_n)]dxdt\\ &= \int_{Q_T} a(T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)dxdt\\ &\quad - \int_{Q_T} a(T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt.\end{aligned}$$

According to (45), we pass to the limit as n tends to $+\infty$ for fixed $m > 0$ and we obtain

$$\begin{aligned}&\lim_{n\to+\infty} \int_{\{m\leq|u_n|\leq m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt\\ &= \int_{Q_T} a(T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)dxdt\\ &\quad - \int_{Q_T} a(T_m(u),\nabla T_m(u))\nabla T_m(u)dxdt\\ &= \int_{\{m\leq|u|\leq m+1\}} a(u,\nabla u)\nabla u dxdt,\end{aligned}$$

with (35), we easily obtain (17).

Similarly we deduce

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}}\Phi(u)\nabla u dxdt=0. \tag{47}$$

Let $S\in W^{2,\infty}(I\!R)$ which is a piecewise C^1-function and such that S' has a compact support, Let $K>0$ such that $\mathrm{supp}(S')\subset[-K,K]$. Pointwise multiplication of the approximate problem (25) by $S'(u_n)$, we get

$$\begin{cases}\dfrac{\partial S(b(u_n))}{\partial t}+\mathrm{div}\Big(S'(b(u_n))(a(x,t,u_n,\nabla u_n)-\Phi(x,t,u_n))\Big)\\ +S''(b(u_n))\Big(a(x,t,u_n,\nabla u_n)-\Phi(x,t,u_n)\Big)\nabla b(u_n)\\ =fS'(b(u_n)).\end{cases} \tag{48}$$

Now we will pass to the limit as $n\to+\infty$ of each term of (48),

- *Limit of* $\frac{\partial S(b(u_n))}{\partial t}$

  since S is bounded, and $S(b(u_n))$ converges to $S(b(u))$ a.e. in Q_T and weakly in $L^\infty(Q_T)$, then $\frac{\partial S(b(u_n))}{\partial t}$ converges to $\frac{\partial S(b(u))}{\partial t}$ in $\mathcal{D}'(Q_T)$.
- *Limit of* $S'(b(u_n))a(u_n,\nabla u_n)$

  since $\mathrm{supp}(S')\subset[-K,K]$ and (8) we have

$$S'(b(u_n))a(u_n,\nabla u_n)=S'(b(u_n))a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))\quad \text{a.e. in}\quad Q_T.$$

  Owing to the pointwise convergence of u_n to u, the bounded character of S', and by Lemma 7 and Proposition 2, we conclude $a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))$ converges to $a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))$ weakly in $(L_{\overline{M}}(Q_T))^N$. This allows us to obtain $S'(b(u_n))a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))$ converges to

  $S'(b(u))a(T_{\frac{k}{b_0}}(u),\nabla T_{\frac{k}{b_0}}(u))$ weakly for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$, and $S'(b(u))a(T_{\frac{k}{b_0}}(u),\nabla T_{\frac{k}{b_0}}(u)=S'(b(u))a(u,\nabla u)\quad$ a.e. in $\quad Q_T$.
- *Limit of* $S''(b(u_n))a(u_n,\nabla u_n)\nabla b(u_n)$
  since $\mathrm{supp}(S')\subset[-K,K]$ and (8), we get

  $S''(b(u_n))a(u_n,\nabla u_n)\nabla b(u_n)=S''(b(u_n))a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))\nabla b(u_n)$ a.e. in Q_T. Owing to the pointwise convergence of $S''(b(u_n))$ to $S''(b(u_n))$ as n tends to $+\infty$, the bounded character of S'' and by Lemma 7 and Proposition 2, we conclude $S''(b(u_n))a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))\nabla b(u_n)\rightharpoonup S''(b(u))a(T_{\frac{k}{b_0}}(u),\nabla T_{\frac{k}{b_0}}(u))\nabla b(u)$ weakly in $L^1(Q_T)$ as $n\to+\infty$, and $S''(b(u))a(T_{\frac{k}{b_0}}(u),\nabla T_{\frac{k}{b_0}}(u))\nabla b(u)=S''(b(u))a(u,\nabla u)\nabla b(u)\quad$ a.e. in $\quad Q_T$.

– *Limit of* $S'(b(u_n))\Phi(u_n)$

since $\operatorname{supp}(S') \subset [-K, K]$ and (8) we have $S'(b(u_n))\Phi(u_n) = S'(b(u_n))\Phi(T_{\frac{k}{b_0}}(u_n))$ a.e. in Q_T.

In a similar way, we obtain $S'(b(u_n))\Phi(u_n) \to S'(b(u))\Phi(u)$ weakly for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$.

– *Limit of* $S''(b(u_n))\Phi(x, t, u_n)\nabla b(u_n)$

Also we have

$$S''(b(u_n))\Phi(u_n)\nabla b(u_n) = S''(b(u_n))\Phi(T_{\frac{k}{b_0}}(u_n))\nabla T_{\frac{k}{b_0}}(u_n)b'(u_n)$$

using the weakly convergence of truncation, it is possible to prove that,

$$S''(b(u_n))\Phi(u_n)\nabla b(u_n) \to S''(b(u))\Phi(u)\nabla b(u) \text{ strongly in } L^1(Q_T).$$

– *Limit of* $f_n S'(b(u_n))$

we have $u_n \to u$ a.e. in Q_T, S' and b are piecewise C^1. It is enough to use (23) to get that $f_n S'(b(u_n)) \to f S'(b(u))$ strongly in $L^1(Q_T)$.

Finally, to show (19), remark that S being bounded, $S(b(u_n))$ is bounded in $L^\infty(Q_T)$. The Eq. (48) allows to show that $\frac{\partial S(b(u_n))}{\partial t}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q_T)+L^1(Q_T)$. By Lemma 5 in [16] this implies that $S(b(u_n))$ lies in a compact set of $C^0([0, T]; L^\infty(\Omega))$. It follows that, on one hand, $S(b(u_n)(t = 0))$ converges to $S(b(u)(t = 0))$ strongly in $L^1(Q_T)$. On the other hand, the smoothness of S imply that $S(b(u)(t = 0)) = S(b(u_0))$ in Ω. This complete the proof of the existence result.

Example 2 As an example of equations to which the present result on the existence of renormalized solutions can be applied, we give:

1. For $M(t) = \dfrac{1}{p}|u|^p$, $b(u) = |u|^{p-2}u$, $a(x, t, u, \nabla u) = |\nabla u|^{p-2}\nabla u$ and $\Phi(x, t, u) = \exp\left(\dfrac{\eta}{\|x\| + t + 2}\right)\beta(\dfrac{\alpha_0}{\lambda})^{\frac{p}{q}}|u|^{\frac{p}{q}}$.

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \triangle_M - \operatorname{div}\left(\exp\left(\dfrac{\eta}{\|x\| + t + 2}\right)\beta\left(\dfrac{\alpha_0}{\lambda}\right)^{\frac{p}{q}}|u|^{\frac{p}{q}}\right) = f & \text{in } Q_T, \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ b(u(x, 0)) = b(u_0)(x) & \text{in } \Omega. \end{cases}$$

2. For $-\triangle_M = -\operatorname{div}(\dfrac{m(|\nabla u|)}{|\nabla u|}.\nabla u)$ where m is the derivative of M, $b(u) = u$ and $\overline{c} \in (L^\infty(Q_T))^N$.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle_M - div(\overline{c}(x,t)\overline{M}^{-1}M(\dfrac{\alpha_0}{\lambda}|b(u)|)) = f \text{ in } Q_T, \\ u(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \qquad \text{in } \Omega. \end{cases}$$

3. For $M(t) = t\log(1+t)$, $a(x,t,u,\nabla u) = (1+|u|)^2\nabla u\dfrac{\log(1+|\nabla u|)}{|\nabla u|}$ and $\overline{c} \in (L^\infty(Q_T))^N$.

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - div(1+|u|)^2\nabla u\dfrac{log(1+|\nabla u|)}{|\nabla u|} - \text{div}(\overline{c}(x,t)\overline{M}^{-1}M(\dfrac{\alpha_0}{\lambda}b(u))) = f \quad \text{in} \quad Q_T, \\ u(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \\ b(u(x,0)) = b(u_0)(x) \qquad \text{in} \quad \Omega. \end{cases}$$

5 Uniqueness result

Before showing the uniqueness of the solution of the problem (1), we will give the following technical lemma.

Let u and v be two renormalized solutions of the problem (1) and let us define for any $0 < k < s$,

$$\begin{aligned} \Gamma(u,v,s,k) = &\int_{\substack{\{b(s)-k<b(u)<b(s)+k\} \\ \cup\{b(-s)-k<b(u)<b(-s)+k\}}} b'(u)\Big(a(u,\nabla u)\nabla u + |\Phi(u)||\nabla u|\Big)dxdt \\ &+ \int_{\substack{\{b(s)-k<b(v)<b(s)+k\} \\ \cup\{b(-s)-k<b(v)<b(-s)+k\}}} b'(v)\Big(a(v,\nabla v)\nabla v + |\Phi(v)||\nabla v|\Big)dxdt. \end{aligned} \tag{49}$$

Lemma 8 *Assume that (8)–(14) hold. Then for any $r > 0$ we have*

$$\liminf_{s\to+\infty}\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k) = 0. \tag{50}$$

Proof See "Appendix". □

Theorem 3 *Assume that Assumptions (8)–(14) hold true and moreover that for any compact set $D \subset I\!R$, there exists $L_D \in E_{\overline{M}}(Q_T)$ and $\rho_D > 0$ such that $\forall s, \overline{s} \in D$,*

$$|a(x,t,s,\xi) - a(x,t,\overline{s},\xi)| \le \Big(L_D(x,t) + \rho_D\overline{P}^{-1}P(|\xi|)\Big)|s-\overline{s}|, \tag{51}$$

$$|\Phi(x,t,s) - \Phi(x,t,\overline{s})| \le L_D(x,t)|s-\overline{s}|, \tag{52}$$

$$|b'(s) - b'(\overline{s})| \le \beta_D|s-\overline{s}|, \tag{53}$$

for almost every $(x,t) \in Q_T$ *and for every* $\xi \in I\!R^N$*. Then the problem* (1) *has a unique renormalized solution.*

Proof We define a smooth approximation of $\widetilde{T}_n(r) = \min(b(n), \max(r, b(-n)))$ by $\widetilde{T}_n^\sigma$ where $\widetilde{T}_n^\sigma(0) = 0$ and

$$(\widetilde{T}_n^\sigma)'(r) = \begin{cases} 0 & \text{for} \quad r \geq b(n)+\sigma, \\ \frac{b(n)+\sigma-r}{\sigma} & \text{for} \quad b(n) \leq r \leq b(n)+\sigma, \\ 1 & \text{for} \quad b(-n) \leq r \leq b(n), \\ \frac{r+\sigma-b(-n)}{\sigma} & \text{for} \quad b(-n)-\sigma \leq r \leq b(-n), \\ 0 & \text{for} \quad r \leq b(-n)-\sigma. \end{cases} \tag{54}$$

For a fixed $n > 0$, we have for any $z \in L^1(Q_T)$,

$$\lim_{\sigma \to 0} (\widetilde{T}_n^\sigma)'(b(z)) = \chi_{\{|z| \leq n\}} \quad \text{a.e. in} \quad Q_T, \tag{55}$$

and

$$\lim_{\sigma \to 0} \widetilde{T}_n^\sigma(b(z)) = \widetilde{T}_n(b(z)) \quad \text{a.e. in} \quad Q_T. \tag{56}$$

Consider now two renormalized solutions u and v of (15)–(18) for the data f and $b(u_0)$. Since $\widetilde{T}_n^\sigma \in W^{2,\infty}(I\!R)$ and $supp((\widetilde{T}_n^\sigma)') \subset [b(-n)-\sigma, b(n)+\sigma]$, then we take $S = \widetilde{T}_n^\sigma$ and we use $\frac{1}{k}T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v)))$ as a test function in the difference of equations (18) for u and v, we get

$$\begin{aligned} &\frac{1}{k}\int_0^T \int_0^t \left\langle \frac{\partial\left(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))\right)}{\partial t}; T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))\right\rangle dsdt \\ &+I_{1,n}^\sigma + I_{2,n}^\sigma + I_{3,n}^\sigma + I_{4,n}^\sigma = I_{5,n}^\sigma, \end{aligned} \tag{57}$$

where

$$\begin{aligned} I_{1,n}^\sigma = \frac{1}{k}\int_0^T \int_0^t \int_\Omega &\Big[(\widetilde{T}_n^\sigma)'(b(u))a(u,\nabla u) \\ &-(\widetilde{T}_n^\sigma)'(b(v))a(v,\nabla v)\Big]\nabla T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v)))dxdsdt, \\ I_{2,n}^\sigma = \frac{1}{k}\int_0^T \int_0^t \int_\Omega &\Big[(\widetilde{T}_n^\sigma)''(b(u))a(u,\nabla u)\nabla b(u) \\ &-(\widetilde{T}_n^\sigma)''(b(v))a(v,\nabla v)\nabla b(v)\Big]T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v)))dxdsdt, \end{aligned}$$

$$I^{\sigma}_{3,n} = \frac{1}{k}\int_0^T\int_0^t\int_{\Omega}\Big[(\widetilde{T}^{\sigma}_n)'(b(u))\Phi(u) - (\widetilde{T}^{\sigma}_n)'(b(v))\Phi(v)\Big]\nabla T_k(\widetilde{T}^{\sigma}_n(b(u))$$
$$-\widetilde{T}^{\sigma}_n(b(v)))dxdsdt,$$
$$I^{\sigma}_{4,n} = \frac{1}{k}\int_0^T\int_0^t\int_{\Omega}\Big[(\widetilde{T}^{\sigma}_n)''(b(u))\Phi(u)\nabla b(u)$$
$$-(\widetilde{T}^{\sigma}_n)''(b(v))\Phi(v)\nabla b(v)\Big]T_k(\widetilde{T}^{\sigma}_n(b(u)) - \widetilde{T}^{\sigma}_n(b(v)))dxdsdt,$$
$$I^{\sigma}_{5,n} = \frac{1}{k}\int_0^T\int_0^t\int_{\Omega} f\Big[(\widetilde{T}^{\sigma}_n)'(b(u)) - (\widetilde{T}^{\sigma}_n)'(b(v))\Big]T_k(\widetilde{T}^{\sigma}_n(b(u))$$
$$-\widetilde{T}^{\sigma}_n(b(v)))dxdsdt,$$

for any $k > 0, n > 0, \sigma > 0$.

The following lemma will be useful in the sequel,

Lemma 9

$$\lim_{n\to+\infty}\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}\int_0^T\int_0^t\left\langle\frac{\partial\Big(\widetilde{T}^{\sigma}_n(b(u)) - \widetilde{T}^{\sigma}_n(b(v))\Big)}{\partial t}; T_k(\widetilde{T}^{\sigma}_n(b(u)) - \widetilde{T}^{\sigma}_n(b(v)))\right\rangle dsdt$$
$$= \int_{Q_T}|b(u) - b(v)|dxdt. \tag{58}$$

Proof Notice that

$$\widetilde{T}^{\sigma}_n(b(u))(t=0) = \widetilde{T}^{\sigma}_n(b(v))(t=0) = \widetilde{T}^{\sigma}_n(b(u_0)) \quad \text{a.e. in} \quad \Omega,$$

and

$$\frac{1}{k}\int_0^T\int_0^t\left\langle\frac{\partial\Big(\widetilde{T}^{\sigma}_n(b(u)) - \widetilde{T}^{\sigma}_n(b(v))\Big)}{\partial t}; T_k(\widetilde{T}^{\sigma}_n(b(u)) - \widetilde{T}^{\sigma}_n(b(v))\Big)\right\rangle dsdt$$
$$= \frac{1}{k}\int_{Q_T}\overline{T}_k(\widetilde{T}^{\sigma}_n(b(u)) - \widetilde{T}^{\sigma}_n(b(v)))dxdt,$$

where $\overline{T}_k(r) = \int_0^r T_k(z)dz$.

Passing to the limit we obtain

$$\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}\int_{Q_T}\overline{T}_k(\widetilde{T}^{\sigma}_n(b(u)) - \widetilde{T}^{\sigma}_n(b(v)))dxdt$$
$$= \int_{Q_T}|\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))|dxdt,$$

and letting $n \to +\infty$ in this last equality, we deduce (58). □

Now, we analyze the limit of $I^\sigma_{1,n}$, $I^\sigma_{2,n}$, $I^\sigma_{3,n}$, $I^\sigma_{4,n}$ and $I^\sigma_{5,n}$ one by one.

The limit of $I^\sigma_{1,n}$: Notice that

$$I^\sigma_{1,n} = \frac{1}{k}\int_0^T \int_0^t \int_\Omega Q^\sigma_n dxdsdt = \frac{1}{k}\int_{Q_T} (T-t) Q^\sigma_n dxdt,$$

where $Q^\sigma_n = (\widetilde{T}^\sigma_n)'(b(u))a(u,\nabla u) - (\widetilde{T}^\sigma_n)'(b(v))a(v,\nabla v)]\nabla T_k(\widetilde{T}^\sigma_n(b(u)) - \widetilde{T}^\sigma_n(b(v)))$.

Since $\operatorname{supp}((\widetilde{T}^\sigma_n)') \subset [b(-n)-\sigma, b(n)+\sigma]$,
then

$$(\widetilde{T}^\sigma_n)'(b(u))a(u,\nabla u) = (\widetilde{T}^\sigma_n)'(b(u))a(T_{n+1}(u), \nabla T_{n+1}(u))$$

and

$$(\widetilde{T}^\sigma_n)'(b(v))a(v,\nabla v) = (\widetilde{T}^\sigma_n)'(b(u))a(T_{n+1}(v), \nabla T_{n+1}(v)).$$

Then by (55), (56) and (54) one has

$$\begin{cases} Q^\sigma_n \quad \text{converges to} \quad [\chi_{\{|u|\le n\}}a(u,\nabla u) - \chi_{\{|v|\le n\}}a(v,\nabla v)]\nabla T_k(\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))), \\ |Q^\sigma_n| \le C_n^2 b_1[|a(T_{n+1}(u),\nabla T_{n+1}(u))| + |a(T_{n+1}(v),\nabla T_{n+1}(v))|] \\ \qquad\qquad \times(|\nabla T_{n+1}(u)| + |\nabla T_{n+1}(v)|)\chi_{\{|\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))|\le k\}} = R_n. \end{cases}$$

where $C_n = \max(|b(-n)-\sigma|, b(n)+\sigma)$.
Since $R_n \in L^1(Q_T)$ we use the Lebesgue dominated convergence theorem to have

$$\begin{aligned} \lim_{\sigma\to 0} I^\sigma_{1,n} &= \lim_{\sigma\to 0}\frac{1}{k}\int_{Q_T}(T-t)Q^\sigma_n dxdt \\ &= \frac{1}{k}\int_{Q_T}(T-t)[\chi_{\{|u|\le n\}}a(u,\nabla u) \\ &\qquad -\chi_{\{|v|\le n\}}a(v,\nabla v)]\nabla T_k(T_n(b(u)) - T_n(b(v)))dxdt \\ &= J_1+J_2+J_3+J_4+J_5, \end{aligned} \tag{59}$$

where

$$J_1 = \frac{1}{k}\int_{\{|b(u)-b(v)|\le k, |u|\le n, |v|\le n\}}(T-t)\Big(a(u,\nabla u) - a(u,\nabla v)\Big)(\nabla u - \nabla v)b'(u)dxdt,$$
$$J_2 = \frac{1}{k}\int_{\{|b(u)-b(v)|\le k, |u|\le n, |v|\le n\}}(T-t)\Big(a(u,\nabla v) - a(v,\nabla v)\Big)(\nabla u - \nabla v)b'(u)dxdt,$$
$$J_3 = \frac{1}{k}\int_{\{|b(u)-b(v)|\le k, |u|\le n, |v|\le n\}}(T-t)\Big(a(u,\nabla u) - a(v,\nabla v)\Big)\nabla v(b'(u) - b'(v))dxdt,$$
$$J_4 = \frac{1}{k}\int_{\{|\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))|\le k, |u|>n, |v|\le n\}}(T-t)a(v,\nabla v)\nabla b(v)dxdt,$$

$$J_5 = \frac{1}{k}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k,|u|\le n,|v|>n\}}(T-t)a(u,\nabla u)\nabla b(u)dxdt.$$

Since $a(u, \nabla u)$ satisfies the condition (11), one has

$$J_1 \ge 0. \tag{60}$$

On the other hand by (51) we have

$$|J_2| \le \frac{b_1}{k}T\int_{Q_T}\chi_{\{|b(u)-b(v)|\le k\},|u|\le n,|v|\le n\}}|u-v|\Big(L_D(x,t)+\rho_D\overline{P}^{-1}P(|v|)\Big)$$
$$\times(|\nabla u|+|\nabla v|)dxdt.$$

Using (8), one has $|u-v| \le \frac{1}{b_0}|b(u)-b(v)| \le \frac{1}{b_0}$, then

$$|J_2| \le \frac{b_1}{b_0}T\int_{\{|b(u)-b(v)|\le k\}}\Big(L_D(x,t)+\rho_D\overline{P}^{-1}P(|v|)\Big)(|\nabla u|+|\nabla v|)dxdt.$$

Since $L_D(x,t) \in E_{\overline{M}}(Q_T)$, u and v in $W^{1,x}L_M(Q_T)$ and using (4), one has $(L_D(x,t)+\rho_D\overline{P}^{-1}P(|v|))(|\nabla u|+|\nabla v|) \in L^1(Q_T)$ and the Lebesgue dominated convergence theorem allows us to conclude that for all $n \ge 1$

$$\limsup_{k\to 0} J_2 = 0. \tag{61}$$

We denote by C_n the compact subset $[-n-1, n+1]$. Due to (53), there exists a positive number β_n such that $|b'(r_1)-b'(r_2)| \le \beta_n|r_1-r_2|$ for all r_1 and r_2 lying in C_n. Using (8) and Rolle's theorem, we get

$$|b'(r_1)-b'(r_2)| \le \frac{\beta_n}{b_0}|b(r_1)-b(r_2)|.$$

Then $|b'(r_1)-b'(r_2)| \le k\frac{\beta_n}{b_0}$ on $\{|b(u)-b(v)| \le k, |u| \le n, |v| \le n\}$ and we get

$$|J_3| \le \frac{\beta_n}{b_0}T\int_{Q_T}\Big(a(T_n(u),\nabla T_n(u))-a(T_n(v),\nabla T_n(v))\Big)|\nabla v|\chi_{\{|b(u)-b(v)|\le k,|u|\le n,|v|\le n,u\ne v\}}dxdt.$$

$$\text{Since}\ \begin{cases}\lim\limits_{k\to 0}\chi_{\{|b(u)-b(v)|\le k,|u|\le n,|v|\le n,u\ne v\}} = 0 \quad \text{a.e. in} \quad Q_T,\\ |a(T_n(u),\nabla T_n(u))-a(T_n(v),\nabla T_n(v))||\nabla T_n(v)| \in L^1(Q_T),\end{cases}$$

we use the Lebesgue dominated convergence theorem to conclude that, for all $n \ge 1$,

$$\lim_{k\to 0} J_3 = 0. \tag{62}$$

In view of the definition of $\widetilde{T}_n$, we have

$$J_4 = \frac{1}{k} \int_{\substack{\{b(n)-k \le b(v) \le b(n)\} \\ \cup\{b(-n) \le b(v) \le b(-n)+k\}}} (T-t) a(v, \nabla v) \nabla b(v) dx dt,$$

and using (11) we deduce

$$\liminf_{n\to+\infty} \limsup_{k\to 0} J_4 \ge 0. \tag{63}$$

Similarly we have

$$J_5 = \frac{1}{k} \int_{\substack{\{b(n)-k \le b(u) \le b(n)\} \\ \cup\{b(-n) \le b(u) \le b(-n)+k\}}} (T-t) a(u, \nabla u) \nabla b(u) dx dt,$$

and

$$\liminf_{n\to+\infty} \limsup_{k\to 0} J_5 \ge 0. \tag{64}$$

Now from (59)–(64) we obtain

$$\liminf_{n\to+\infty} \limsup_{k\to 0} \lim_{\sigma\to 0} I_{1,n}^{\sigma} \ge 0. \tag{65}$$

The limit of $I_{2,n}^{\sigma}$ *and* $I_{4,n}^{\sigma}$: Now we claim that

$$|I_{2,n}^{\sigma}| + |I_{4,n}^{\sigma}| \le \frac{T}{\sigma} \Gamma(u, v, n, \sigma). \tag{66}$$

From a simple derivation of the function $(\widetilde{T}_n^{\sigma})'$ it yields that for any $\sigma > 0$ and $k > 0$

$$\begin{aligned} |I_{2,n}^{\sigma}| \le \frac{T}{\sigma} & \int_{\substack{\{b(n)-\sigma \le b(u) \le b(n)\} \\ \cup\{b(-n) \le b(u) \le b(-n)+\sigma\}}} a(u, \nabla u)) \nabla b(u) dx dt \\ & + \frac{T}{\sigma} \int_{\substack{\{b(n)-\sigma \le b(v) \le b(n)\} \\ \cup\{b(-n) \le b(v) \le b(-n)+\sigma\}}} a(v, \nabla v)) \nabla b(v) dx dt. \end{aligned} \tag{67}$$

Similarly we have

$$\begin{aligned} |I_{4,n}^{\sigma}| \le \frac{T}{\sigma} & \int_{\substack{\{b(n)-\sigma \le b(u) \le b(n)\} \\ \cup\{b(-n) \le b(u) \le b(-n)+\sigma\}}} \Phi(u) \nabla b(u) dx dt \\ & + \frac{T}{\sigma} \int_{\substack{\{b(n)-\sigma \le b(v) \le b(n)\} \\ \cup\{b(-n) \le b(v) \le b(-n)+\sigma\}}} \Phi(v) \nabla b(v) dx dt. \end{aligned} \tag{68}$$

By combining (67) and (68) we readily deduce (66).

The limit of $I^{\sigma}_{3,n}$: There we prove that

$$\limsup_{\sigma\to 0}|I^{\sigma}_{3,n}| \le \frac{T}{k}\Gamma(u,v,n,k)+\epsilon(k), \tag{69}$$

where $\epsilon(k)$ is a positive function such that $\lim_{k\to 0}\epsilon(k)=0$.

For $n\ge 0$ we have

$$\begin{aligned}\limsup_{\sigma\to 0}|I^{\sigma}_{3,n}| = &\Big|\frac{1}{k}\int_{Q_T}(T-t)(\chi_{\{|u|\le n\}}\Phi(u)\\ &-\chi_{\{|v|\le n\}}\Phi(v))\nabla T_k(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v)))dxdt\Big|.\\ \le\ & K_1+K_2+K_3,\end{aligned}$$

where

$$\begin{aligned}K_1 &= \frac{T}{k}\int_{Q_T}\chi_{\{|u|\le n,|v|>n\}}|\Phi(u)||\nabla T_k(\widetilde{T}_n(b(u))-b(n.\operatorname{sgn}(v))|dxdt,\\ K_2 &= \frac{T}{k}\int_{Q_T}\chi_{\{|u|>n,|v|\le n\}}|\Phi(v)||\nabla T_k(\widetilde{T}_n(b(v))-b(n.\operatorname{sgn}(u))|dxdt,\\ K_3 &= \frac{T}{k}\int_{Q_T}\chi_{\{|u|\le n,|v|\le n\}}|\Phi(u)-\Phi(v)||\nabla T_k(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v)))|dxdt.\end{aligned}$$

We estimate K_1 and K_2 by (12) we have

$$\begin{aligned}K_1 &\le \frac{T}{k}\int_{Q_T}\chi_{\{|u|\le n,|v|>n\}}\chi_{\{|b(u)-b(n.\operatorname{sgn}(v))|\le k\}}|\Phi(u)||\nabla b(u)|dxdt\\ &\le \frac{T}{k}\int_{\substack{\{b(n)-k\le b(u)\le b(n)\}\\ \cup\{b(-n)\le b(u)\le b(-n)+k\}}}|\Phi(u)||\nabla b(u)|dxdt,\end{aligned} \tag{70}$$

and similarly

$$K_2 \le \frac{T}{k}\int_{\substack{\{b(n)-k\le b(v)\le b(n)\}\\ \cup\{b(-n)\le b(v)\le b(-n)+k\}}}|\Phi(v)||\nabla b(v)|dxdt. \tag{71}$$

On the other hand, by (53) one have since $L_D\in L_{\overline{M}}(Q_T)$,

$$K_3 \le \frac{T}{k}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k\}\cap\{|u|\le n,|v|\le n\}}L_D(x,t)|u-v||\nabla T_k(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v)))|dxdt$$

by (8) we obtain

$$K_3 \le \frac{T}{k}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k\}\cap\{|u|\le n,|v|\le n\}}$$

$$\frac{1}{b_0} L_D(x,t)|b(u)-b(v)||\nabla T_k(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v)))|dxdt$$
$$= \frac{T}{k}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k\}\cap\{|u|\le n,|v|\le n\}} \frac{T}{b_0} L_D(x,t)|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))||\nabla T_k(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v)))|dxdt$$
$$\le \frac{T}{b_0}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k\}\cap\{|u|\le n,|v|\le n\}} L_D(x,t)(|\nabla\widetilde{T}_n(b(u))|+|\nabla\widetilde{T}_n(b(v))|)dxdt = \epsilon(k).$$

Since L_D in $L_{\overline{M}}(Q_T)$ and due to (16), the function $L_D(x,t)(|\nabla T_n(b(u))| + |\nabla T_n(b(v))|) \in L^1(Q_T)$. Using the Lebesgue dominated convergence theorem we obtain $\lim_{k\to 0}\epsilon(k) = 0$ and

$$\lim_{k\to 0}|K_3| = 0. \tag{72}$$

Estimates (70)–(72) imply (69).

The limit of $I^{\sigma}_{5,n}$:

Using the Lebesgue theorem and (55) and (56), we obtain

$$\lim_{\sigma\to 0}|I^{\sigma}_{5,n}| \le \frac{T}{k}\int_{Q_T}|T_k(\widetilde{T}_n(b(u)-\widetilde{T}_n(b(v))| \times |f||\chi_{\{|u|\le n\}}-\chi_{\{|v|\le n\}}|dxdt.$$

Since $\lim_{k\to 0}\frac{T_k(z)}{k} = \operatorname{sgn}(z)$ in $I\!R$ and weakly-* in L^∞ then

$$\lim_{k\to 0}\lim_{n\to+\infty}\lim_{\sigma\to 0}|I^{\sigma}_{5,n}| \le T\lim_{n\to+\infty}\left(\int_{\{|u|\ge n\}}|f|dxdt + \int_{\{|v|\ge n\}}|f|dxdt\right) = 0.$$

Then

$$\lim_{k\to 0}\liminf_{n\to+\infty}\lim_{\sigma\to 0} I^{\sigma}_{5,n} = 0. \tag{73}$$

Finally, going back to (57) and using Lemma 8, one may deduce that $u = v$ a.e. in Q_T.

Appendix

Proof of Lemma 8 Define the functions

$$L_1(s) = \int_{\{0<b(u)<s\}}\Big(a(u,\nabla u)\nabla u + |\Phi(u)||\nabla u|\Big)dxdt + \int_{\{0<b(v)<s\}}\Big(a(v,\nabla v)\nabla v + |\Phi(v)||\nabla v|\Big)dxdt, \tag{74}$$

and

$$L_2(s) = \int_{\{-s<b(u)<0\}} \Big(a(u,\nabla u)\nabla u + |\Phi(u)||\nabla u|\Big)dxdt + \int_{\{-s<b(v)<0\}} \Big(a(v,\nabla v)\nabla v + |\Phi(v)||\nabla v|\Big)dxdt. \tag{75}$$

Due to (10) the functions L_1 and L_2 are monotone increasing. L_1 and L_2 are differentiable almost everywhere (see [28]), with L_1' and L_2' measurable and so we have for any $s > \eta > 0$

$$L_1(s) - L_1(\eta) \geq \int_\eta^s L_1'(\xi)d\xi \quad \text{and} \quad L_2(s) - L_2(\eta) \geq \int_\eta^s L_2'(\xi)d\xi, \tag{76}$$

and for almost any $s > 0$

$$L_1'(s) = \frac{1}{2}\limsup_{k\to 0}\frac{1}{k}\Bigg[\int_{\{s-k<b(u)<s+k\}} \Big(a(u,\nabla u)\nabla u + |\Phi(u)||\nabla u|\Big)dxdt + \int_{\{s-k<b(v)<s+k\}} \Big(a(v,\nabla v)\nabla v + |\Phi(v)||\nabla v|\Big)dxdt\Bigg], \tag{77}$$

and

$$L_2'(s) = \frac{1}{2}\limsup_{k\to 0}\frac{1}{k}\Bigg[\int_{\{-s-k<b(u)<-s+k\}} \Big(a(u,\nabla u)\nabla u + |\Phi(u)||\nabla u|\Big)dxdt + \int_{\{-s-k<b(v)<-s+k\}} \Big(a(v,\nabla v)\nabla v + |\Phi(v)||\nabla v|\Big)dxdt\Bigg]. \tag{78}$$

If the thesis of the lemma is not true, let $\epsilon_0 > 0$ and let $n_0 > 0$ be a real number such that for every real number $s \geq n_0$ we have

$$\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k) \geq \epsilon_0. \tag{79}$$

Since b' is a continuous and positive function, we have for almost $\xi \geq n_0$,

$$\begin{aligned}
&\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k) \\
&\leq b'(s)\limsup_{k\to 0}\frac{1}{k}\Bigg[\int_{\{b(s)-k<b(u)<b(s)+k\}} \Big(a(u,\nabla u)\nabla u + |\Phi(u)||\nabla u|\Big)dxdt \\
&\quad + \int_{\{b(s)-k<b(v)<b(s)+k\}} \Big(a(v,\nabla v)\nabla v + |\Phi(v)||\nabla v|\Big)dxdt\Bigg]
\end{aligned}$$

$$+b'(-s)\limsup_{k\to 0}\frac{1}{k}\left[\int_{\{b(-s)-k<b(u)<b(-s)+k\}}\Big(a(u,\nabla u)\nabla u+|\Phi(u)||\nabla u|\Big)dxdt\right.$$
$$\left.+\int_{\{b(-s)-k<b(v)<b(-s)+k\}}\Big(a(v,\nabla v)\nabla v+|\Phi(v)||\nabla v|\Big)dxdt\right]. \tag{80}$$

From (77), (79) and (78) it follows that

$$b'(\xi)L_1'(b(\xi))+b'(-\xi)L_2'(-b(-\xi))\geq\frac{\epsilon_0}{2}.$$

In view of (76), we deduce that for any $s>\eta>n_0$ we have

$$L_1(b(s))-L_1(b(\eta))+L_2(-b(-s))-L_2(-b(-\eta))\geq\frac{\epsilon_0}{2}(s-\eta). \tag{81}$$

Taking $s=n+1$ and $\eta=n$ with $n>n_0$ we have

$$\int_{\{n\leq|u|\leq n+1\}}\Big(a(u,\nabla u)\nabla u+|\Phi(u)||\nabla u|\Big)dxdt$$
$$+\int_{\{n\leq|v|\leq n+1\}}\Big(a(v,\nabla v)\nabla v+|\Phi(v)||\nabla v|\Big)dxdt\geq\frac{\epsilon_0}{2}.$$

The last inequality contradicts (17) and (47). □

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