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On the modeling of the flow of the Antarctic Circumpolar Current

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Abstract A recent model for the flow of the Antarctic Circumpolar Current, formulated in spherical coordinates as a Dirichlet boundary-value problem for a nonlinear elliptic partial differential equation, reduces for flows with no azimuthal variations to a two-point boundary-value problem for a second-order ordinary differential equation. We provide some general settings for which these apparently simpler solutions are the unique solutions, due to an inherent symmetry of the model.

Keywords Geophysical flow · Elliptic equation · Maximum principle

Mathematics Subject Classification 4B15 · 35J15 · 37N10

1 Introduction

The Antarctic Circumpolar Current (ACC) is one of the most significant ocean currents and the only current that completely encircles the polar axis, flowing eastward through the southern regions of the Atlantic, Indian, and Pacific Oceans (see [3]), being about 23,000 km long, and with a width in excess of 800 km (minimum attained in the region of the Drake Passage). Unlike other ocean currents, the effect of the ACC is felt from the ocean surface down to the ocean floor (with depths as much as 4–5 km). Because the ACC is linked to the three major oceans, it is important in global ocean circulation and climate (see the discussion in [1]). The wave-current interactions in the Southern

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Ocean create some of the largest surface ocean waves on Earth ('monster waves'—see the discussion in [14]).

A recent model for ocean gyres in spherical coordinates, developed in [4], expresses the horizontal velocity components of the ACC flow as

$$\frac{1}{\sin\theta}\,\psi_\varphi \quad \text{and} \quad -\,\psi_\theta, \tag{1}$$

in terms of the stream function $\psi(\theta,\varphi)$, where $\theta \in [0,\pi)$ is the polar angle (with $\theta = 0$ corresponding to the North Pole) and $\varphi \in [0, 2\pi)$ is the angle of longitude (or azimuthal angle), the vertical velocity component being much smaller (by a factor of about 10^{-4}). The governing equation is

$$\frac{1}{\sin^2\theta}\,\psi_{\varphi\varphi} + \psi_\theta \cot\theta + \psi_{\theta\theta} - 2\omega\cos\theta = F(\psi), \tag{2}$$

where $\omega > 0$ is the non-dimensional form of the Coriolis parameter (generating the spin vorticity $2\omega\cos\theta$) and where $F(\psi)$ is the oceanic vorticity. In (2), the type of ocean flow dictates the nature of the function F; for example, wind-driven flows are typically modeled by constant (non-zero) oceanic vorticity F (see [6] for data for wind-driven flows, [2] for a general discussion of vorticity in water flows, and [5] for some recent developments on geophysical wave-current interactions). The governing Eq. (2) holds in a region whose boundary consists of a streamline (a level set of ψ).

Using the stereographic projection of the unit sphere centered at at origin from the North Pole to the equatorial plane,

$$\xi = r\,e^{i\,\varphi} \quad \text{with} \quad r = \cot\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1-\cos\theta}, \tag{3}$$

where (r,φ) are the polar coordinates in the equatorial plane, the model (2) in spherical coordinates can be transformed into the equivalent planar elliptic partial differential equation

$$\Delta\psi + 8\omega\,\frac{1-(x^2+y^2)}{(1+x^2+y^2)^3} - \frac{4F(\psi)}{(1+x^2+y^2)^2} = 0, \tag{4}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator expressed in terms of the Cartesian coordinates (x, y) in the equatorial plane (see the discussion in [10]). Since the ACC lies in the region between the 56th and 60th parallel south, which crosses nothing but ocean, and this region is mapped by the stereographic projection into an annular region of the equatorial plane, to (4) we associate the Dirichlet boundary conditions

$$\begin{cases} \psi = \psi_e \ \text{ on } r = r_-, \\ \psi = \psi_p \ \text{ on } r = r_+, \end{cases} \tag{5}$$

for suitable constants r_- and r_+ with $0 < r_- < r_+ < 1$; here $r = \sqrt{x^2+y^2} \in (0,1)$ is the radius of the stereographic projection of a circle of latitude in the southern hemisphere. In (5), the constants ψ_i and ψ_e are the values assigned to the two streamlines that represent the boundary of the ACC (one closer to the Equator and the other closer to the South Pole). The aim of this paper is to show that for a large class of oceanic vorticities F, the solution of (4)–(5) is radially symmetric. The physical interpretation of this result is that the flow is uniform in the azimuthal direction (independent of φ in spherical coordinates).

2 Flows with no azimuthal variations

A flow with no variation in the azimuthal direction corresponds to a radially symmetric solution $\psi = \psi(r)$ of the elliptic boundary-value problem (4)–(5). Setting

$$0 < t_1 = -\ln(r_+) < t_2 = -\ln(r_-),$$

the change of variables $r = e^{-t/2}$ and

$$\psi(r) = u(t), \qquad t_1 < t < t_2, \tag{6}$$

transforms (4) to the second-order differential equation

$$u''(t) - \frac{e^t}{(1+e^t)^2}\, F(u(t)) + \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} = 0, \qquad t_1 < t < t_2, \tag{7}$$

with the boundary conditions

$$\begin{cases} u(t_1) = \psi_p, \\ u(t_2) = \psi_e. \end{cases} \tag{8}$$

Explicit solutions to the boundary-value problem (7)–(8) were provided for constant oceanic vorticity F in [10], and some existence result for a large class of nonlinear functions F were recently obtained in [11].

3 A general symmetry result

In this section we will prove the main result of this paper.

Theorem *Assume that the function $F : \mathbb{R} \to \mathbb{R}$ is continuous and non-decreasing. Then, given ψ_p and ψ_e, if ψ solves (4)–(5) then ψ is radially symmetric, with $\psi(r) = u(t)$ for $r = e^{-t/2}$ and u solution to (7)–(8).*

Proof Let ψ_0 be the solution of (4)–(5) related by means of (6) to the solution $u(t)$ of (7)–(8) and set $\Psi = \psi - \psi_0$. Then

$$\Delta\Psi - \frac{4[F(\psi) - F(\psi_0)]}{(1+x^2+y^2)^2} = 0, \qquad r_- < \sqrt{x^2+y^2} < r_+, \tag{9}$$

and

$$\begin{cases} \Psi = 0 \text{ for } x^2+y^2 = r_-^2, \\ \Psi = 0 \text{ for } x^2+y^2 = r_+^2. \end{cases} \tag{10}$$

Multiplying (9) by Ψ and integrating over the annulus

$$\mathcal{A} = \{(x, y) \in \mathbb{R}^2 : r_- < \sqrt{x^2 + y^2} < r_+\}, \tag{11}$$

Green's first identity (see [13]) yields

$$4 \iint_{\mathcal{A}} \frac{[F(\psi) - F(\psi_0)]\Psi}{(1 + x^2 + y^2)^2}\, dxdy + \iint_{\mathcal{A}} |\nabla \Psi|^2\, dxdy = 0.$$

Since by hypothesis

$$[F(\psi) - F(\psi_0)]\Psi = [F(\psi) - F(\psi_0)]\{\psi - \psi_0\} \geq 0,$$

we deduce that Ψ is a constant, and due to (10) that constant can only be zero. The proof is complete. □

Remark (i) If F is continuously differentiable, then the conclusion of the above theorem can be reached alternatively by combining the mean-value theorem with maximum principles since (9) yields

$$\Delta \Psi - \frac{4F'(\tilde{\psi})}{(1 + x^2 + y^2)^2}\, \Psi = 0, \qquad r_- < \sqrt{x^2 + y^2} < r_+, \tag{12}$$

after determining $\tilde{\psi}$ with $F(\psi) - F(\psi_0) = F'(\tilde{\psi})[\psi - \psi_0)]$. Indeed, regarding (12) as a linear equation in Ψ, the maximum principle yields $\psi = \psi_0$ throughout the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ (see the discussion of elliptic boundary-value problems in the appendix of Chapter 3 in [2]).

(ii) Since for linear functions $F(u) = au + b$ we have $F(\psi) - F(\psi_0) = a\Psi$, the above considerations indicate what may go wrong if the monotonicity assumption on F is not satisfied. Note that the rotation symmetry of the circular annulus $\mathcal{A}$ allows us use polar coordinates and separation of variables to obtain an sequence of positive eigenvalues $\lambda_{kn} > 0$ for homogeneous boundary conditions on the boundary $\partial\mathcal{A}$ of $\mathcal{A}$:

$$\begin{cases} \Delta U_{kn} + \lambda_{kn} U_{kn} = 0 & \text{in } \mathcal{A}, \\ U_{kn} = 0 & \text{on } \partial\mathcal{A}. \end{cases}$$

For example, an explicit representation of U_{kn} in terms of a linear combination of the first and second kind Bessel functions of order $n \geq 1$, multiplied by $\cos(n\varphi)$ and $\sin(n\varphi)$, with the index $k \geq 1$ keeping track of the coefficients and scales of the Bessel functions (dependent on the radial variable r), is available in [8]. By means of a variational approach (see [13]), one can show that there are positive eigenvalues $\lambda > 0$ of the problem

$$\begin{cases} \Delta U + \dfrac{4\lambda}{(1 + r^2)^2}\, U = 0 & \text{in } \mathcal{A}, \\ U = 0 & \text{on } \partial\mathcal{A}. \end{cases}$$

Consequently, for $F(u) = -\lambda u$, the problem (4)–(5) does not have a unique solution that is radially symmetric. □

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