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Discrepancy estimates for generalized polynomials

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Abstract We obtain an upper bound for the discrepancy of the sequence $([p(n)\alpha]\beta)_{n\geq 0}$ generated by the generalized polynomial $[p(x)\alpha]\beta$, where $p(x)$ is a monic polynomial with real coefficients, α and β are irrational numbers satisfying certain conditions.

Keywords Discrepancy · Generalized polynomial · Irrationality measure

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1 Introduction

A sequence $(x_n)_{n\geq 0}$ of real numbers is said to be uniformly distributed modulo 1 if

$$\lim_{N\to\infty} \frac{\#\{n \leq N : \{x_n\} \in [a, b)\}}{N} = b - a \tag{1}$$

holds for all real numbers a, b satisfying $0 \leq a < b \leq 1$. Here and in what follows, $\{x\}$ denotes the fractional part of x. Weyl [10] proved that if $P(x) \in \mathbb{R}[x]$ is any polynomial in which at least one of the coefficients other than the constant term is irrational, then the sequence $(P(n))_{n\geq 0}$ is uniformly distributed modulo 1.

A natural extension of the family of real valued polynomials arises by adding the operation integral part, denoted by $[\cdot]$, to the arithmetic operations addition and multiplication. Polynomials which can be obtained in this way are called generalized polynomials. For example $[a_0+a_1x]$, $a_0+[a_1x+[a_2x^2]]$ are generalized polynomials.

In the spirit of Weyl's result it is natural to consider the uniform distribution of generalized polynomials. The case $([n\alpha]\beta)_{n\geq 0}$ is treated in [8] (see Theorem 1.8, p. 310) and it follows from a result of Veech (see Theorem 1, [9]) that the sequence $([p(n)]\beta)_{n\geq 0}$, $p(x)$ is a polynomial with real coefficients, is uniformly distributed under certain conditions on the coefficients of $p(x)$ and β. Håland [4,5] showed that if the coefficients of a generalized polynomial $q(x)$ are sufficiently independent then the sequence $(q(n))_{n\geq 0}$ is uniformly distributed.

In order to quantify the convergence in (1) the notion of discrepancy has been introduced. Let $(x_n)_{n\geq 0}$ be a sequence of real numbers and N be any positive integer. The discrepancy of this sequence, denoted by $D_N(x_n)$, is defined by

$$D_N(x_n) = \sup_{0\leq a<b\leq 1} \left| \frac{\#\{n \leq N : \{x_n\} \in [a, b)\}}{N} - (b - a) \right|.$$

Now we have the following definition.

Definition 1 Let $t \geq 1$ be a real number. We say that a pair (α, β) of real numbers is of *finite type t* if for each $\epsilon > 0$ there is a positive constant $c = c(\epsilon, \alpha, \beta)$ such that for any pair of rational integers $(m, n) \neq (0, 0)$, we have

$$(\max(1, |m|))^{t+\epsilon}(\max(1, |n|))^{t+\epsilon}\|m\alpha + n\beta\| \geq c$$

where $\|x\|$ denotes the distance of x from the nearest integer.

The corresponding definition for a single real number α is the one of *irrationality measure*. The precise definition is the following.

Definition 2 Let $t \geq 1$ be a real number. We say that an irrational number γ has *irrationality measure* $t + 1$ if for any integer n and $\epsilon > 0$, we have

$$\max(1, |n|)^{t+\epsilon}\|n\gamma\| \gg_{\epsilon,\gamma} 1.$$

It is well known that when γ has irrationality measure $t+1$, the discrepancy $D_N(n\gamma)$ of the sequence $(n\gamma)_{n\geq 0}$ satisfies

$$D_N(n\gamma) \ll_{\gamma,\epsilon} N^{\frac{-1}{t}+\epsilon}$$

for each $\epsilon > 0$.

The discrepancy of non-trivial generalized polynomials was first considered by Hofer and Ramaré [6]. More precisely, they considered the discrepancy of the sequence $([n\alpha]\beta)_{n\geq 0}$ and proved that for each $\epsilon > 0$

$$D_N([n\alpha]\beta) \ll_{\epsilon,\alpha,\beta} N^{\frac{-1}{3t-2}+\epsilon}$$

when $(\alpha, \alpha\beta)$ and $(\beta, \frac{1}{\alpha})$ are of finite type t.

Let $p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a monic polynomial of degree $d \geq 2$. In this paper we consider the discrepancy of the sequence $([p(n)\alpha]\beta)_{n\geq 0}$. We prove the following theorem.

Theorem 1 *Let α, β and $N > 1$ be non-zero real numbers. Suppose that the pair $(\alpha, \alpha\beta)$ is of finite type t for a real number $t \geq 1$. Then for any $\epsilon > 0$,*

$$D_N([p(n)\alpha]\beta) \ll_{\epsilon,\alpha,\beta,d} N^{-\frac{2-2^{-d+2}}{2^{d-1}(2t+1)+7t+2}+\epsilon}.$$

We use a modified version of the method of Hofer and Ramaré [6] for the proof of the above theorem.

Remark 1 The above theorem, in particular, shows that the sequence $([p(n)\alpha]\beta)$ is uniformly distributed if $(\alpha, \alpha\beta)$ is of finite type t for $t \geq 1$. This fact also follows from a theorem of Carlson (see Theorem 2, [2]). Theorem 1 of [9] also implies uniform distribution of this sequence under certain conditions on the coefficients of the polynomial $p(x)$.

2 Preliminaries

For any real number τ, let $f_\tau(x) = e(\tau\{x\})$ where $e(x)$ denotes $e^{2\pi i x}$. Let $\delta > 0$ be a real number. We are going to approximate f_τ by a function $g_{\tau,\delta}$. Here $g_{\tau,\delta}$ is defined by

$$g_{\tau,\delta}(x) = \frac{1}{(2\delta)^r} 1_{[-\delta,\delta]} * \cdots * 1_{[-\delta,\delta]} * f_\tau(x), \tag{2}$$

where we have r copies of $1_{[-\delta,\delta]}$ each denoting the indicator function of the interval $[-\delta, \delta]$.

We have the following analog of Lemma 3.1 in [6].

Lemma 1 *For any sequence $\{u_n\}_{n\geq 0}$ of real numbers, and any positive integer N we have*

$$\sum_{n=0}^{N-1} |f_\tau(u_n) - g_{\tau,\delta}(u_n)| \ll Nr\delta + Nr^2\delta|\tau| + ND_N(u_n). \tag{3}$$

Using Fourier inversion formula, we have

$$g_{\tau,\delta}(x) \;=\; \sum_{k\in\mathbb{Z}} \hat{g}_{\tau,\delta}(k)e(-kx)\;,$$

with

$$\hat{g}_{\tau,\delta}(k) \;=\; \left(\frac{\sin 2\pi k\delta}{2\pi k\delta}\right)^r \frac{e(\tau+k)-1}{2\pi i(k+\tau)}.$$

Since $\left|\frac{\sin 2\pi x}{x}\right|^r \ll_r \min\left(1, \frac{1}{|x|^r}\right)$, and for any irrational τ, $|e(\tau)-1| \ll \|\tau\|$, we have the following lemma which holds trivially.

Lemma 2 *For any irrational number τ, we have*

$$|\hat{g}_{\tau,\delta}(k)| \ll_r \frac{\|\tau+k\|}{|\tau+k|}\min\left(1, \frac{1}{(|k|\delta)^r}\right).$$

We will state Lemmas 3 and 4 for arbitrary real number τ but we keep in mind that we will use these lemmas with $\tau = -h\beta$, for some positive integer h. The next lemma gives an upper bound for the tail of the Fourier series of $g_{\tau,\delta}$.

Lemma 3 *Let K be sufficiently large real number such that $|\tau+k| \geq \frac{k}{2}$ for all $k \in \mathbb{Z}$ with $|k| > K$. Then we have*

$$\sum_{|k|>K} \hat{g}_{\tau,\delta}(k) \ll_r (\delta K)^{-r}.$$

The following lemma shows that for any $p > 1$ the L^p-norm of $\hat{g}_{\tau,\delta}$ is bounded.

Lemma 4 *Let τ be a real number and $0 < \delta < \min\left(\frac{1}{2|\tau|}, 1\right)$. Then for any real number $p > 1$, we have*

$$\sum_{k\in\mathbb{Z}} |\hat{g}_{\tau,\delta}(k)|^p \ll 1\;,$$

where the implied constant depends only on p.

Proof We can assume the sum is running over $k \geq 1$. Using Lemma 2, we get

$$\begin{aligned}\sum_{k\geq 1} |\hat{g}_{\tau,\delta}(k)|^p &\leq \sum_{k\geq 1} \frac{||\tau+k||^p}{|\tau+k|^p}\min\left(1, \frac{1}{(k\delta)^{pr}}\right)\\ &= \sum_{k\leq \delta^{-1}} \frac{||\tau+k||^p}{|\tau+k|^p} + \delta^{-pr}\sum_{k>\delta^{-1}} \frac{1}{|\tau+k|^p k^{pr}}\;.\end{aligned}$$

Note that $k > \delta^{-1} > 2|\tau|$ implies $|\tau + k| \geq k/2$, hence

$$\sum_{k>\delta^{-1}} \frac{1}{|\tau + k|^p k^{pr}} \ll \delta^{p(r+1)-1} .$$

Hence we have

$$\sum_{k\geq 1} |\hat{g}_{\tau,\delta}(k)|^p \ll \sum_{k\leq \delta^{-1}} \frac{||\tau + k||^p}{|\tau + k|^p} + 1.$$

When τ is a non-negative real number, sum on the right hand side is clearly $\ll 1$. Hence we can assume that τ is a negative real number. The contributions for the sum above from the terms with $k = [-\tau]$ and $k = [-\tau] + 1$ are ≤ 1. Hence we have

$$\sum_{k\geq 1} |\hat{g}_{\tau,\delta}(k)|^p \ll S_1 + S_2 + 1 ,$$

where

$$S_1 = \sum_{k=1}^{[-\tau]-1} \frac{1}{|\tau + k|^p} \quad \text{and} \quad S_2 = \sum_{k=[-\tau]+2}^{\delta^{-1}} \frac{1}{|\tau + k|^p}.$$

Now the summand in S_1 is monotonically increasing, hence

$$S_1 = \int_1^{[-\tau]-1} \frac{dx}{(\tau + x)^p} + O\left(\frac{1}{(\tau + [-\tau] - 1)^p}\right) + O\left(\frac{1}{(\tau + 1)^p}\right).$$

It is easy to see that

$$\int_1^{[-\tau]-1} \frac{dx}{(\tau + x)^p} \ll 1,$$

as $p > 1$. Thus we conclude

$$S_1 \ll 1.$$

In a similar way, with only difference being the summand is monotonically decreasing, one can show that

$$S_2 \ll 1$$

which finishes the proof. □

Now we need a variant of a lemma of Weyl–van der Corput (see Lemma 2.7, [1]) as given by Granville and Ramaré (see Lemma 8.3 of [3]).

Lemma 5 *Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_N$ is a sequence of complex numbers, each with $|\lambda_i| \leq 1$, and define $\Delta\lambda_m = \lambda_m$, $\Delta_r\lambda_m = \lambda_{m+r}\overline{\lambda}_m$ and*

$$\Delta_{r_1,\ldots,r_k,s}\lambda_m \;=\; (\Delta_{r_1,\ldots,r_k}\lambda_{m+s})\overline{(\Delta_{r_1,\ldots,r_k}\lambda_m)}.$$

Then for any given $k \geq 1$, and real number $Q \in [1, N]$,

$$\left|\frac{1}{8N}\sum_{m=1}^{N}\lambda_m\right|^{2^k} \leq \frac{1}{8Q} + \frac{1}{8Q^{2-2^{-k+1}}}\sum_{r_1=1}^{Q}\sum_{r_2=1}^{Q^{\frac{1}{2}}}\cdots\sum_{r_k=1}^{Q^{2^{-k+1}}}\left|\frac{1}{N}\sum_{m=1}^{N-r_1-\cdots-r_k}\Delta_{r_1,\ldots,r_k}\lambda_m\right|.$$

The following lemma, often called as Erdős–Turán inequality, is very useful to estimate the discrepancy of a given sequence (see Theorem 2.5, p. 112 of [8]).

Lemma 6 (Erdős–Turán) *Let $(x_n)_{n\geq 0}$ be any sequence of real numbers and $N \geq 1$. The discrepancy $D_N(x_n)$ of the sequence $(x_n)_{n\geq 0}$ satisfies the following:*

$$D_N(x_n) \leq \frac{6}{H+1} + \frac{4}{\pi}\sum_{h=1}^{H}\frac{1}{h}\left|\frac{1}{N}\sum_{n=0}^{N-1}e(hx_n)\right|, \tag{4}$$

where H is any arbitrary positive integer.

The above lemma shows that the exponential sums play an important role not only in showing the uniform distribution of a sequence, but also in estimating the discrepancy of a given sequence.

The following lemma is an easy consequence of Lemma 6.

Lemma 7 *Let θ be an irrational number. Then the discrepancy $D_L(\ell\theta)$ of the sequence $\{\ell\theta : 1 \leq \ell \leq L\}$ satisfies the following upper bound.*

$$D_L(\ell\theta) \leq C\left(\frac{1}{H} + \frac{1}{L}\sum_{j=1}^{H}\frac{1}{j\|j\theta\|}\right)$$

for any $H > 1$ and for some absolute constant $C > 0$.

If α is of irrationality measure $t+1$ for $t \geq 1$, it is known that the discrepancy of $(n^2\alpha)$ satisfies the following upper bound.

$$D_N(n^2\alpha) \ll_{\epsilon,t} N^{-\frac{1}{t+1}+\epsilon} + N^{-\frac{2}{5}}\sqrt{\log N}$$

for any $\epsilon > 0$ (see equation (50) p. 113 in [7]). To estimate the discrepancy of $([p(n)\alpha]\beta)_{n\geq 0}$, we need the following general version.

Proposition 1 *Let α be a non-zero real number of irrationality measure $t+1$ for a real $t \geq 1$. Then the discrepancy $D_N(p(n)\alpha)$ of the sequence $(p(n)\alpha)_{n\geq 0}$ satisfies*

$$D_N(p(n)\alpha) \ll_{\epsilon,d,t} N^{-\frac{2-2^{-d+2}}{2^{d-1}(2t+1)}+\epsilon}$$

for any $\epsilon > 0$.

Proof Let $x_n = p(n)\alpha$ in Lemma 6. Then

$$D_N(p(n)\alpha) \ll \frac{1}{H} + \frac{1}{N}\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{n=0}^{N-1} e(p(n)h\alpha)\right|. \tag{5}$$

To estimate the exponential sum on the right hand side we use Lemma 5 with $Q = N$ and $k = d-1$. Hence we get that

$$\left|\sum_{n=0}^{N-1} e(p(n)h\alpha)\right|^{2^{d-1}} \ll N^{2^{d-1}-1}$$
$$+ N^{2^{d-1}+2^{-d+2}-3}\sum_{r_1=1}^{N}\cdots\sum_{r_{d-1}=1}^{N^{2^{-d+2}}}\left|\sum_{n=0}^{N-r_1-\cdots-r_{d-1}} e(d!hr_1\cdots r_{d-1}n\alpha)\right|.$$

Using the bound $|\sum_{n=0}^{N-1} e(n\lambda)| \ll \min(N, \frac{1}{\|\lambda\|})$ gives

$$\left|\sum_{n=0}^{N-1} e(p(n)h\alpha)\right|^{2^{d-1}}$$
$$\ll N^{2^{d-1}-1} + N^{2^{d-1}+2^{-d+2}-3}\sum_{r_1=1}^{N}\cdots\sum_{r_{d-1}=1}^{N^{2^{-d+2}}}\min\left(N, \frac{1}{\|d!hr_1\cdots r_{d-1}\alpha\|}\right)$$
$$\ll N^{2^{d-1}-1} + N^{2^{d-1}+2^{-d+2}-3}\sum_{m=1}^{N^{2-2^{-d+2}}} T(m)\min\left(N, \frac{1}{\|d!hm\alpha\|}\right), \tag{6}$$

where in the second line of the above inequality

$$T(m) = \left|\left\{(r_1,\ldots,r_{d-1}) \in [1,N]\times\cdots\times[1,N^{2^{-d+2}}] : r_1\cdots r_{d-1} = m\right\}\right|.$$

Hence $T(m) \ll \tau_{d-1}(m)$. Let $\epsilon_1 = \frac{\epsilon}{(2-2^{-d+2})}$. Using the fact that $\tau_{d-1}(m) \ll_{\epsilon_1} m^{\epsilon_1}$ we get that

$$\left|\sum_{n=0}^{N-1} e(p(n)h\alpha)\right|^{2^{d-1}} \ll_{\epsilon,d} N^{2^{d-1}-1} + N^{2^{d-1}+2^{-d+2}-3+\epsilon} \sum_{m=1}^{N^{2-2^{-d+2}}} \min\left(N, \frac{1}{\|d!hm\alpha\|}\right). \tag{7}$$

Let $L = N^{2-2^{-d+2}}$. We have

$$\sum_{m=1}^{L} \min\left(N, \frac{1}{\|d!mh\alpha\|}\right) = N|E_0| + \sum_{m\notin E_0} \frac{1}{\|d!mh\alpha\|} ,$$

where

$$E_k = \left\{m \leq L : \frac{k}{N} < \|d!mh\alpha\| \leq \frac{k+1}{N}\right\}.$$

With this notation we have

$$\sum_{m=1}^{L} \min\left(N, \frac{1}{\|d!mh\alpha\|}\right) \ll N|E_0| + \sum_{k=1}^{N-1} \frac{N}{k}|E_k|.$$

Observe that

$$|E_k| = \frac{2L}{N} + O(LD_L(d!mh\alpha)).$$

Hence we have

$$\sum_{m=1}^{L} \min\left(N, \frac{1}{\|d!mh\alpha\|}\right) \ll L\log N + NLD_L(d!mh\alpha)\log N. \tag{8}$$

Since α has irrationality measure $t+1$, $\|d!mh\alpha\| \geq_\epsilon (d!mh)^{-(t+\epsilon)}$. Then by Lemma 7

$$\begin{aligned} D_L(d!mh\alpha) &\ll_\epsilon \frac{1}{H} + \frac{1}{L}\sum_{j=1} \frac{1}{j\|d!hj\alpha\|} \\ &\ll_{\epsilon,d,t} \frac{1}{H} + \frac{(d!h)^{t+\epsilon}}{L}\sum_{j=1}^{H} j^{t-1+\epsilon} \\ &\ll_{\epsilon,d,t} \frac{1}{H} + L^{-1}H^{t+\epsilon}h^{t+\epsilon}. \end{aligned}$$

Choose $H = \left[L^{\frac{1}{t+1}} h^{-\frac{t}{t+1}}\right]$ to get

$$D_L(d!mh\alpha) \ll_{\epsilon,d,t} L^{-\frac{1}{t+1}+\epsilon} h^{\frac{t}{t+1}+\epsilon}. \tag{9}$$

Using this estimate in (8) gives us

$$\sum_{m=1}^{L} \min\left(N, \frac{1}{\|d!mh\alpha\|}\right) \ll_{\epsilon,d,t} NL^{1-\frac{1}{t+1}+\epsilon} h^{\frac{t}{t+1}+\epsilon}. \tag{10}$$

The above estimate when $L = N^{2-2^{-d+2}}$ together with (7) gives

$$\left|\sum_{n=0}^{N-1} e(p(n)h\alpha)\right|^{2^{d-1}} \ll_{\epsilon,d,t} N^{2^{d-1}-1} + N^{2^{d-1}-\frac{2-2^{-d+2}}{t+1}+\epsilon}. \tag{11}$$

In the above estimate clearly the second term dominates. Hence we get

$$\left|\sum_{n=0}^{N-1} e(p(n)h\alpha)\right| \ll_{\epsilon,d,t} N^{1-\frac{2-2^{-d+2}}{2^{d-1}(t+1)}+\epsilon}. \tag{12}$$

Now (5) and (12) together gives

$$D_N(p(n)\alpha) \ll_{\epsilon,d,t} \frac{1}{H} + N^{-\frac{2-2^{-d+2}}{2^{d-1}(t+1)}+\epsilon} H^{\frac{t}{t+1}+\epsilon}.$$

Finally we choose $H = \left[N^{\frac{2-2^{-d+2}}{2^{d-1}(2t+1)}}\right]$ to get

$$D_N(p(n)\alpha) \ll_{\epsilon,d,t} N^{-\frac{2-2^{-d+2}}{2^{d-1}(2t+1)}+\epsilon}.$$

□

3 Proof of the theorem

Let H be any positive integer which will be chosen later. By Lemma 6, we have

$$D_N([p(n)\alpha]\beta) \leq \frac{2}{H+1} + \frac{2}{N}\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{n=0}^{N-1} e(h[p(n)\alpha]\beta)\right|. \tag{13}$$

Recall that $f_\tau(x) = e(\tau\{x\})$ and $g_{\tau,\delta}$ is defined as in (2) with $\delta := \delta(h) = h^{-1}N^{-\theta}$ for some $0 < \theta < 1$. Writing $[x] = x - \{x\}$ we have

$$\sum_{n=0}^{N-1} e(h[p(n)\alpha]\beta) = \sum_{n=0}^{N-1} e(hp(n)\alpha\beta) f_{-h\beta}(p(n)\alpha)$$
$$= \sum_{n=0}^{N-1} e(hp(n)\alpha\beta) g_{-h\beta,\delta}(p(n)\alpha)$$
$$+ O\left(\sum_{n=0}^{N-1} |f_{-h\beta}(p(n)\alpha) - g_{-h\beta,\delta}(p(n)\alpha)|\right). \qquad (14)$$

By Lemma 1 for the O-term on the right hand side of (14) and substituting it in the inequality (13) we have

$$D_N([p(n)\alpha]\beta) \ll \frac{1}{H} + \frac{1}{N}\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{n=0}^{N-1} e(hp(n)\alpha\beta) g_{-h\beta,\delta}(p(n)\alpha)\right|$$
$$+ r\sum_{h=1}^{H}\frac{\delta}{h} + |\beta| r^2 \sum_{h=1}^{H}\delta + D_N(p(n)\alpha)\log H.$$

The Fourier inversion formula for $g_{\tau,\delta}$ gives us

$$D_N([p(n)\alpha]\beta) \ll \frac{1}{N}\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{k\in\mathbb{Z}} \hat{g}_{-h\beta,\delta}(k)\sum_{n=0}^{N-1} e(p(n)\alpha(h\beta - k))\right| + \frac{1}{H}$$
$$+ r\sum_{h=1}^{H}\frac{\delta}{h} + |\beta| r^2 \sum_{h=1}^{H}\delta + D_N(p(n)\alpha)\log H. \qquad (15)$$

Let

$$S_N = \frac{1}{N}\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{k\in\mathbb{Z}} \hat{g}_{-h\beta,\delta}(k)\sum_{n=0}^{N-1} e(p(n)\alpha(h\beta - k))\right|. \qquad (16)$$

Let ρ be a real number such that $\rho \in [1, 2]$, which will be chosen later. We also suppose $N^\theta > 2|\beta|$. Splitting the first sum inside the modulus into $|k| > h^\rho N^\theta$ and $|k| \le h^\rho N^\theta$ gives us

$$S_N \ll \frac{1}{N}\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{|k|\le h^\rho N^\theta} \hat{g}_{-h\beta,\delta}(k)\sum_{n=0}^{N-1} e(p(n)\alpha(h\beta - k))\right|$$
$$+ \sum_{h=1}^{H}\frac{1}{h}\sum_{|k|>h^\rho N^\theta} |\hat{g}_{-h\beta,\delta}(k)|.$$

Lemma 3 with $K = h^\rho N^\theta$ shows that the second term on the right hand side is $\ll H^{r(1-\rho)}$.

Hence we have

$$S_N \ll \frac{1}{N}\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{|k|\le h^\rho N^\theta}\hat{g}_{-h\beta,\delta}(k)\sum_{n=0}^{N-1}e(p(n)\alpha(h\beta-k))\right| + H^{r(1-\rho)}. \tag{17}$$

Using Hölder's inequality

$$\begin{aligned}
&\left|\sum_{|k|\le h^\rho N^\theta}\hat{g}_{-h\beta,\delta}(k)\sum_{n=0}^{N-1}e(p(n)\alpha(h\beta-k))\right| \\
&\ll \left(\sum_{|k|\le h^\rho N^\theta}|\hat{g}_{-h\beta,\delta}(k)|^{\frac{2^{d-1}}{2^{d-1}-1}}\right)^{\frac{2^{d-1}-1}{2^{d-1}}}\left(\sum_{|k|\le h^\rho N^\theta}\left|\sum_{n=0}^{N-1}e(p(n)\alpha(h\beta-k))\right|^{2^{d-1}}\right)^{\frac{1}{2^{d-1}}} \\
&\ll \left(\sum_{|k|\le h^\rho N^\theta}\left|\sum_{n=0}^{N-1}e(p(n)\alpha(h\beta-k))\right|^{2^{d-1}}\right)^{\frac{1}{2^{d-1}}}.
\end{aligned} \tag{18}$$

Here we have used Lemma 4 to get the last inequality.

Let $\xi = \alpha(h\beta - k)$. Using Lemma 5, with $k = d-1$ and $\lambda_m = e(p(m)\xi)$ we get that the following inequalities hold for any $Q \in [1, N]$:

$$\begin{aligned}
&\left|\sum_{n=0}^{N-1}e(p(n)\xi)\right|^{2^{d-1}} \\
&\ll \frac{N^{2^{d-1}}}{Q} + \frac{N^{2^{d-1}-1}}{Q^{2-2^{-d+2}}}\sum_{r_1=1}^{Q}\sum_{r_2=1}^{Q^{\frac{1}{2}}}\cdots\sum_{r_{d-1}=1}^{Q^{2^{-d+2}}}\left|\sum_{n=0}^{N-1-r_1-\cdots-r_{d-1}}e(d!r_1\cdots r_{d-1}n\xi)\right| \\
&\ll \frac{N^{2^{d-1}}}{Q} + \frac{N^{2^{d-1}-1}}{Q^{2-2^{-d+2}}}\sum_{r_1=1}^{Q}\sum_{r_2=1}^{Q^{\frac{1}{2}}}\cdots\sum_{r_{d-1}=1}^{Q^{2^{-d+2}}}\left|\min\left(N, \frac{1}{\|d!r_1\cdots r_{d-1}\xi\|}\right)\right|,
\end{aligned}$$

where we have used $\sum_{n=0}^{N-1} e(n\lambda) \ll \min(N, \frac{1}{\|\lambda\|})$ to get the last inequality.

Let $T(m) = |\{(r_1,\dots,r_{d-1}) \in [1, Q]\times\cdots\times[1, Q^{2^{-d+2}}] : r_1\cdots r_{d-1} = m\}|$. With this notation the above inequality will be

$$\left|\sum_{n=0}^{N-1}e(p(n)\xi)\right|^{2^{d-1}} \ll \frac{N^{2^{d-1}}}{Q} + \frac{N^{2^{d-1}-1}}{Q^{2-2^{-d+2}}}\sum_{m=1}^{Q^{2-2^{-d+2}}}T(m)\min\left(N, \frac{1}{\|d!\xi m\|}\right).$$

Let $\epsilon > 0$ be any real number. Let $\epsilon_2 = \frac{\epsilon}{(2-2^{-d+2})}$. Since $T(m) \leq \tau_{d-1}(m) \ll_{\epsilon_2} m^{\epsilon_2}$, we get

$$\left|\sum_{n=0}^{N-1} e(p(n)\xi)\right|^{2^{d-1}} \ll_{\epsilon,d} \frac{N^{2^{d-1}}}{Q} + \frac{N^{2^{d-1}-1}}{Q^{2-2^{-d+2}-\epsilon}} \sum_{m=1}^{Q^{2-2^{-d+2}}} \min\left(N, \frac{1}{\|d!\xi m\|}\right). \tag{19}$$

Now we prove the following lemma which will be used to estimate the right hand side of the above equation.

Lemma 8 *Let $\xi = \alpha(h\beta - k)$. Then for any $\epsilon > 0$ we have*

$$\sum_{\ell=1}^{L} \min\left(N, \frac{1}{\|d!\ell\xi\|}\right) \ll_{\alpha,\beta,\epsilon,d} L\log N + NL^{1-\frac{1}{2t+1}+\epsilon}(h|k|)^{\frac{t}{2t+1}+\epsilon}\log N.$$

Proof For $0 \leq m \leq N-1$, define

$$E_m = \left\{\ell \leq L : \frac{m}{N} < \|d!\ell\xi\| \leq \frac{m+1}{N}\right\}.$$

We have

$$\begin{aligned}\sum_{\ell=1}^{L} \min\left(N, \frac{1}{\|d!\ell\xi\|}\right) &= N|E_0| + \sum_{l\notin E_0} \frac{1}{\|d!\ell\xi\|} \\ &\leq N|E_0| + \sum_{m=1}^{N-1} \frac{N}{m}|E_m|.\end{aligned}$$

Observe that

$$|E_k| = \frac{2L}{N} + O(LD_L(d!\ell\xi)).$$

Thus

$$\sum_{\ell=1}^{L} \min\left(N, \frac{1}{\|d!\ell\xi\|}\right) \ll L\log N + NLD_L(d!\ell\xi)\log N. \tag{20}$$

Using Lemma 7 and the fact that

$$\|d!\ell\xi\| = \|d!\ell\alpha(h\beta - k)\| \geq \frac{C(\alpha,\beta,\epsilon)}{((d!\ell)^2h|k|)^{t+\epsilon}}$$

for any positive integer $\ell \geq 1$, we get

$$D_L(d!\ell\xi) \ll_{\alpha,\beta,\epsilon} \frac{1}{m} + \frac{1}{L}(h|k|(d!m)^2)^{t+\epsilon}$$

for any positive integer m. Now we choose $m = L^{1/(2t+1)}(h|k|)^{-t/(2t+1)}$ to get

$$D_L(d!\ell\xi) \ll_{\alpha,\beta,\epsilon,d} (h^t|k|^t)^{\frac{1}{2t+1}+\epsilon} L^{-\frac{1}{2t+1}+\epsilon}. \tag{21}$$

Substituting the above estimate in (20) gives us

$$\sum_{\ell=1}^{L} \min\left(N, \frac{1}{\|d!\ell\xi\|}\right) \ll_{\alpha,\beta,\epsilon,d} L\log N + NL^{1-\frac{1}{2t+1}+\epsilon}(h|k|)^{\frac{t}{2t+1}+\epsilon} \log N.$$

□

Apply Lemma 8 in (19) with $L = Q^{2-2^{-d+2}}$ and let $Q = N$ to get

$$\left|\sum_{n=0}^{N-1} e(p(n)\xi)\right|^{2^{d-1}} \ll_{\alpha,\beta,\epsilon,d} N^{2^{d-1}-1} + N^{2^{d-1}-\left(\frac{2-2^{-d+2}}{2t+1}\right)+\epsilon} h^{\frac{t}{2t+1}+\epsilon}|k|^{\frac{t}{2t+1}+\epsilon}. \tag{22}$$

Summing both sides of the above inequality over k we get that

$$\sum_{|k|\le h^\rho N^\theta} \left|\sum_{n=0}^{N-1} e(p(n)\xi)\right|^{2^{d-1}} \ll_{\alpha,\beta,\epsilon,d} N^{2^{d-1}-1+\theta} h^\rho + N^{2^{d-1}-(\frac{2-2^{-d+2}}{2t+1})+\theta(\frac{3t+1}{2t+1})+\epsilon} h^{\frac{t}{2t+1}+\rho(\frac{3t+1}{2t+1})+\epsilon}. \tag{23}$$

Clearly the first term on the right hand side is dominated by the second term. Putting this inequality in (18) we get that

$$\left|\sum_{|k|\le h^\rho N^\theta} \hat{g}_{-h\beta}(k) \sum_{n=0}^{N-1} e(p(n)\alpha(h\beta-k))\right| \ll_{\alpha,\beta,\epsilon,d} N^{1-(\frac{2-2^{-d+2}}{2^{d-1}(2t+1)})+\theta(\frac{3t+1}{2^{d-1}(2t+1)})+\epsilon} h^{\frac{t}{2^{d-1}(2t+1)}+\rho(\frac{3t+1}{2^{d-1}(2t+1)})+\epsilon}.$$

Hence we have

$$\sum_{h=1}^{H} \frac{1}{h}\left|\sum_{|k|\le h^\rho N^\theta} \hat{g}_{-h\beta}(k) \sum_{n=0}^{N-1} e(p(n)\alpha(h\beta-k))\right| \ll_{\alpha,\beta,\epsilon,d} N^{1-(\frac{2-2^{-d+2}}{2^{d-1}(2t+1)})+\theta(\frac{3t+1}{2^{d-1}(2t+1)})+\epsilon} H^{\frac{t}{2^{d-1}(2t+1)}+\rho(\frac{3t+1}{2^{d-1}(2t+1)})+\epsilon}.$$

From (17) and above inequality we have

$$S_N \ll_{\alpha,\beta,\epsilon,d} N^{-(\frac{2-2^{-d+2}}{2^{d-1}(2t+1)})+\theta(\frac{3t+1}{2^{d-1}(2t+1)})+\epsilon} H^{\frac{t}{2^{d-1}(2t+1)}+\rho(\frac{3t+1}{2^{d-1}(2t+1)})+\epsilon} + O(H^{r(1-\rho)}).$$

Hence we have from (15) with $\delta^{-1} = hN^{\theta}$ that

$$D_N([p(n)\alpha]\beta) \ll_{\alpha,\beta,\epsilon,d} H^{\frac{t}{2^{d-1}(2t+1)}+\rho(\frac{3t+1}{2^{d-1}(2t+1)})+\epsilon} N^{-(\frac{2-2^{-d+2}}{2^{d-1}(2t+1)})+\theta(\frac{3t+1}{2^{d-1}(2t+1)})+\epsilon} + H^{r(1-\rho)} + \frac{1}{H} + N^{-\theta} + N^{-\theta}\log H + D_N(p(n)\alpha)\log H\ .$$

We choose $\rho = 1 + \epsilon_1$ with $\epsilon_1 = \epsilon_1(\epsilon, t) > 0$ sufficiently small real number, and r is an integer satisfying $r > \frac{1}{\epsilon_1}$. Hence the second term on the right hand side is $\ll H^{-1}$.

Now we choose $H = [N^{\theta}]$ with $\theta = \frac{2-2^{-d+2}}{2^{d-1}(2t+1)+(4t+1)+\rho(3t+1)}$. With these choices we have

$$D_N([p(n)\alpha]\beta) \ll_{\alpha,\beta,\epsilon,d} N^{-\frac{2-2^{-d+2}}{2^{d-1}(2t+1)+7t+2}+\epsilon} + D_N(p(n)\alpha)\log N. \qquad (24)$$

By Lemma 1, we have

$$D_N(p(n)\alpha) \ll_{\epsilon,d,t} N^{-\frac{2-2^{-d+2}}{2^{d-1}(2t+1)}+\epsilon}.$$

Putting this in (24), we get

$$D_N([p(n)\alpha]\beta) \ll_{\alpha,\beta,\epsilon,d} N^{-\frac{2-2^{-d+2}}{2^{d-1}(2t+1)+7t+2}+\epsilon}.$$

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