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Logarithmic coefficients for certain subclasses of close-to-convex functions

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Abstract Let $\mathcal{S}$ denote the class of functions analytic and univalent (i.e. one-to-one) in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. The logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined by $\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n$. Let $\mathcal{F}_1(\mathcal{F}_2$ and $\mathcal{F}_3$ resp.) denote the class of functions $f \in \mathcal{A}$ such that $\operatorname{Re}(1-z)f'(z) > 0$ ($\operatorname{Re}(1-z^2)f'(z) > 0$ and $\operatorname{Re}(1-z+z^2)f'(z) > 0$ resp.) in $\mathbb{D}$. The classes $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are subclasses of the class of close-to-convex functions. In the present paper, we determine the sharp upper bound for $|\gamma_1|$, $|\gamma_2|$ and $|\gamma_3|$ for functions f in the classes $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$.

Keywords Analytic · Univalent · Starlike · Convex and close-to-convex functions · Coefficient estimates · Logarithmic coefficients

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1 Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk in the complex plane $\mathbb{C}$. A single-valued function f is said to be univalent in a domain $\Omega \subseteq \mathbb{C}$ if it never takes the same value twice, that is, if $f(z_1) = f(z_2)$ for $z_1, z_2 \in \Omega$ then $z_1 = z_2$. Let $\mathcal{A}$ denote the class of analytic functions f in $\mathbb{D}$ normalized by $f(0) = 0 = f'(0) - 1$. If $f \in \mathcal{A}$ then $f(z)$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. It is pertinent to mention that recently, Aleman and Constantin [1] have provided a nice connection between the theory of univalent function to fluid dynamics. Indeed, Aleman and Constantin [1] provided an amicable approach towards obtaining explicit solutions to the incompressible two-dimensional Euler equations by means of univalent harmonic map. More precisely, the problem of finding all solutions which in Lagrangian variables describing the particle paths of the flow present a labelling by harmonic functions is reduced to solving an explicit nonlinear differential system in $\mathbb{C}^n$ with $n = 3$ or $n = 4$ (see also [4]).

A domain $\Omega \subseteq \mathbb{C}$ is said to be a starlike domain with respect to a point $z_0 \in \Omega$ if the line segment joining z_0 to any point in Ω lies in Ω. If z_0 is the origin then we say that Ω is a starlike domain. A function $f \in \mathcal{A}$ is said to be a starlike function if $f(\mathbb{D})$ is a starlike domain. We denote by $\mathcal{S}^*$ the class of starlike functions f in $\mathcal{S}$. It is well-known that [6] a function $f \in \mathcal{A}$ is in $\mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

A domain Ω is said to be convex if it is starlike with respect to each point of Ω. A function $f \in \mathcal{A}$ is said to be convex if $f(\mathbb{D})$ is a convex domain. We denote the class of convex univalent functions in $\mathbb{D}$ by $\mathcal{C}$. A function $f \in \mathcal{A}$ is in $\mathcal{C}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

It is well-known that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$.

A function $f \in \mathcal{A}$ is said to be close-to-convex (having argument $\alpha \in (-\pi/2, \pi/2)$) with respect to $g \in \mathcal{S}^*$ if

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

We denote the class of all such functions by $\mathcal{K}_\alpha(g)$. Let

$$\mathcal{K}(g) := \bigcup_{\alpha \in (-\pi/2, \pi/2)} \mathcal{K}_\alpha(g) \quad \text{and} \quad \mathcal{K}_\alpha := \bigcup_{g \in \mathcal{S}^*} \mathcal{K}_\alpha(g)$$

be the classes of close-to-convex functions with respect to g and close-to-convex functions with argument α, respectively. Let

$$\mathcal{K} := \bigcup_{\alpha\in(-\pi/2,\pi/2)} \mathcal{K}_\alpha = \bigcup_{g\in\mathcal{S}^*} \mathcal{K}(g)$$

denote the class of close-to-convex functions in $\mathcal{A}$. It is well-known that every close-to-convex function is univalent in $\mathbb{D}$ [13]. A domain $\Omega \subseteq \mathbb{C}$ is said to be linearly accessible if its complement is the union of a family of non-intersecting half-lines. A function $f \in \mathcal{S}$ whose range is linearly accessible is called a linearly accessible function. Kaplan's theorem [13] makes it seem plausible that the class of linearly accessible family and the class $\mathcal{K}$ coincide. In fact, Lewandowski [14] has observed that the class $\mathcal{K}$ is the same as the class of linearly accessible functions introduced by Biernacki [3] in 1936. In 1962, Bielecki and Lewandowski [2] proved that every function in the class $\mathcal{K}$ is linearly accessible.

Let $\mathcal{P}$ denote the class of analytic functions $h(z)$ of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.2}$$

such that $\operatorname{Re} h(z) > 0$ in $\mathbb{D}$. To prove our main results we need the following results.

Lemma 1.3 [15] *Let $h \in \mathcal{P}$ be of the form (1.2). Then*

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2) \\ 4c_3 &= c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)t. \end{aligned}$$

for some complex valued x and t with $|x| \leq 1$ and $|t| \leq 1$.

Lemma 1.4 [17, pp 166] *Let $h \in \mathcal{P}$ be of the form (1.2). Then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

The inequality is sharp for functions $L_{t,\theta}(z)$ of the form

$$L_{t,\theta}(z) = t\left(\frac{1 + e^{i\theta} z}{1 - e^{i\theta} z}\right) + (1 - t)\left(\frac{1 + e^{i2\theta} z^2}{1 - e^{i2\theta} z^2}\right).$$

Lemma 1.5 [16] *Let $h \in \mathcal{P}$ be of the form (1.2) and μ be a complex number. Then*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}.$$

The result is sharp for the functions given by $p(z) = \frac{1+z^2}{1-z^2}$ and $p(z) = \frac{1+z}{1-z}$.

Given a function $f \in \mathcal{S}$, the coefficients γ_n defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n \tag{1.6}$$

are called the logarithmic coefficients of $f(z)$. The logarithmic coefficients are central to the theory of univalent functions for their role in the proof of Bieberbach conjecture. Milin conjectured that for $f \in \mathcal{S}$ and $n \geq 2$,

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0.$$

Since Milin's conjecture implies Bieberbach conjecture, in 1985, De Branges proved Milin conjecture to give an affirmative proof of the Bieberbach conjecture [5].

By differentiating (1.6) and equating coefficients we obtain

$$\gamma_1 = \frac{1}{2}a_2 \tag{1.7}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) \tag{1.8}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \tag{1.9}$$

It is evident from (1.7) that $|\gamma_1| \leq 1$ if $f \in \mathcal{S}$. An application of Fekete–Szegö inequality [6, Theorem 3.8] in (1.8) yields the following sharp estimate

$$|\gamma_2| \leq \frac{1}{2}(1 + 2e^{-2}) = 0.635\ldots \quad \text{for } f \in \mathcal{S}.$$

The problem of finding the sharp upper bound for $|\gamma_n|$ for $f \in \mathcal{S}$ is still open for $n \geq 3$. The sharp upper bounds for modulus of logarithmic coefficients are known for functions in very few subclasses of $\mathcal{S}$. For the Koebe function $k(z) = z/(1-z)^2$, the logarithmic coefficients are $\gamma_n = 1/n$. Since the Koebe function $k(z)$ plays the role of extremal function for most of the extremal problems in the class $\mathcal{S}$, it is expected that $|\gamma_n| \leq \frac{1}{n}$ holds for functions in the class $\mathcal{S}$. However, this is not true in general. Indeed, there exists a bounded function f in the class $\mathcal{S}$ with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$ (see [6, Theorem 8.4]). A simple exercise shows that $|\gamma_n| \leq 1/n$ for functions in $\mathcal{S}^*$ and the equality holds for the Koebe function. Consequently, attempts have been made to find bounds for logarithmic coefficients for close-to-convex functions in the unit disk $\mathbb{D}$. Elhosh [8] attempted to extend the result $|\gamma_n| \leq 1/n$ to the class $\mathcal{K}$. However Girela [11] pointed out an error in the proof and proved that for every $n \geq 2$ there exists a function f in $\mathcal{K}$ such that $|\gamma_n| \geq 1/n$. Ye [23] provided an estimate for $|\gamma_n|$ for functions f in the class $\mathcal{K}$, showing that $|\gamma_n| \leq An^{-1}\log n$ where A is a constant. The sharp inequalities are known for sums involving logarithmic coefficients (see [6,7]). For $f \in \mathcal{S}$, Roth [21] proved the following sharp inequality

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^2|\gamma_n|^2 \le 4\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^2\frac{1}{n^2} = \frac{2\pi^2-12}{3}.$$

Recently, it has been proved that $|\gamma_3| \le 7/12$ for functions in the class $\mathcal{K}_0$ with the additional assumption that the second coefficient of the corresponding starlike function $g(z)$ is real [22]. However this bound is not sharp. Enough emphasis cannot be laid on this fact as it highlights nature of complexity involved in obtaining the sharp upper bound for $|\gamma_3|$. More recently Firoz and Vasudevarao [10] improved the bound on $|\gamma_3|$ by proving $|\gamma_3| \le \frac{1}{18}(3+4\sqrt{2}) = 0.4809$ for functions f in the class $\mathcal{K}_0$ without the assumption requiring the second coefficient of the corresponding starlike function $g(z)$ be real. However, this improved bound is still not sharp. Consequently, the problem of finding the sharp upper bound for $|\gamma_3|$ for the classes $\mathcal{K}_0$ as well as $\mathcal{K}$ is still open. Recently, the sharp logarithmic coefficients ($|\gamma_n|$ for $n = 1, 2, 3$) for close-to-convex functions (with argument 0) with respect to odd starlike functions have been studied by Firoz and Vasudevarao [9].

In the present paper we consider the following three familiar subclasses of close-to-convex functions

$$\mathcal{F}_1 := \left\{f \in \mathcal{A} : \operatorname{Re}(1-z)f'(z) > 0 \quad \text{for } z \in \mathbb{D}\right\}$$
$$\mathcal{F}_2 := \left\{f \in \mathcal{A} : \operatorname{Re}(1-z^2)f'(z) > 0 \quad \text{for } z \in \mathbb{D}\right\}$$
$$\mathcal{F}_3 := \left\{f \in \mathcal{A} : \operatorname{Re}(1-z+z^2)f'(z) > 0 \quad \text{for } z \in \mathbb{D}\right\}.$$

The region of variability for the classes $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ have been extensively studied by Ponnusamy, Vasudevarao and Yanagihara ([19,20]). The classes $\mathcal{F}_1$, $\mathcal{F}_2$ have been generalized to the class of harmonic close-to-convex functions in $\mathbb{D}$ by Ponnusamy, Rasila and Kaliraj [18]. In fact the harmonic analogue of the class $\mathcal{F}_2$ contains convex functions in the vertical direction [18] (see also references therein).

The main aim of this paper is to determine the sharp upper bounds for $|\gamma_1|$, $|\gamma_2|$ and $|\gamma_3|$ for functions f in the classes $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$.

2 Main results

Throughout the remainder of this paper, we assume that $f \in \mathcal{K}_0$ and $h \in \mathcal{P}$ have the series representations (1.1) and (1.2) respectively. Further, assume that $g \in \mathcal{S}^*$ has the following series representation:

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{2.1}$$

It is not difficult to see that the function $H_{t,\mu}(z)$ given by

$$H_{t,\mu}(z) = (1-2t)\left(\frac{1+z}{1-z}\right) + t\left(\frac{1+\mu z}{1-\mu z}\right) + t\left(\frac{1+\overline{\mu}z}{1-\overline{\mu}z}\right)$$

belongs to the class $\mathcal{P}$ for $0 \leq t \leq 1/2$ and $|\mu| = 1$. Since $f \in \mathcal{K}_0$, there exists an $h \in \mathcal{P}$ such that

$$zf'(z) = g(z)h(z). \tag{2.2}$$

Using the representations (1.1), (1.2) and (2.1) in (2.2) we obtain

$$z + \sum_{n=2}^{\infty} na_n z^n = \left(z + \sum_{n=2}^{\infty} b_n z^n\right)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right). \tag{2.3}$$

Comparing the coefficients on both the sides of (2.3), we obtain

$$2a_2 = b_2 + c_1 \tag{2.4}$$

$$3a_3 = b_3 + b_2c_1 + c_2 \tag{2.5}$$

$$4a_4 = b_4 + c_1b_3 + c_2b_2 + c_3. \tag{2.6}$$

A substitution of (2.4) in (1.7) gives

$$\gamma_1 = \frac{1}{4}\left(b_2 + c_1\right). \tag{2.7}$$

An application of the triangle inequality to (2.7) gives

$$4|\gamma_1| \leq |b_2| + |c_1|. \tag{2.8}$$

Substituting (2.4) and (2.5) in (1.8), we obtain

$$\gamma_2 = \frac{1}{48}\left(8b_3 + 2b_2c_1 + 8c_2 - 3b_2^2 - 3c_1^2\right). \tag{2.9}$$

Let $c_1 = de^{i\alpha}$ and $q = \cos\alpha$ with $0 \leq d \leq 2$ and $0 \leq \alpha < 2\pi$. Applying the triangle inequality in conjunction with Lemma 1.4 allows us to rewrite (2.9) as

$$6|\gamma_2| \leq 2 - \frac{d^2}{2} + \frac{1}{8}\left|\left(dq + b_2 + id\sqrt{1-q^2}\right)^2 + (8b_3 - 4b_2^2)\right|. \tag{2.10}$$

Substituting (2.4), (2.5) and (2.6) in (1.9), we obtain

$$\gamma_3 = \frac{1}{48}\left(6c_3 - b_2^2c_1 - b_2c_1^2 + 2b_2c_2 + 2b_3c_1 + b_2^3 - 4b_3b_2 + 6b_4 + c_1^3 - 4c_1c_2\right). \tag{2.11}$$

A simple application of Lemma 1.3 to (2.11) shows that

$$\begin{aligned} 96\gamma_3 = {} & 6t(1-|x|^2)(4-c_1^2) + c_1^3 + (4b_3 - 2b_1^2)c_1 + (2b_2^3 - 8b_2b_3 + 2b_4) \\ & + x(4-c_1^2)(2b_2 + 2c_1 - 3c_1x). \end{aligned} \tag{2.12}$$

Let b_n be real for all $n \in \mathbb{N}$. Let $c_1 = c$ and assume that $0 \le c \le 2$. Let $x = re^{i\theta}$ and $p = \cos\theta$ with $0 \le r \le 1$ and $0 \le \theta < 2\pi$. Taking modulus on both the sides of (2.12) and applying the triangle inequality we obtain

$$96|\gamma_3| \le 6(1-r^2)(4-c^2) + |\phi(c,r,p)| \tag{2.13}$$

where

$$\begin{aligned}\phi(c,r,p) &= c^3 + (4b_3 - 2b_1^2)c + (2b_2^3 - 8b_2b_3 + 2b_4)\\ &\quad + re^{i\theta}(4-c^2)(2b_2 + 2c - 3cre^{i\theta}).\end{aligned}$$

Theorem 2.14 *Let $f \in \mathcal{F}_1$ be given by (1.1). Then*

(i) $|\gamma_1| \le \frac{3}{4}$,
(ii) $|\gamma_2| \le \frac{4}{9}$.
(iii) *If* $1/2 \le a_2 \le 3/2$ *then* $|\gamma_3| \le \frac{1}{288}\left(11 + 15\sqrt{30}\right)$.

The inequalities are sharp.

Proof Let $f \in \mathcal{F}_1$. Then f is a close-to-convex function with respect to the starlike function $g(z) = z/(1-z)$. In view of (2.2) the function $f(z)$ can be written as

$$zf'(z) = \frac{z}{1-z}\, h(z). \tag{2.15}$$

As $|c_1| \le 2$ for $h \in \mathcal{P}$ (see [12, Ch 7, Theorem 3]) a comparison of the R.H.S. of (2.2) and (2.15), shows that (2.8) reduces to

$$4|\gamma_1| \le 1 + |c_1| \le 3. \tag{2.16}$$

A function $p \in \mathcal{P}$ having $|c_1| = 2$ is given by $p(z) = L_{1,\theta}(z)$ for $0 \le \theta < 2\pi$ and substituting $p(z)$ in place of $h(z)$ in (2.15) determines a function $f \in \mathcal{F}_1$ for which the upper bound on $|\gamma_1|$ is sharp.

In view of (2.2) and (2.15), we can rewrite (2.10) as

$$6|\gamma_2| \le 2 - \frac{|c_1|^2}{2} + \frac{1}{8}\sqrt{(d^2+5+2dq)^2 - 16d^2(1-q^2)} =: g(d,q). \tag{2.17}$$

In view of (2.17) it suffices to find points in the square $S := [0,2] \times [-1,1]$ where $g(d,q)$ attains the maximum value to determine the maximum value of $|\gamma_2|$. Solving $\frac{\partial g(d,q)}{\partial d} = 0$ and $\frac{\partial g(d,q)}{\partial q} = 0$ shows that there is no real valued solution to the pair of equations. Thus $g(d,q)$ does not attain maximum in the interior of S.

On the side $d = 0$, $g(d,q)$ reduces to $g(0,q) = 21/8$. On the side $d = 2$, $g(d,q)$ can be written as $g(2,q) = \frac{1}{8}\sqrt{80t^2 + 72t + 17}$. An elementary calculation shows that $\max_{-1\le q\le 1} g(2,q) = g(2,1) = 1.625$.

On the side $q = -1$, $g(d,q)$ maybe simplified to $g(d,-1) = (21-2d-3d^2)/8$. It is not difficult to see that $g(d,1)$ is decreasing for $c \in [0,2]$. Thus $\max_{0\le d\le 2} g(d,-1) = d(0,-1) = 21/8 = 2.625$.

On the side $q = 1$, $g(d, q)$ becomes $g(d, 1) = (21 + 2d - 3d^2)/8$. An elementary computation shows that $\max_{0 \le d \le 2} g(d, 1) = d(1/3, 1) = 8/3$.

Thus the maximum value of $g(d, q)$ and consequently that of $|\gamma_2|$ is attained at $(d, q) = (1/3, 1)$, i.e., at $c_1 = 1/3$. Thus, from (2.17) we obtain $|\gamma_2| \le 4/9$. Therefore in view of (2.15) and Lemma 1.4 the equality holds in (ii) for the function $\widetilde{F}_1 \in \mathcal{F}_1$ such that $z\widetilde{F_1}'(z) = z(1-z)^{-1} L_{t,\theta}(z)$ with $t = 1/6$ and $\theta = 0$.

In view of (2.15), we may rewrite (2.13) as

$$48|\gamma_3| \le 3(4 - c^2)(1 - r^2) + \sqrt{\phi_1(c, r, p)}, \tag{2.18}$$

where

$$\begin{aligned}\phi_1(c, r, p) = &\left(\frac{c^3}{2} + c + 3\right)^2 + (4 - c^2)^2 r^2 \left(-3c^2 pr + \frac{9}{4} c^2 r^2\right.\\ &\left.+ c^2 - 3cpr + 2c + 1\right)\\ &+ 2\left(\frac{c^3}{2} + c + 3\right)(4 - c^2) r \left(\frac{3}{2} cr - 3cp^2 r - 1 + cp + p\right).\end{aligned}$$

Let $G(c, r, p) = 3(4 - c^2)(1 - r^2) + \sqrt{\phi_1(c, r, p)}$. Thus it suffices to find points in the closed cuboid $R := [0, 2] \times [0, 1] \times [-1, 1]$ where $G(c, r, p)$ attains the maximum value. We accomplish this by finding the maximum values in the interior of the six faces, on the twelve edges and in the interior of R.

On the face c = 0, it can be seen that $G(c, r, p)$ reduces to

$$G(0, r, p) = \sqrt{24pr + 16r^2 + 9} + 12(1 - r^2). \tag{2.19}$$

To determine the points on this face where the maxima occur, we solve $\frac{\partial G(0,r,p)}{\partial r} = 0$ and $\frac{\partial G(0,r,p)}{\partial p} = 0$. The only solution for this pair of equations is $(r, p) = (0, 0)$. Thus, no maxima occur in the interior of the face $c = 0$.

On the face c = 2, $G(c, r, p)$ becomes $G(2, r, p) = 9$ and hence

$$\max_{0<r<1,\ -1<p<1} G(2, r, p) = 9.$$

On the face $r = 0$, $G(c, r, p)$ reduces to

$$G(c, 0, p) = 12 - 3c^2 + \frac{1}{2}(c^3 + 2c + 6). \tag{2.20}$$

To determine points where maxima occur, it suffices to find points where $\frac{\partial G(c,0,p)}{\partial c} = 0$ because $G(c, 0, p)$ is independent of p. The set of all such points is $\{\frac{1}{3}(6 - \sqrt{30})\} \times \{0\} \times [-1, 1]$ and hence $G(\frac{1}{3}(6 - \sqrt{30}), 0, p) = \frac{10\sqrt{10}}{3\sqrt{3}} + 9 = 15.0858$. Thus

$$\max_{0<c<2,\ -1<p<1} G(c,0,p) = \frac{10\sqrt{10}}{3\sqrt{3}} + 9 = 15.0858.$$

On the face $r = 1$, $G(c, r, p)$ reduces to

$$G(c,1,p) = \sqrt{\psi_1(c,p) + \frac{1}{2}(c^2-4)(c^3+2c+6)(6cp^2-2cp-2p-3c)} \tag{2.21}$$

where

$$\psi_1(c,p) = \left(\frac{c^3}{2}+c+3\right)^2 + (c^2-4)^2\left(\frac{1}{4}(c^2-12pc+8c)+1\right).$$

A computation shows that $\frac{\partial G(c,1,p)}{\partial p} = 0$ yields

$$p = \frac{2c^4+2c^3-5c^2-2c+3}{3c\left(c^3+2c+6\right)}. \tag{2.22}$$

A more involved computation shows that $\frac{\partial G(c,1,p)}{\partial c} = 0$ implies

$$(9c^5-12c^3+27c^2-24c-36)p^2-(12c^5+10c^4-52c^3-30c^2+46c+8)p$$
$$+(6c^5+5c^4-42c^3-33c^2+57c+37) = 0. \tag{2.23}$$

Substituting (2.22) in (2.23) and performing a lengthy computation gives

$$\frac{(c^3-7c-3)\zeta_1(c)}{3c^2\left(c^3+2c+6\right)^2} = 0 \tag{2.24}$$

where

$$\zeta_1(c) = 6c^{10}-5c^9+20c^8+86c^7-49c^6+257c^5$$
$$+623c^4-629c^3-1095c^2-60c+36.$$

The numerical solutions of (2.24) such that $0 < c < 2$ are $c \approx 0.151355$ and $c \approx 1.30718$. Substituting these values of c in (2.22) gives $p \approx 0.904769$ and $p \approx 0.050509$. The corresponding values of $G(c,1,p)$ are $G(0.151355, 1, 0.904769) = 6.83676$ and $G(1.30718, 1, 0.050509) = 11.2488$ respectively.

As $G(c,1,p)$ is uniformly continuous on $[0,2]\times\{1\}\times[-1,1]$, the difference between extremum values of $G(c,1,p)$ and either of 6.83676 or 11.2488 can be made smaller than an $\epsilon \ll 1$. Therefore

$$\max_{0<c<2,\ -1<p<1} G(c,1,p) \approx 11.2488. \tag{2.25}$$

On the face $p=-1$, $G(c,r,p)$ reduces to

$$G(c,r,-1)=\frac{1}{2}(3r^2+2r+1)c^3+(3r^2+r-3)c^2-(6r^2+4r-1)c-(12r^2+4r-15).$$

Now we show that $\frac{\partial G(c,r,-1)}{\partial c}=0$ and $\frac{\partial G(c,r,-1)}{\partial r}=0$ have no solution in the interior of this face. On the contrary, assume that $\frac{\partial G(c,r,-1)}{\partial c}=0$ and $\frac{\partial G(c,r,-1)}{\partial r}=0$ have a solution in the interior of the face $p=-1$. Then $\frac{\partial G(c,r,-1)}{\partial r}=0$ gives

$$r=\frac{c+1}{3(2-c)}. \tag{2.26}$$

By substituting (2.26) in $\frac{\partial G(c,r,-1)}{\partial c}=0$, we obtain $c=\frac{1}{6}(-4\pm\sqrt{190})$, both of which lie outside the range of $c\in[0,2]$.

On the face $p=1$, $G(c,r,p)$ reduces to

$$G(c,r,1)=\frac{1}{2}(3r^2-2r+1)c^3+(3r^2-r-3)c^2-(6r^2+4r-1)c-(12r^2-4r-15).$$

At the points where $G(c,r,1)$ attains the maximum value, $\frac{\partial G(c,r,1)}{\partial c}$ and $\frac{\partial G(c,r,1)}{\partial r}$ necessarily vanish. The solution to the pair of equations $\frac{\partial G(c,r,1)}{\partial c}=0$ and $\frac{\partial G(c,r,1)}{\partial r}=0$ is $(c,r)=(\frac{1}{2}(60-\sqrt{30}),\frac{1}{105}(25-\sqrt{30}))$ and subsequently

$$G\left(\frac{1}{2}(6-\sqrt{30}),\frac{1}{105}(25-\sqrt{30}),1\right)=5\sqrt{\frac{15}{2}}+\frac{11}{6}=15.5264.$$

Further computations show that

$$\max_{0<c<2,\ 0<r<1} G(c,r,1)=\sqrt{\frac{15}{2}}+\frac{11}{6}=15.5264.$$

Now we find out the maximum values attained by $G(c,r,p)$ on the edges of R. Evaluating (2.19) on the edge $c=0$, $p=1$ we obtain $G(0,r,1)=12(1-r^2)+4r+3$. A simple computation shows that the maximum of $G(0,r,1)$ is $46/3$ which occurs at $r=1/6$. At the end points of this edge, we have $G(0,0,1)=15$ and $G(0,1,1)=7$. Hence

$$\max_{0\le r\le 1} G(0,r,1)=\frac{46}{3}.$$

In view of (2.19), we obtain by a series of straightforward computations the maximum value of $G(c,r,p)$ on the edges $c=0$, $r=0$; $c=0$, $r=1$ and $c=0$, $p=-1$ as

$$\max_{-1\le p\le 1} G(0,0,p)=15,\quad \max_{-1\le p\le 1} G(0,1,p)=7 \quad\text{and}\quad \max_{0\le r\le 1} G(0,r,-1)=15.$$

A simple observation shows that $G(2, r, p) = 9$ implies

$$\max_{-1\le p\le 1} G(2, 0, p) = \max_{-1\le p\le 1} G(2, 1, p) = \max_{0\le r\le 1} G(2, r, -1) = \max_{0\le r\le 1} G(2, r, 1) = 9.$$

As (2.20) is independent of p, the maximum value of $G(c, r, p)$ on the edges $r = 0$, $p = -1$ and $r = 0$, $p = 1$ is

$$\max_{0\le c\le 2} G(c, 0, -1) = \max_{0\le c\le 2} G(c, 0, 1) = 15.0858.$$

On the edge $r = 1$, $p = -1$, (2.21) can be simplified to $G(c, 1, -1) = |3c^3 + c^2 - 9c - 1|$. A straightforward calculation shows that

$$\max_{0\le c\le 2} G(c, 1, -1) = 9.$$

On the edge $r = 1$, $p = 1$, (2.21) reduces to $G(c, 1, 1) = c^3 - c^2 - c + 7$. A simple computation shows that

$$\max_{0\le c\le 2} G(c, 1, 1) = 9.$$

Now we show that $G(c, r, p)$ does not attain maximum value in the interior of the cuboid R. In order to find the points where the maximum value is obtained in the interior of R, we solve $\frac{\partial G(c,r,p)}{\partial c} = 0$, $\frac{\partial G(c,r,p)}{\partial r} = 0$ and $\frac{\partial G(c,r,p)}{\partial p} = 0$. A computation shows that $\frac{\partial G(c,r,p)}{\partial p} = 0$ implies

$$p = \frac{3c^4r^2 + c^4 + 3c^3r^2 + c^3 - 12c^2r^2 + 2c^2 - 12cr^2 + 8c + 6}{6c(c^3 + 2c + 6)r}. \tag{2.27}$$

By substituting (2.27) in $\frac{\partial G(c,r,p)}{\partial r} = 0$, we get

$$r = \frac{\sqrt{c^3 + 2c + 6}}{\sqrt{3}\sqrt{c^3 - 4c}}. \tag{2.28}$$

It is easy to see that $\frac{c^3+2c+6}{3(c^3-4c)}$ is negative for all values of $c \in [0, 2]$. Hence there cannot be an extremum inside the cuboid R. This shows that the maximum value of $|\gamma_3|$ is $\frac{1}{48}(5\sqrt{\frac{15}{2}} + \frac{11}{6})$ for $(c, r, p) = (\frac{1}{2}(6 - \sqrt{30}), \frac{1}{105}(25 - \sqrt{30}), 1)$.

Let $c = c_1$ and $(c, r, p) = (\frac{1}{2}(6 - \sqrt{30}), \frac{1}{105}(25 - \sqrt{30}), 1)$. Then in view of Lemma 1.3 we obtain $c_2 = \frac{1}{12}(76 - 13\sqrt{30})$ and $c_3 = \frac{1}{72}(554 - 75\sqrt{30})$. It is not difficult to see that a function $G^* \in \mathcal{P}$ having

$$(c_1, c_2, c_3) = \left(\frac{1}{2}(6 - \sqrt{30}), \frac{1}{12}(76 - 13\sqrt{30}), \frac{1}{72}(554 - 75\sqrt{30})\right)$$

is given by $G^*(z) = H_{t_1,\mu_1}(z)$ where $\mu_1 = \frac{1}{12}(-1-\sqrt{30}) + i\frac{1}{12}\sqrt{113-2\sqrt{30}}$, and $t_1 = \frac{3}{278}(15\sqrt{30}-56)$. Therefore the bound in (iii) is sharp for the function $F_1(z)$ such that

$$zF_1'(z) = \frac{z}{1-z}G^*(z).$$

□

Theorem 2.29 *Let $f \in \mathcal{F}_2$ be given by (1.1). Then*

(i) $|\gamma_1| \le \frac{1}{2}$,
(ii) $|\gamma_2| \le \frac{1}{2}$.
(iii) *If* $0 \le a_2 \le 1$ *then* $|\gamma_3| \le \frac{1}{972}(95 + 23\sqrt{46})$.

The inequalities are sharp.

Proof Let $f \in \mathcal{F}_2$. It is evident that f is close-to-convex with respect to the starlike function $g(z) = z/(1-z^2)$. From (2.2), $f(z)$ can be written as

$$zf'(z) = \frac{z}{1-z^2}h(z). \tag{2.30}$$

Thus in view of (2.30), (2.8) reduces to

$$4|\gamma_1| \le |c_1|. \tag{2.31}$$

Noting that $|c_1| \le 2$, (2.31) then implies that $|\gamma_1| \le 1/2$. It is easy to see that A function $p \in \mathcal{P}$ having $|c_1| = 2$ is given by $p(z) = L_{1,\theta}(z)$ for $0 \le \theta < 2\pi$. Substituting $L_{1,\theta}(z)$ in place of $h(z)$ in (2.30) shows that (i) is sharp.

A comparison of (2.30) and (2.2) shows that (2.9) reduces to

$$6\gamma_2 \le \left(c_2 - \frac{3}{8}c_1^2\right) + 1.$$

Applying the triangle inequality in conjunction with Lemma 1.5 with $\mu = 3/8$ shows that $|\gamma_2| \le 1/2$. It is evident from Lemma 1.5 that the equality holds in (ii) for the function $\widetilde{F_2}(z)$ such that $z\widetilde{F_2}'(z) = z(1-z^2)^2 L_{0,0}(z)$.

Considering (2.30) as an instance of (2.2), (2.13) can be simplified to

$$96|\gamma_3| \le 6(4-c^2)(1-r^2) + c\sqrt{\phi_2\,(c,r,p)}, \tag{2.32}$$

where

$$\begin{aligned}\phi_2(c,r,p) = {} & (c^2+4)^2 + 2r(4-c^2)(4+c^2)(2p+3r-6p^2r)\\ & + r^2(4-c^2)^2(4+9r^2-12rp).\end{aligned}$$

Let $F(c, r, p) = 6(1 - r^2)(4 - c^2) + c\sqrt{\phi_2(c, r, p)}$. We find points where $F(c, r, p)$ attains the maximum value by finding its local maxima on the six faces and in the interior of R. On the face $c = 0$, $F(c, r, p)$ becomes

$$F(0, r, p) = 24(1 - r^2). \tag{2.33}$$

As $F(0, r, p)$ is a decreasing function of r, the maximum value of $F(0, r, p)$ is attained on the edge $c = 0, r = 0$. Consequently, we have

$$\max_{0 \le r \le 1,\ -1 \le p \le 1} F(0, r, p) = 24.$$

On the face $c = 2$, $F(c, r, p)$ becomes $F(2, r, p) = 16$ and hence

$$\max_{0 \le r \le 1,\ -1 \le p \le 1} F(2, r, p) = 16.$$

On the face $r = 0$, we can simplify $F(c, r, p)$ as

$$F(c, 0, p) = 24 - 6c^2 + c\left(c^2 + 4\right). \tag{2.34}$$

Since $F(c, 0, p)$ is independent of p, we find the set of all points where $\frac{\partial F(c,0,p)}{\partial c}$ vanishes as $\{\frac{2}{3}\left(3 - \sqrt{6}\right)\} \times \{0\} \times [-1, 1]$ and hence $F\left(\frac{2}{3}\left(3 - \sqrt{6}\right), 0, p\right) = \frac{16}{9}(9 + 2\sqrt{6}) = 24.7093$. Evaluating (2.34) on the edges $c = 0, r = 0$ and $c = 2, r = 0$, we obtain

$$\max_{0 \le c \le 2,\ -1 \le p \le 1} F(c, 0, p) = 24.7093.$$

On the face $r = 1$, $F(c, r, p)$ reduces to

$$F(c, 1, p) = 2c\sqrt{24c^2(p - 1) - 16(p - 1)(5 + 3p) + c^4(2 - 4p + 3p^2)}. \tag{2.35}$$

We solve $\frac{\partial F(c,1,p)}{\partial c} = 0$ and $\frac{\partial F(c,1,p)}{\partial p} = 0$ to determine points where maxima occur in the face $r = 1$. A computation shows that $\frac{\partial F(c,1,p)}{\partial p} = 0$ implies

$$p = \frac{2\left(c^2 - 2\right)}{3\left(c^2 + 4\right)}. \tag{2.36}$$

A slightly involved computation shows that $\frac{\partial F(c,1,p)}{\partial c} = 0$ gives

$$(18c^4 - 96)p^2 - 8(3c^4 - 12c^2 + 8)p + (12c^4 - 96c^2 + 160) = 0. \tag{2.37}$$

Substituting (2.36) in (2.37) followed by a computation gives

$$\frac{4(3c^8 - 160c^4 - 512c^2 + 2048)}{3(c^2+4)^2} = 0. \tag{2.38}$$

The numerical solution of (2.38) in $0 < c < 2$ is $c \approx 1.54836$. Using (2.36) we then obtain $p \approx 0.414152$. Therefore $F(1.54836, 1, 0.414152) = 18.0595$.

Using uniform continuity of $F(c, 1, p)$ on $[0, 2] \times \{1\} \times [-1, 1]$ we infer that the difference between the maximum value of $F(c, 1, p)$ and 18.0595 can be made smaller than an $\epsilon \ll 1$. On the edge $c = 0$, $r = 1$, $F(c, r, p)$ becomes $F(0, 1, p) = 0$. On the edge $c = 2$, $r = 1$, $F(c, r, p)$ becomes $F(2, 1, p) = 16$. On the edge $r = 1$, $p = -1$, (2.35) can be simplified to $F(c, 1, -1) = 2c|3c^2 - 8|$. It is easy to see that $F(c, 1, -1)$ has the maximum value 16 on $[0, 2]$.

A simple computation shows that the maximum value of $F(c, r, p)$ on the edge $r = 1$, $p = 1$ is 16. Therefore,

$$\max_{0\le c\le 2,\ -1\le p\le 1} F(c, 1, p) \approx 18.0595.$$

On the face $p = -1$, $F(c, r, p)$ reduces to

$$F(c, r, -1) = 6(4 - c^2)(1 - r^2) + c|c^2 + 4 - (2r - 3r^2)(4 - c^2)|.$$

A computation similar to the one on the face $p = -1$ in Theorem 2.14 shows that $\frac{\partial F(c,r,-1)}{\partial c} = 0$ and $\frac{\partial F(c,r,-1)}{\partial r} = 0$ have no solution in the interior of the face $p = -1$. Thus the maximum value is attained on the edges.

On the edge $c = 0$, $p = -1$, $F(c, r, p)$ becomes $F(0, r, -1) = 24(1 - r^2)$. The maximum value of $F(0, r, -1)$ is clearly 24. On the edge $r = 0$, $p = -1$, $F(c, r, p)$ becomes

$$F(c, 0, -1) = 6(4 - c^2) + c(4 + c^2).$$

The maximum value of $F(c, 0, -1)$ is $\frac{16}{9}(9 + 2\sqrt{6}) = 24.7093$ (see the face $r = 0$). The maximum values of $F(c, r, p)$ on the edges $c = 2$, $p = -1$ and $r = 1$, $p = -1$ are 16 and 10.0566 respectively (see the faces $c = 2$ and $r = 1$). Therefore

$$\max_{0\le c\le 2,\ 0\le r\le 1} F(c, r, -1) = \frac{16}{9}(9 + 2\sqrt{6}) = 24.7093.$$

On the face $p = 1$, $F(c, r, p)$ reduces to

$$F(c, r, 1) = 6(4 - c^2)(1 - r^2) + c|c^2 + 4 + (2r + 3r^2)(4 - c^2)|.$$

Solving $\frac{\partial F(c,r,1)}{\partial c} = 0$ and $\frac{\partial F(c,r,1)}{\partial r} = 0$ we obtain $(c, r) = \left(\frac{1}{3}(8 - \sqrt{46}), \frac{1}{75}(11 - \sqrt{46})\right)$ and hence $F\left(\frac{1}{3}(8 - \sqrt{46}), \frac{1}{75}(11 - \sqrt{46}), 1\right) = \frac{8}{81}(95 + 23\sqrt{46}) = 24.7895$. It is not difficult to see that the maximum value of $F(c, r, 1)$ on the edges is 24.7093, which occurs on the edge $r = 0$, $p = 1$ (see the face $r = 0$) as the computations for the edges have been done on earlier faces. Therefore

$$\max_{0\le c\le 2,\ 0\le r\le 1} F(c, r, 1) = \frac{8}{81}(95 + 23\sqrt{46}) = 24.7895.$$

We now show that $F(c, r, p)$ cannot attain a maximum in the interior of the cuboid R. To determine points in the interior of R where the maxima occurs (if any), we solve $\frac{\partial F(c,r,p)}{\partial c} = 0$, $\frac{\partial F(c,r,p)}{\partial r} = 0$ and $\frac{\partial F(c,r,p)}{\partial p} = 0$. A computation shows that $\frac{\partial F(c,r,p)}{\partial p} = 0$ implies

$$p = \frac{3c^2r^2 + c^2 - 12r^2 + 4}{6(c^2 + 4)r}. \tag{2.39}$$

Using (2.39) in $\frac{\partial F(c,r,p)}{\partial r} = 0$ and then solving for r yields

$$r = \frac{\sqrt{c^2 + 4}}{\sqrt{3}\sqrt{c^2 - 4}}.$$

As $\frac{c^2+4}{3(c^2-4)}$ is negative for all values of $c \in [0, 2]$, there cannot be an extremum in the interior of R. This proves that the maximum value of $|\gamma_3|$ is $\frac{1}{972}(95 + 23\sqrt{46})$ for $(c, r, p) = \left(\frac{1}{3}(8 - \sqrt{46}), \frac{1}{75}(11 - \sqrt{46}), 1\right)$.

Let $c = c_1$ and $(c, r, p) = \left(\frac{1}{3}(8 - \sqrt{46}), \frac{1}{75}(11 - \sqrt{46}), 1\right)$. Then in view of Lemma 1.3, we obtain $c_2 = \frac{1}{27}(134 - 19\sqrt{46})$ and $c_3 = \frac{2}{243}(721 - 71\sqrt{46})$. It is not difficult to see that a function $F^* \in \mathcal{P}$ having

$$(c_1, c_2, c_3) = \left(\frac{1}{3}(8 - \sqrt{46}), \frac{1}{27}(134 - 19\sqrt{46}), \frac{2}{243}(721 - 71\sqrt{46})\right)$$

is given by $F^*(z) = H_{t_2,\mu_2}(z)$ where $\mu_2 = \frac{1}{18}(-1-\sqrt{46})+i\,\frac{1}{18}\sqrt{277 - 2\sqrt{46}}$ and $t_2 = \frac{1}{10}(\sqrt{46}-4)$. This shows that the bound in (iii) is sharp for the function $F_2(z)$ such that $zF_2'(z) = z(1 - z^2)^{-1}F^*(z)$. □

Theorem 2.40 *Let $f \in \mathcal{F}_3$ be given by (1.1). Then*

(i) $|\gamma_1| \le \frac{3}{4}$,
(ii) $|\gamma_2| \le \frac{2}{5}$.
(iii) *If* $1/2 \le a_2 \le 3/2$ *then* $|\gamma_3| \le \frac{743+131\sqrt{262}}{7776}$.

The inequalities are sharp.

Proof Let $f \in \mathcal{F}_3$. Then f is close-to-convex with respect to the starlike function $g(z) = z/(1 - z + z^2)$. In view of (2.2), $f(z)$ can be written as

$$zf'(z) = \frac{z}{1 - z + z^2}h(z). \tag{2.41}$$

Therefore (2.8) reduces to

$$4|\gamma_1| \le 1 + |c_1|. \tag{2.42}$$

Thus from (2.42) we obtain $|\gamma_1| \le 3/4$ as $|c_1| \le 2$ for $h \in \mathcal{P}$. A function in $\mathcal{P}$ having $|c_1| = 2$ is given by $L_{1,\theta}(z)$, $0 \le \theta < 2\pi$ The equality in (i) is attained for a function $\widetilde{f}(z)$ such that $z\widetilde{f}'(z) = z(1 - z + z^2)^{-1} L_{1,\theta}(z)$.

In view of (2.41), (2.10) becomes

$$6|\gamma_2| \le 2 - \frac{|c_1|^2}{2} + \frac{1}{8}\sqrt{(d^2 + 1 - 2dt)(d^2 + 9 + 6dt)} =: k(d, q). \tag{2.43}$$

It is evident from (2.43) that it is sufficient to find the maximum value of $k(d, q)$ in the square S to obtain the same for $|\gamma_2|$.

To obtain points where $k(d, q)$ attains maximum, we solve $\frac{\partial k(d,q)}{\partial d} = 0$ and $\frac{\partial k(d,q)}{\partial q} = 0$. The solutions obtained are complex, showing that $k(d, q)$ does not attain maximum in the interior of S.

On the side $d = 0$, $k(d, q)$ reduces to $k(d, q) = 2.375$. On the side $d = 2$, we see that $k(d, q) = (\sqrt{65 + 8t} - 48t)/8$. An elementary computation shows that $\max_{-1 \le q \le 1} k(2, q) = k(2, 1/12) = 1.01036$.

On the side $q = -1$, $k(d, q)$ becomes $k(d, -1) = (19 + 2d - 5d^2)/8$. A straightforward computation shows that $\max_{0 \le d \le 2} k(d, -1) = k(1/5, -1) = 12/5 = 2.4$.

On the side $q = 1$, $k(d, q)$ may be simplified as $k(d, 1) = (19 - 2d - 5d^2)/8$. As $k(d, 1)$ is a decreasing function for $d \in [0, 2]$, we see that $\max_{0 \le d \le 2} k(d, 1) = k(0, 1) = 19/8 = 2.375$.

Thus the maximum value of $k(d, q)$ in S is $12/5$ and occurs at $(d, q) = (1/5, -1)$. Consequently, (2.43) implies that $|\gamma_2| \le 2/5$, with the equality occurring for $c_1 = -1/5$.

Therefore, in view of Lemma 1.4, the equality in (ii) holds for the function $\widetilde{F_2}(z)$ such that $z\widetilde{F_2}'(z) = z(1 - z + z^2)^{-1} L_{t,\theta}(z)$ where $t = 1/10$ and $\theta = \pi$.

Using (2.41) we may rewrite (2.13) as

$$96|\gamma_3| \le 6(1 - r^2)(4 - c^2) + \sqrt{\phi_3(c, r, p)} \tag{2.44}$$

where

$$\begin{aligned}\phi_3(c, r, p) = {} & (c^3 - 2c - 10)^2 + 2r(4 - c^2)(c^3 - 2c - 10)(2p + 2cp - 6crp^2 + 3rc) \\ & + r^2(4 - c^2)^2(4c^2 + 4 + 9c^2r^2 + 8c - 12c^2rp - 12crp).\end{aligned}$$

Let $K(c, r, p) = 6(1 - r^2)(4 - c^2) + \sqrt{\phi_3(c, r, p)}$. We find the points in the cuboid R where the maxima of $K(c, r, p)$ occur.

On the face $c = 0$, $K(c, r, p)$ takes the following form

$$K(0, r, p) = 24(1 - r^2) + 2\sqrt{25 - 40rp + 16r^2}. \tag{2.45}$$

By solving $\frac{\partial K(0,r,p)}{\partial r} = 0$ and $\frac{\partial K(0,r,p)}{\partial p} = 0$ we obtain $(r, p) = (0, 0)$. Thus $K(c, r, p)$ does not attain maximum in the interior of the face $c = 0$.

On the face $c = 2$, $K(c, r, p)$ reduces to $K(2, r, p) = 6$ and hence

$$\max_{0<r<1,\ -1<p<1} K(2, r, p) = 6.$$

On the face $r = 0$, $K(c, r, p)$ may be simplified as

$$K(c, 0, p) = 6(4 - c^2) + |c^3 - 2c - 10|. \tag{2.46}$$

Since $K(c, 0, p)$ is independent of p, it suffices to find out points such that $\frac{\partial K(c,0,p)}{\partial c} = 0$. The set of all such points is $\{\frac{1}{3}(-6 + \sqrt{42})\} \times \{0\} \times [-1, 1]$ and

$$K\left(\frac{1}{3}(-6 + \sqrt{42}), 0, p\right) = \frac{14}{9}(9 + 2\sqrt{42}) = 34.1623.$$

Therefore

$$\max_{0<c<2,\ -1<p<1} K(c, 0, p) = \frac{14}{9}(9 + 2\sqrt{42}) = 34.1623.$$

On the face $r = 1$, $K(c, r, p)$ becomes

$$K(c, 1, p) = \sqrt{\psi_3(c, p) + 2(c^3 - 2c - 10)(c^2 - 4)(6cp^2 - 2cp - 2p - 3c)} \tag{2.47}$$

where

$$\psi_3(c, p) = (c^3 - 2c - 10)^2 + (c^2 - 4)^2(13c^2 - 12c^2 p + 8c - 12cp + 4).$$

A computation shows that $\frac{\partial K(c,1,p)}{\partial p} = 0$ implies

$$p = \frac{2c^4 + 2c^3 - 7c^2 - 12c - 5}{3c(c^3 - 2c - 10)}. \tag{2.48}$$

A lengthy computation shows that $\frac{\partial K(c,1,p)}{\partial c}$ implies

$$\begin{aligned}(9c^5 - 36c^3 - 45c^2 + 24c + 60)p^2 \\ - (12c^5 + 10c^4 - 60c^3 - 60c^2 + 46c + 48)p \\ + (6c^5 + 5c^4 - 34c^3 - 9c^2 + 33c - 9) = 0.\end{aligned} \tag{2.49}$$

Substituting (2.48) in (2.49) and then performing another lengthy computation gives

$$\frac{(c^3 - 5c + 5)\zeta_2(c)}{3c^2(c^3 - 2c - 10)^2} = 0 \tag{2.50}$$

where

$$\zeta_2(c) = 6c^{10} - 5c^9 - 32c^8 - 104c^7 + 147c^6 + 375c^5 \\ +459c^4 - 375c^3 - 1135c^2 + 140c + 100.$$

The numerical solutions of (2.50) are obtained as $c \approx 0.354278$ and $c \approx 1.27688$. Further computations show that $K(c, 1, p)$ does not attain a maxima at these points even though the partial derivatives vanish. On the face $p = -1$, $K(c, r, p)$ reduces to

$$K(c, r, -1) = 6(1 - r^2)(4 - c^2) - (c^3 - 2c - 10) + 2(4 - c^2)(2 + 2c + 3cr^2).$$

By solving $\frac{\partial K(c,r,-1)}{\partial c} = 0$ and $\frac{\partial K(c,r,-1)}{\partial r} = 0$ we obtain $c = \frac{1}{6}(-14 + \sqrt{262})$ and $r = \frac{1}{69}(3 + \sqrt{262})$. The corresponding maximum value is

$$K\left(\frac{1}{6}(-14 + \sqrt{262}), \frac{1}{69}(3 + \sqrt{262}), -1\right) = \frac{1}{81}(743 + 131\sqrt{262}) = 35.3509.$$

Therefore

$$\max_{0<c<2,\ 0<r<1} K(c, r, -1) = 35.3509.$$

On the face $p = 1$, $K(c, r, p)$ reduces to

$$K(c, r, 1) = 6(1 - r^2)(4 - c^2) - (c^3 - 2c - 10) + (4 - c^2)(3cr^2 - 2 - 2c). \quad (2.51)$$

It is not difficult to see that $\frac{\partial K(c,r,1)}{\partial c} = 0$ and $\frac{\partial K(c,r,1)}{\partial r} = 0$ have no solution in the interior of the face $p = 1$. Thus $K(c, r, p)$ does not attain maximum in the interior of this face.

Now we find the maximum values attained on the edges of R. It is evident from (2.45) that on the edges $c = 0, r = 0$ and $c = 0, r = 1$, the maximum values of $K(c, r, p)$ are

$$\max_{-1\le p\le 1} K(0, 0, p) = 34 \quad \text{and} \quad \max_{-1\le p\le 1} K(0, 1, -1) = 18.$$

On the edge $c = 0$, $p = -1$, (2.45) reduces to $K(0, r, -1) = 24(1 - r^2) + 2(5 + 4r)$. An elementary computation shows that the maximum value of $K(0, r, -1)$ is attained at $\left(0, \frac{1}{6}, -1\right)$ and $\max_{0\le r\le 1} K(0, r, -1) = 104/3$.

On the edge $c = 0$, $p = 1$, (2.45) reduces to $K(0, r, 1) = 24(1 - r^2) + 2(5 - 4r)$. A computation shows that $\max_{0\le r\le 1} K(0, r, 1) = 34$.

It is evident that $K(2, r, p) = 6$ implies

$$\max_{-1\le p\le 1} K(2, 0, p) = \max_{-1\le p\le 1} K(2, 1, p) = \max_{0\le r\le 1} K(2, r, -1) = \max_{0\le r\le 1} K(2, r, 1) = 6.$$

Considering (2.46) and the maximum value of $K(c, 0, p)$ we obtain the maximum values on the edges $r = 0$, $p = -1$ and $r = 0$, $p = 1$ as

$$\max_{0\le c\le 2} K(c, 0, -1) = \max_{0\le c\le 2} K(c, 0, 1) = 34.1623.$$

On the edge $r = 1$, $p = -1$, (2.47) maybe be simplified as

$$K(c, 1, -1) = \sqrt{\left(c^3 - 2c - 10\right)^2 - 2(5c + 2)\left(c^3 - 2c - 10\right)(4 - c^2) + \left(25c^2 + 20c + 4\right)(4 - c^2)^2}.$$

A computation shows that $K(c, 1, -1)$ attains the local maximum at $(1, 1, -1)$ and $\max_{0\le c\le 2} K(c, 1, -1) = 32$.

On the edge $r = 1$, $p = 1$, (2.47) reduces to $K(c, 1, 1) = 2(1 + 3c + c^2 - c^3)$. An elementary computation shows that

$$\max_{0\le c\le 2} K(c, 1, 1) = K\left(\frac{1}{3}(1 + \sqrt{10}), 1, 1\right) = \frac{8}{27}(14 + 5\sqrt{10}) = 8.833.$$

Now we show that $K(c, r, p)$ does not attain maximum in the interior of the cuboid R. At the points where the maxima occur in the cuboid R we have $\frac{\partial K(c,r,p)}{\partial c} = 0$, $\frac{\partial K(c,r,p)}{\partial r} = 0$ and $\frac{\partial K(c,r,p)}{\partial p} = 0$. A computation shows that $\frac{\partial K(c,r,p)}{\partial p} = 0$ implies

$$p = \frac{3c^4r^2 + c^4 + 3c^3r^2 + c^3 - 12c^2r^2 - 2c^2 - 12cr^2 - 12c - 10}{6c\left(c^3 - 2c - 10\right)r}. \tag{2.52}$$

Substituting (2.52) in $\frac{\partial K(c,r,p)}{\partial r} = 0$ and then solving for r we obtain

$$r = \frac{\sqrt{c^3 - 2c - 10}}{\sqrt{3c^3 - 12c}}. \tag{2.53}$$

Substituting (2.53) in (2.52) gives

$$p = \frac{(c + 1)\sqrt{c(c^2 - 4)}}{\sqrt{3}c\sqrt{c^3 - 2c - 10}}. \tag{2.54}$$

Substituting (2.53) and (2.54) in $\frac{\partial K(c,r,p)}{\partial c} = 0$, we obtain

$$\frac{8\left(c^3 - 5c + 5\right)}{c^2 - 4} = 0.$$

It can be seen that the roots to the above equation are either negative or imaginary. This shows that a maximum cannot be attained inside R. Thus we see that the maximum value for $|\gamma_3|$ is attained for

$$(c, r, p) = \left(\frac{1}{6}(-14 + \sqrt{262}), \frac{1}{69}(3 + \sqrt{262}), -1\right)$$

and is equal to $(743 + 131\sqrt{262})/81 = 35.3509$. Using these values of (c, r, p) in Lemma 1.3, we obtain $c_2 = \frac{1}{108}(548 - 37\sqrt{262})$ and $c_3 = \frac{47525\sqrt{262}-698926}{44712}$. Therefore for given

$$(c_1, c_2, c_3) = \left(\frac{1}{6}(-14 + \sqrt{262}), \frac{1}{108}(548 - 37\sqrt{262}), \frac{47525\sqrt{262} - 698926}{44712}\right)$$

there exists a function $K^* \in \mathcal{P}$ given by $K^*(z) = H_{t_3,\mu_3}(z)$, where

$$\mu_3 = \frac{-769 + 35\sqrt{262}}{828} + i\frac{\sqrt{-226727 + 53830\sqrt{262}}}{828}$$
$$\text{and } t_3 = \frac{32352 - 687\sqrt{262}}{64622}.$$

The inequality (iii) is sharp for the function $F_3(z)$ such that

$$zF_3'(z) = \frac{z}{1 - z + z^2}K^*(z).$$

□

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