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Several properties of α-harmonic functions in the unit disk

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Abstract The aim of this paper is to obtain the Schwarz–Pick type inequality for α-harmonic functions f in the unit disk and get estimates on the coefficients of f. As an application, a Landau type theorem of α-harmonic functions is established.

Keywords Laplace differential operator · Coefficient estimate · Schwarz–Pick estimate · Landau type theorem

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1 Introduction and main results

Let $\mathbb{C}$ be the complex plane. For $a \in \mathbb{C}$, let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$ $(r > 0)$ and $\mathbb{D}(0, r) = \mathbb{D}_r$. Also, we use the notations $\mathbb{D} = \mathbb{D}_1$ and $\mathbb{T} = \partial\mathbb{D}$, the boundary of $\mathbb{D}$.

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Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

We will consider the matrix norm

$$|A| = \sup\{|Az| : \ z \in \mathbb{C}, \ |z| = 1\}$$

and the matrix function

$$l(A) = \inf\{|Az| : \ z \in \mathbb{C}, \ |z| = 1\}.$$

Let D and Ω be domains in $\mathbb{C}$, and let $f = u + iv \colon D \to \Omega$ be a function that has both partial derivatives at $z = x + iy$ in D, where u and v are real functions. The Jacobian matrix of f at z is denoted by

$$Df(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Set

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

Then

$$|Df(z)| = \sup\{|Df(z)\varsigma| : \ |\varsigma| = 1\} = |f_z(z)| + |f_{\overline{z}}(z)|, \tag{1.1}$$
$$l(Df(z)) = \inf\{|Df(z)\varsigma| : \ |\varsigma| = 1\} = \big||f_z(z)| - |f_{\overline{z}}(z)|\big| \tag{1.2}$$

and

$$|J_f(z)| = |Df(z)| \cdot l(Df(z)),$$

where $J_f(z)$ stands for the Jacobian of f at z.

We denote by Δ_α the weighted Laplace operator corresponding to the so-called standard weight $w_\alpha = (1 - |z|^2)^\alpha$, that is,

$$\Delta_{\alpha,z} = \frac{\partial}{\partial z}(w_\alpha)^{-1}\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial z}(1 - |z|^2)^{-\alpha}\frac{\partial}{\partial \overline{z}}$$

in $\mathbb{D}$, where $\alpha > -1$ (see [20, Proposition 1.5] for the reason for this constraint).

Olofsson and Wittsten [20] introduced this operator Δ_α and a counterpart of the classical Poisson integral formula was given.

We remark that in the study of Bergman spaces of $\mathbb{D}$, one often considers the weights w_α in $\mathbb{D}$ ($\alpha > -1$). For an account of recent developments in Bergman space theory, we mention the monograph by Hedenmalm et al. [13]. The case $\alpha = 0$ is commonly referred to as the unweighted case, whereas the case $\alpha = 1$ has attracted special attention recently with contributions by Hedenmalm, Shimorin and others (see for instance [14–16,22] etc).

Of particular interest to us is the following α-harmonic equation in $\mathbb{D}$:

$$\Delta_\alpha(f) = 0. \tag{1.3}$$

Denote the associated *Dirichlet boundary value problem* of functions f satisfying the Eq. (1.3) by

$$\begin{cases} \Delta_\alpha(f) = 0 & \text{in } \mathbb{D}, \\ f = f^* & \text{on } \mathbb{T}. \end{cases} \tag{1.4}$$

Here the boundary data f^* is a distribution on $\mathbb{T}$, i.e. $f^* \in \mathcal{D}'(\mathbb{T})$, and the boundary condition in the Eq. (1.4) is to be understood as $f_r \to f^* \in \mathcal{D}'(\mathbb{T})$ as $r \to 1^-$, where

$$f_r(e^{i\theta}) = f(re^{i\theta}) \tag{1.5}$$

for $\theta \in [0, 2\pi]$ and $r \in [0, 1)$.

For simplicity, we introduce the following definition.

Definition 1.1 For $\alpha > -1$, a complex-valued function f is said to be α-*harmonic* if f is twice continuously differentiable in $\mathbb{D}$ and satisfies the condition (1.3).

Olofsson and Wittsten [20] showed that if an α-harmonic function f satisfies

$$\lim_{r\to 1^-} f_r = f^* \in \mathcal{D}'(\mathbb{T}) \ \ (\alpha > -1),$$

then it has the form of a *Poisson type integral*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}_\alpha(ze^{-i\theta}) f^*(e^{i\theta}) d\theta \tag{1.6}$$

in $\mathbb{D}$, where

$$\mathcal{P}_\alpha(z) = \frac{(1-|z|^2)^{\alpha+1}}{(1-z)(1-\overline{z})^{\alpha+1}}.$$

In the following, we always assume that any α-harmonic function has such a representation which plays a key role in the discussions of this paper.

Obviously, α-harmonicity coincides with harmonicity when $\alpha = 0$. See [12] and the references therein for the properties of harmonic mappings. Particularly, Colonna proved the following Schwarz–Pick type inequality.

Theorem A [11, Theorems 3 and 4] *Let f be a harmonic function of $\mathbb{D}$ into $\mathbb{D}$. Then for $z \in \mathbb{D}$,*

$$|Df(z)| \le \frac{4}{\pi} \cdot \frac{1}{1-|z|^2}.$$

This estimate is sharp and all the extremal functions are

$$f(z)=\frac{2\delta}{\pi}\arg\Big(\frac{1+\psi(z)}{1-\psi(z)}\Big),$$

where $\delta\in\mathbb{C}$, $|\delta|=1$ *and* ψ *is a conformal automorphism of* $\mathbb{D}$.

For the related discussions on this topic, see [2,4,7,10,17,21] etc.

As the first aim of this paper, we shall generalize Theorem A to the case of α-harmonic functions. Our first result is as follows.

Theorem 1.1 *Suppose that* f *is an* α*-harmonic function in* $\mathbb{D}$ *with* $\alpha>-1$, *that* $f^*\in C(\mathbb{T})$ *and that* $\sup_{z\in\overline{\mathbb{D}}}|f(z)|\leq M$, *where* M *is a constant. Then for* $z\in\mathbb{D}$,

$$|Df(z)|\leq\frac{M(\alpha+2)}{c_\alpha}\cdot\frac{1}{1-|z|}\leq\frac{2M(\alpha+2)}{c_\alpha}\cdot\frac{1}{1-|z|^2},$$

where $c_\alpha=\frac{(\Gamma(\frac{\alpha}{2}+1))^2}{\Gamma(\alpha+1)}$ *and* $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$ $(s>0)$ *is the Gamma function.*

In particular, if f *maps* $\mathbb{D}$ *into* $\mathbb{D}$, *then*

$$|Df(z)|\leq\frac{2(\alpha+2)}{c_\alpha}\cdot\frac{1}{1-|z|^2}.$$

Let $\lambda_D(z)|dz|$ be the hyperbolic metric of the domain D having constant Gaussian curvature -1. The hyperbolic distance $d_{h_D}(z_1,z_2)$ between two points z_1 and z_2 in D is defined by

$$\inf_\gamma\Big\{\int_\gamma\lambda_D(z)|dz|\Big\},$$

where the infimum is taken over all rectifiable curves γ in D connecting z_1 and z_2.

We have known that if $D=\mathbb{D}$, then (cf. [1])

$$\lambda_{\mathbb{D}}(z)=\frac{2}{1-|z|^2}\quad\text{and}\quad d_{h_{\mathbb{D}}}(z_1,z_2)=\log\frac{|1-z_1\overline{z}_2|+|z_1-z_2|}{|1-z_1\overline{z}_2|-|z_1-z_2|}.$$

As a consequence of Theorem 1.1, we have

Corollary 1.1 *Under the assumptions of Theorem 1.1, if* f *maps* $\mathbb{D}$ *into* $\mathbb{D}$, *then for* z_1 *and* $z_2\in\mathbb{D}$,

$$|f(z_1)-f(z_2)|\leq\frac{\alpha+2}{c_\alpha}d_{h_{\mathbb{D}}}(z_1,z_2).$$

In [20], the authors got the following homogeneous expansion of α-harmonic functions (see [20, Theorem 1.2]):

A function f in $\mathbb{D}$ is α-harmonic if and only if it has the following convergent power series expansion:

$$f(z) = \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} P_{\alpha,k}(|z|^2)\overline{z}^k, \tag{1.7}$$

where $P_{\alpha,k}(x) = \int_0^1 t^{k-1}(1-tx)^{\alpha} dt$ $(-1 < x < 1)$ and $\{c_k\}_{k=-\infty}^{\infty}$ denotes a sequence of complex numbers with $\lim_{|k|\to\infty} \sup |c_k|^{\frac{1}{|k|}} \leq 1$.

The second aim of this paper is to prove the following estimates on coefficients c_k and c_{-k}.

Theorem 1.2 *Suppose that f is an α-harmonic function in $\mathbb{D}$ with $\alpha > -1$ and that $\sup_{z\in\mathbb{D}} |f(z)| \leq M$, where M is a constant. If f has the series expansion (1.7), then for $k \in \{0, 1, 2, ...\}$,*

$$|c_k| \leq M, \tag{1.8}$$

and for $k \in \{1, 2, ...\}$,

$$|c_k| + |c_{-k}| B(k, \alpha + 1) \leq \frac{4M}{\pi}, \tag{1.9}$$

where $B(p, q)$ denotes the Beta function.

By [20, Definition 2.1], we find that

$$\mathcal{P}_{\alpha}(ze^{-i\theta}) = \sum_{k=0}^{\infty} e^{-ik\theta} z^k + \sum_{k=1}^{\infty} \frac{\Gamma(k+\alpha+1)}{\Gamma(k)\Gamma(\alpha+1)} P_{\alpha,k}(|z|^2) e^{ik\theta} \overline{z}^k.$$

If $|f^*(z)| \leq M$, then by (1.6), we get

$$|c_{-k}| = \left| \frac{\Gamma(k+\alpha+1)}{\Gamma(k)\Gamma(\alpha+1)} \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} f^*(e^{i\theta}) d\theta \right| \leq M \frac{\Gamma(k+\alpha+1)}{\Gamma(k)\Gamma(\alpha+1)} \to \infty$$

as $k \to \infty$.

Moreover, from the proof of [20, Theorem 1.2], we see that

$$(1-|z|^2)^{-\alpha} \frac{\partial}{\partial \overline{z}} f(z) = \overline{h(z)},$$

where $h(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in \mathbb{D}$ and $c_{-k} = \overline{a_{k-1}}$ for $k \geq 1$. Note that if $h(z)$ is a normalized (in the sense that $h(0) = h'(0) - 1 = 0$) univalent analytic function in $\mathbb{D}$, then by Louis de Branges's theorem it is well-known that $|a_k| \leq k$ for all $k \geq 2$ so that

$$c_{-1} = 0, \quad c_{-2} = 1 \quad \text{and} \quad |c_{-k}| = |a_{k-1}| \leq k-1 \quad \text{for all} \quad k \geq 3. \tag{1.10}$$

The classical Landau theorem says that there is a $\rho = \frac{1}{M+\sqrt{M^2-1}}$ such that every function f, analytic in $\mathbb{D}$ with $f(0) = f^{'}(0) - 1 = 0$ and $|f(z)| < M$, is univalent in the disk $\mathbb{D}_\rho$. Moreover, the range $f(\mathbb{D}_\rho)$ contains a disk of radius $M\rho^2$, where $M \geq 1$ is a constant (see [18]). Recently, many authors considered Landau type theorem for α-harmonic functions f when $\alpha = 0$ (see [3,5–9] etc).

As an application of Theorems 1.1 and 1.2, we get the following Landau type theorem for α-harmonic functions.

Theorem 1.3 *Suppose that f is an α-harmonic function in $\mathbb{D}$ with $\alpha \geq 0$, that $f^* \in C(\mathbb{T})$, that $\sup_{z\in\overline{\mathbb{D}}} |f(z)| \leq M$, where M is a constant, and that $f(0) = |J_f(0)| - \beta = 0$. If f satisfies (1.10), then we have the following:*

(1) *f is univalent in $\mathbb{D}_{\rho_0}$, where ρ_0 satisfies the following equation*

$$\frac{\beta c_\alpha}{M(\alpha+2)} - (M+5)\frac{\rho_0(2-\rho_0)}{(1-\rho_0)^2} = 0; \tag{1.11}$$

(2) *$f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{R_0}$ with*

$$R_0 \geq (M+5)\left(\frac{\rho_0}{1-\rho_0}\right)^2.$$

The arrangement of the rest of this paper is as follows. In Sect. 2, we shall prove Theorem 1.1 and Corollary 1.1. Section 3 will be devoted to the proof of Theorem 1.2. In Sect. 4, Theorem 1.3 will be demonstrated.

2 Schwarz–Pick type inequality

The aim of this section is to prove Theorem 1.1 and Corollary 1.1. The proofs need a result from [19]. Before the statement of this result, we do some preparation.

In [19], the author considered the following integral means:

$$\mathcal{M}_\alpha(r) = \frac{1}{2\pi}\int_0^{2\pi} K_\alpha(re^{i\theta})d\theta, \tag{2.1}$$

where $r \in [0, 1)$ and

$$K_\alpha(z) = c_\alpha|\mathcal{P}_\alpha(z)| = c_\alpha\frac{(1-|z|^2)^{\alpha+1}}{|1-z|^{\alpha+2}}$$

in $\mathbb{D}$.

Let us recall the following result from [19].

Theorem B [19, Theorem 3.1] *Let $\alpha > -1$. The integral means function $\mathcal{M}_\alpha(r)$ given by (2.1) satisfies the following assertions.*

(1) $\lim_{r\to 1^-} \mathcal{M}_\alpha(r) = 1$;
(2) $\mathcal{M}_\alpha^{(n)}(r) \geq 0$ *for* $r \in [0, 1)$ *and* $n \geq 0$.

The following result also plays a key role in the proof of Theorem 1.1.

Lemma 2.1 *If $\alpha > -1$ and $f^* \in C(\mathbb{T})$, then*

$$\frac{\partial}{\partial z}\int_0^{2\pi} \mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta = \int_0^{2\pi} \frac{\partial}{\partial z}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta$$

and

$$\frac{\partial}{\partial \overline{z}}\int_0^{2\pi} \mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta = \int_0^{2\pi} \frac{\partial}{\partial \overline{z}}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta.$$

Proof By elementary calculations we see that the following equalities hold:

$$\frac{\partial}{\partial z}\mathcal{P}_\alpha(ze^{-i\theta}) = \frac{(1-|z|^2)^\alpha\left[e^{-i\theta}(1-|z|^2)-(\alpha+1)\overline{z}(1-ze^{-i\theta})\right]}{(1-ze^{-i\theta})^2(1-\overline{z}e^{i\theta})^{\alpha+1}} \tag{2.2}$$

and

$$\frac{\partial}{\partial \overline{z}}\mathcal{P}_\alpha(ze^{-i\theta}) = \frac{(\alpha+1)(1-|z|^2)^\alpha e^{i\theta}}{(1-\overline{z}e^{i\theta})^{\alpha+2}}. \tag{2.3}$$

Then we know that functions

$$\frac{\partial}{\partial z}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta}) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})$$

are continuous on $\overline{\mathbb{D}}_r \times [0, 2\pi]$, where $r \in [0, 1)$.

Let $z = \rho e^{i\varphi} \in \overline{\mathbb{D}}_r$. It follows from

$$\frac{\partial}{\partial \rho}\mathcal{P}_\alpha(ze^{-i\theta}) = \frac{\partial}{\partial z}\mathcal{P}_\alpha(ze^{-i\theta})e^{i\varphi} + \frac{\partial}{\partial \overline{z}}\mathcal{P}_\alpha(ze^{-i\theta})e^{-i\varphi}$$

and

$$\frac{\partial}{\partial \varphi}\mathcal{P}_\alpha(ze^{-i\theta}) = \frac{\partial}{\partial z}\mathcal{P}_\alpha(ze^{-i\theta})iz - \frac{\partial}{\partial \overline{z}}\mathcal{P}_\alpha(ze^{-i\theta})i\overline{z}$$

that both

$$\frac{\partial}{\partial \rho}\mathcal{P}_\alpha(ze^{-i\theta})f(e^{i\theta}) \quad \text{and} \quad \frac{\partial}{\partial \varphi}\mathcal{P}_\alpha(ze^{-i\theta})f(e^{i\theta})$$

are continuous in $\overline{\mathbb{D}}_r \times [0, 2\pi]$. Hence

$$\int_0^{\rho}\int_0^{2\pi}\frac{\partial}{\partial\rho}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta d\rho=\int_0^{2\pi}\int_0^{\rho}\frac{\partial}{\partial\rho}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\rho d\theta$$
$$=\int_0^{2\pi}\big(\mathcal{P}_\alpha(ze^{-i\theta})-\mathcal{P}_\alpha(0)\big)f^*(e^{i\theta})d\theta$$

and

$$\int_0^{\varphi}\int_0^{2\pi}\frac{\partial}{\partial\varphi}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta d\varphi=\int_0^{2\pi}\int_0^{\varphi}\frac{\partial}{\partial\varphi}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\varphi d\theta$$
$$=\int_0^{2\pi}\big(\mathcal{P}_\alpha(ze^{-i\theta})-\mathcal{P}_\alpha(\rho e^{-i\theta})\big)f^*(e^{i\theta})d\theta.$$

By differentiating with respect to ρ and φ, respectively, we get

$$\int_0^{2\pi}\frac{\partial}{\partial\rho}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta=\frac{\partial}{\partial\rho}\int_0^{2\pi}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta \tag{2.4}$$

and

$$\int_0^{2\pi}\frac{\partial}{\partial\varphi}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta=\frac{\partial}{\partial\varphi}\int_0^{2\pi}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta. \tag{2.5}$$

Since

$$\frac{\partial}{\partial z}\mathcal{P}_\alpha(ze^{-i\theta})=\frac{e^{-i\varphi}}{2}\left(\frac{\partial}{\partial\rho}\mathcal{P}_\alpha(ze^{-i\theta})-\frac{i}{\rho}\frac{\partial}{\partial\varphi}\mathcal{P}_\alpha(ze^{-i\theta})\right)$$

and

$$\frac{\partial}{\partial\overline{z}}\mathcal{P}_\alpha(ze^{-i\theta})=\frac{e^{i\varphi}}{2}\left(\frac{\partial}{\partial\rho}\mathcal{P}_\alpha(ze^{-i\theta})+\frac{i}{\rho}\frac{\partial}{\partial\varphi}\mathcal{P}_\alpha(ze^{-i\theta})\right),$$

it follows from (2.4) and (2.5) that the proof of the lemma is complete. □

Now, we are ready to present the proofs of Theorem 1.1 and Corollary 1.1.

Proof of Theorem 1.1 From (2.2) and (2.3), we can easily get

$$\left|\frac{\partial}{\partial z}\mathcal{P}_\alpha(ze^{-i\theta})\right|\leq\frac{1}{\mathrm{c}_\alpha}\cdot\frac{(\alpha+2)|z|+1}{1-|z|^2}K_\alpha(ze^{-i\theta})$$

and

$$\left|\frac{\partial}{\partial\overline{z}}\mathcal{P}_\alpha(ze^{-i\theta})\right|=\frac{\alpha+1}{\mathrm{c}_\alpha}\cdot\frac{1}{1-|z|^2}K_\alpha(ze^{-i\theta}).$$

In the first inequality above, the fact "$1-|z| \leq |1-ze^{-i\theta}|$" is applied. By (1.1), (1.6) and Lemma 2.1 yield

$$|Df(z)| = \left|\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial z}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta\right| + \left|\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial \overline{z}}\mathcal{P}_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta\right|,$$

we see from (2.1) and Theorem B that

$$|Df(z)| \leq \frac{M(\alpha+2)}{\mathrm{c}_\alpha}\cdot\frac{1}{1-|z|}\mathcal{M}_\alpha(|z|) \leq \frac{M(\alpha+2)}{\mathrm{c}_\alpha}\cdot\frac{1}{1-|z|},$$

and so the proof of Theorem 1.1 is complete. □

Proof of Corollary 1.1 For any z_1 and $z_2 \in \mathbb{D}$, let γ be the hyperbolic geodesic connecting z_1 and z_2. It follows from Theorem 1.1 that

$$|f(z_1)-f(z_2)| \leq \int_\gamma |Df(z)|\cdot|dz| \leq \frac{\alpha+2}{\mathrm{c}_\alpha}\int_\gamma \frac{2}{1-|z|^2}|dz| = \frac{\alpha+2}{\mathrm{c}_\alpha}d_{h_{\mathbb{D}}}(z_1,z_2),$$

as required. □

3 Estimates on coefficients

The aim of this paper is to prove Theorem 1.2. We start with a lemma.

Lemma 3.1 *Under the assumptions of Theorem 1.2, if f has the series expansion (1.7), then*

(1) $|c_k| \leq M$ *for* $k \geq 0$*;*
(2) $(|c_k| + |c_{-k}|P_{\alpha,k}(r^2))r^k \leq \frac{4}{\pi}M$ *for* $k > 0$ *and* $r \in (0,1)$.

Proof If $k \neq 0$, let $z = re^{i\theta} \in \mathbb{D}$. Then by (1.7), we have

$$c_k r^k = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{-ik\theta}d\theta \quad \text{and} \quad c_{-k}P_{\alpha,k}(r^2)r^k = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{ik\theta}d\theta.$$

Letting $c_k = |c_k|e^{i\mu_k}$ and $c_{-k} = |c_{-k}|e^{i\nu_k}$ leads to

$$\begin{aligned}(|c_k| + |c_{-k}|P_{\alpha,k}(r^2))r^k &= \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\left(e^{-i(k\theta+\mu_k)} + e^{i(k\theta-\nu_k)}\right)d\theta\\ &\leq \frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\cdot\left|e^{-i(k\theta+\mu_k)} + e^{i(k\theta-\nu_k)}\right|d\theta\\ &\leq \frac{M}{\pi}\int_0^{2\pi}\left|\cos\left(k\theta + \frac{\mu_k-\nu_k}{2}\right)\right|d\theta,\end{aligned}$$

and so [4, Lemma 1] gives

$$(|c_k| + |c_{-k}|P_{\alpha,k}(r^2))r^k \le \frac{4M}{\pi}.$$

Thus the assertion (2) in the lemma is true.

To prove the assertions (1), we first recall from [20, Definition 2.1] that

$$\mathcal{P}_\alpha(ze^{-i\theta}) = \sum_{k=0}^{\infty} e^{-ik\theta}z^k + \sum_{k=1}^{\infty} \frac{\Gamma(k+\alpha+1)}{\Gamma(k)\Gamma(\alpha+1)} P_{\alpha,k}(|z|^2)e^{ik\theta}\overline{z}^k.$$

Then by (1.6), we get

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} z^k \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} f^*(e^{i\theta})d\theta \\ &\quad + \sum_{k=1}^{\infty} \frac{\Gamma(k+\alpha+1)}{\Gamma(k)\Gamma(\alpha+1)} P_{\alpha,k}(|z|^2)\overline{z}^k \frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta} f^*(e^{i\theta})d\theta, \end{aligned}$$

which implies

$$|c_k| = \left| \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} f^*(e^{i\theta})d\theta \right| \le M$$

as required. □

Proof of Theorem 1.2 To prove this theorem, by Lemma 3.1, we only need to check (1.9) in the theorem. By letting $r \to 1^-$ in Lemma 3.1(2), we see that the inequalities (1.9) easily follows. □

4 Landau type theorem

This section consists of two subsections. In the first subsection, we shall prove an auxiliary result. In the second subsection, Theorem 1.3 will be checked.

4.1 A lemma

Lemma 4.1 *For constants* $\alpha > -2$, $\beta > 0$ *and* $M > 0$, *let*

$$\varphi(x) = \frac{\beta c_\alpha}{M(\alpha+2)} + (M+5)\frac{x(x-2)}{(1-x)^2}$$

in $[0, 1)$. *Then*

(1) φ *is continuous in* $[0, 1)$ *and strictly decreasing in* $(0, 1)$;

(2) *there is a unique* $x_0 \in (0, 1)$ *such that* $\varphi(x_0) = 0$.

Proof For $x \in [0, 1)$, obviously,

$$\varphi'(x) = -\frac{2(M+5)}{(1-x)^3} < 0.$$

Hence $\varphi(x)$ is continuous and strictly decreasing in $[0, 1)$. It follows from

$$\varphi(0) = \frac{\beta c_\alpha}{M(\alpha+2)} > 0 \quad \text{and} \quad \lim_{x\to 1^-} \varphi(x) = -\infty < 0$$

that there is a unique $x_0 \in (0, 1)$ such that $\varphi(x_0) = 0$. The proof of this lemma is complete. □

4.2 Proof of Theorem 1.3

To prove this theorem, we need estimates on two quantities $|f_z(z) - f_z(0)| + |f_{\overline{z}}(z) - f_{\overline{z}}(0)|$ and $l(Df(0))$. First, we estimate $|f_z(z) - f_z(0)| + |f_{\overline{z}}(z) - f_{\overline{z}}(0)|$. Obviously, by (1.7), we see that

$$f_z(z) - f_z(0) = \sum_{k=2}^{\infty} kc_k z^{k-1} + \sum_{k=2}^{\infty} c_{-k} \frac{d}{dw} P_{\alpha,k}(w)\overline{z}^{k+1}$$

and

$$f_{\overline{z}}(z) - f_{\overline{z}}(0) = \sum_{k=2}^{\infty} kc_{-k} P_{\alpha,k}(w)\overline{z}^{k-1} + \sum_{k=2}^{\infty} c_{-k} \frac{d}{dw} P_{\alpha,k}(w) z\overline{z}^{k},$$

where $w = |z|^2$.

Since

$$\frac{d}{dw} P_{\alpha,k}(w) = -\int_0^1 t^k \alpha(1-tw)^{\alpha-1} dt \le 0,$$

we get that

$$P_{\alpha,k}(w) \le P_{\alpha,k}(0) = \frac{1}{k}. \tag{4.1}$$

Moreover, since

$$P_{\alpha,k}(w) = \frac{1}{w^k} \int_0^w x^{k-1}(1-x)^\alpha dx,$$

we easily get

$$\frac{d}{dw}P_{\alpha,k}(w) = -\frac{k}{w}P_{\alpha,k}(w) + \frac{(1-w)^{\alpha}}{w}. \tag{4.2}$$

Then (1.9), (1.10), (4.1) and (4.2) guarantee that

$$\begin{aligned}|f_z(z) - f_z(0)| + |f_{\overline{z}}(z) - f_{\overline{z}}(0)| &\le \sum_{k=2}^{\infty} k\big(|c_k| + |c_{-k}|P_{\alpha,k}(w)\big)|z|^{k-1} \\ &\quad +2\sum_{k=2}^{\infty}|c_{-k}|\big(kP_{\alpha,k}(w)+1\big)|z|^{k-1} \\ &\le (M+5)\sum_{k=2}^{\infty}k|z|^{k-1} \\ &= (M+5)\frac{|z|(2-|z|)}{(1-|z|)^2},\end{aligned} \tag{4.3}$$

which is what we want.

Next, we estimate $l(Df(0))$. Applying Theorem 1.1 leads to

$$\beta = |J_f(0)| = |Df(0)|\, l(Df(0)) \le \frac{M(\alpha+2)}{\mathrm{c}_\alpha}\, l(Df(0)),$$

which gives

$$l(Df(0)) \ge \frac{\beta \mathrm{c}_\alpha}{M(\alpha+2)}. \tag{4.4}$$

Now, we are ready to finish the proof of the theorem. First, we demonstrate the univalence of f in $\mathbb{D}_{\rho_0}$, where ρ_0 is determined by Eq. (1.11). For this, let z_1, z_2 be two points in $\mathbb{D}_{\rho_0}$ with $z_1 \ne z_2$, and denote the segment from z_1 to z_2 with the endpoints z_1 and z_2 by $[z_1, z_2]$. Since

$$\begin{aligned}|f(z_2) - f(z_1)| &= \left|\int_{[z_1,z_2]} f_z(z)dz + f_{\overline{z}}(z)d\overline{z}\right| \\ &\ge \left|\int_{[z_1,z_2]} f_z(0)dz + f_{\overline{z}}(0)d\overline{z}\right| \\ &\quad -\left|\int_{[z_1,z_2]}[f_z(z) - f_z(0)]dz + [f_{\overline{z}}(z) - f_{\overline{z}}(0)]d\overline{z}\right|,\end{aligned}$$

we see from (4.3), (4.4) and Lemma 4.1 that

$$
\begin{aligned}
|f(z_2) - f(z_1)| &\geq l(Df(0)) \cdot |z_2 - z_1| - (M+5) \int_0^{|z_2 - z_1|} \frac{|z|(2-|z|)}{(1-|z|)^2} |dz| \\
&> \left[\frac{\beta c_\alpha}{M(\alpha+2)} - (M+5) \frac{\rho_0(2-\rho_0)}{(1-\rho_0)^2} \right] |z_2 - z_1| \\
&= 0.
\end{aligned}
$$

Thus, for arbitrary z_1 and $z_2 \in \mathbb{D}_{\rho_0}$ with $z_1 \neq z_2$, we have

$$
f(z_1) \neq f(z_2),
$$

which implies the univalence of f in $\mathbb{D}_{\rho_0}$.

Next, we prove Theorem 1.3(2). For any $\zeta = \rho_0 e^{i\theta} \in \partial\mathbb{D}_{\rho_0}$, we obtain that

$$
\begin{aligned}
|f(\zeta) - f(0)| &= \left| \int_{[0,\zeta]} f_z(z)dz + f_{\overline{z}}(z)d\overline{z} \right| \\
&\geq \left| \int_{[0,\zeta]} f_z(0)dz + f_{\overline{z}}(0)d\overline{z} \right| \\
&\quad - \left| \int_{[0,\zeta]} [f_z(z) - f_z(0)]dz + [f_{\overline{z}}(z) - f_{\overline{z}}(0)]d\overline{z} \right| \\
&\geq l(Df(0))\rho_0 - (M+5) \int_0^{\rho_0} \frac{|z|(2-|z|)}{(1-|z|)^2} |dz| \quad \text{(by (4.3))} \\
&= \frac{\beta c_\alpha \rho_0}{M(\alpha+2)} - (M+5) \frac{\rho_0^2}{1-\rho_0}. \\
&= (M+5) \left(\frac{\rho_0}{1-\rho_0} \right)^2. \quad \text{(by (1.11))}
\end{aligned}
$$

Hence $f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{R_0}$, where

$$
R_0 \geq (M+5) \left(\frac{\rho_0}{1-\rho_0} \right)^2.
$$

The proof of this theorem is complete. □

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