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Almost Engel linear groups

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Abstract A group G is almost Engel if for every $g \in G$ there is a finite set $\mathcal{E}(g)$ such that for every $x \in G$ all sufficiently long commutators $[x,_n g]$ belong to $\mathcal{E}(g)$, that is, for every $x \in G$ there is a positive integer $n(x, g)$ such that $[x,_n g] \in \mathcal{E}(g)$ whenever $n(x, g) \leq n$. A group G is almost nil if it is almost Engel and for every $g \in G$ there is a positive integer n depending on g such that $[x,_s g] \in \mathcal{E}(g)$ for every $x \in G$ and every $s \geq n$. We prove that if a linear group G is almost Engel, then G is finite-by-hypercentral. If G is almost nil, then G is finite-by-nilpotent.

Keywords Linear groups · Engel condition · Locally nilpotent groups

Mathematics Subject Classification 20E99 · 20F45 · 20H20

1 Introduction

By a linear group we understand here a subgroup of $GL(m, F)$ for some field F and a positive integer m. An element g of a group G is called a (left) Engel element if for any $x \in G$ there exists $n = n(x, g) \geq 1$ such that $[x,_n g] = 1$. As usual, the commutator $[x,_n g]$ is defined recursively by the rule

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$$[x,_n g] = [[x,_{n-1} g], g]$$

assuming $[x,_0 g] = x$. If n can be chosen independently of x, then g is a *(left) n-Engel element*. A group G is called Engel if all elements of G are Engel. It is called n-Engel if all its elements are n-Engel. A group is said to be locally nilpotent if every finite subset generates a nilpotent subgroup. Clearly, any locally nilpotent group is an Engel group. It is a long-standing problem whether any n-Engel group is locally nilpotent. Engel linear groups are known to be locally nilpotent (cf. [2,3]).

We say that a group G is almost Engel if for every $g \in G$ there is a finite set $\mathcal{E}(g)$ such that for every $x \in G$ all sufficiently long commutators $[x,_n g]$ belong to $\mathcal{E}(g)$, that is, for every $x \in G$ there is a positive integer $n(x, g)$ such that $[x,_n g] \in \mathcal{E}(g)$ whenever $n(x, g) \leq n$. (Thus, Engel groups are precisely the almost Engel groups for which we can choose $\mathcal{E}(g) = \{1\}$ for all $g \in G$.) We say that a group G is nil if for every $g \in G$ there is a positive integer n depending on g such that g is n-Engel. The group G will be called almost nil if it is almost Engel and for every $g \in G$ there is a positive integer n depending on g such that $[x,_s g] \in \mathcal{E}(g)$ for every $x \in G$ and every $s \geq n$.

Almost Engel groups were introduced in [6] where it was proved that an almost Engel compact group is necessarily finite-by-(locally nilpotent). The purpose of the present article is to prove the following related result.

Theorem 1.1 *Let G be a linear group.*

1. *If G is almost Engel, then G is finite-by-hypercentral.*
2. *If G is almost nil, then G is finite-by-nilpotent.*

Recall that the union of all terms of the (transfinite) upper central series of G is called the hypercenter. The group G is hypercentral if it coincides with its hypercenter. The hypercentral groups are known to be locally nilpotent (see [10, P. 365]). By well-known results obtained in [2,3], if under the hypotheses of Theorem 1.1 the group G is Engel or nil, then G is hypercentral or nilpotent, respectively.

As a warning to the reader, we mention that in many articles (including some of the author) the expression "the group G is almost an X-group" for a property X means "G has an X-subgroup of finite index". In the present paper, however, the meaning of the term "almost Engel" is different. It is hoped that this discrepancy does not lead to a confusion.

2 Preliminaries

Let G be a group and $g \in G$ an almost Engel element, so that there is a finite set $\mathcal{E}(g)$ such that for every $x \in G$ there is a positive integer $n(x, g)$ with the property that $[x,_n g]$ belongs to $\mathcal{E}(g)$ whenever $n(x, g) \leq n$. If $\mathcal{E}'(g)$ is another finite set with the same property for possibly different numbers $n'(x, g)$, then $\mathcal{E}(g) \cap \mathcal{E}'(g)$ also satisfies the same condition with the numbers $n''(x, g) = \max\{n(x, g), n'(x, g)\}$. Hence there is a *minimal* set with the above property. The minimal set will again be denoted by $\mathcal{E}(g)$ and, following [6], called the *Engel sink for g*, or simply *g-sink* for short. From now on we will always use the notation $\mathcal{E}(g)$ to denote the (minimal) Engel sinks. In particular, it follows that for each $x \in \mathcal{E}(g)$ there exists $y \in \mathcal{E}(g)$ such that $x = [y, g]$. More generally, given a subset $K \subseteq G$ and an almost Engel element $g \in G$, we write $\mathcal{E}(g, K)$ to denote the minimal subset of G with the property that for every $x \in K$ there is a positive integer $n(x, g)$ such that $[x,_n g]$ belongs to $\mathcal{E}(g, K)$ whenever $n(x, g) \leq n$. Throughout the article we use the symbols $\langle X \rangle$ and $\langle X^G \rangle$ to denote the subgroup generated by a set X and the minimal normal subgroup of G containing X, respectively.

A group is said to have a property virtually if some subgroup of finite index has the property. The following lemma can be found in [8, Ch. 12, Lemma 1.2] or in [5, Lemma 21.1.4].

Lemma 2.1 *A virtually abelian group contains a characteristic abelian subgroup of finite index.*

As usual, we write $Z_i(G)$ for the ith term of the upper central series of G and $\gamma_i(G)$ for the ith term of the lower central series. A well-known theorem of Schur states that if G is central-by-finite, then the commutator subgroup G' is finite (see [10, 10.1.4]). Baer proved that if, for a positive integer k, the quotient $G/Z_k(G)$ is finite, then so is $\gamma_{k+1}(G)$ (see [10, 14.5.1]). Recently, the following related result was obtained in [1] (see also [7]).

Theorem 2.2 *Let G be a group and let H be the hypercenter of G. If G/H is finite, then G has a finite normal subgroup N such that G/N is hypercentral.*

We will also require the Dicman Lemma (see [10, 14.5.7]).

Lemma 2.3 *In any group a normal finite subset consisting of elements of finite order generates a finite subgroup.*

In [9] Plotkin proved that if a group G has an ascending series whose quotients locally satisfy the maximal condition, then the Engel elements of G form a locally nilpotent subgroup. In particular we have the following lemma.

Lemma 2.4 *Let G be a group having an ascending series whose quotients locally satisfy the maximal condition and let $a \in G$ be an Engel element. Then $\langle a^G \rangle$ is locally nilpotent.*

Linear groups are naturally equipped with the Zariski topology. If G is a linear group, the connected component of G containing 1 is denoted by G^0. We will use (sometimes implicitly) the following facts on linear groups. All these facts are well-known and are provided here just for the reader's convenience.

- If G is a linear group and N a normal subgroup which is closed in the Zarissky topology, then G/N is linear (see [12, Theorem 6.4]).
- Since finite subsets of G are closed in the Zariski topology, it follows that any finite subgroup of a linear group is closed. Hence G/N is linear for any finite normal subgroup N.
- If G is a linear group, the connected component G^0 has finite index in G (see [12, Lemma 5.3]).

- Each finite conjugacy class in a linear group centralizes G^0 (see [12, Lemma 5.5]).
- In a linear group any descending chain of centralizers is finite.
- A linear group generated by normal nilpotent subgroups is nilpotent (see Gruenberg [3]).
- Tits alternative: A finitely generated linear group either is virtually soluble or contains a subgroup isomorphic to a nonabelian free group (see [11]).
- The Burnside–Schur theorem: A periodic linear group is locally finite (see [12, 9.1]).
- Zassenhaus theorem: A locally soluble linear group is soluble. Every linear group contains a unique maximal soluble normal subgroup (see [12, Corollary 3.8]).
- Since the closure in the Zariski topology of a soluble subgroup is again soluble (see [12, Lemma 5.11]), it follows that the unique maximal soluble normal subgroup of a linear group is closed. In particular, if G is linear and R is the unique maximal soluble normal subgroup of G, then G/R is linear and has no nontrivial normal soluble subgroups.
- A locally nilpotent linear group is hypercentral (see [2] or [3]).
- Gruenberg theorem: The set of Engel elements in a linear group G coincides with the Hirsch–Plotkin radical of G. The set of right Engel elements coincides with the hypercenter of G (see [3]).

Here, as usual, the Hirsch–Plotkin radical of a group is the maximal normal locally nilpotent subgroup. An element $g \in G$ is a right Engel element if for each $x \in G$ there exists a positive integer n such that $[g,_n x] = 1$.

3 Almost Engel elements in virtually soluble groups

In the present section we give certain criteria for a group containing almost Engel elements to be finite-by-nilpotent or finite-by-hypercentral. In particular, we prove that a virtually soluble group generated by finitely many almost Engel elements is finite-by-nilpotent (Theorem 3.3).

Lemma 3.1 *Let $G = H\langle a_1, \ldots, a_s\rangle$, where H is a normal subgroup and a_i are almost Engel elements. Assume that G/H is nilpotent. If $N \le H$ is a finite normal subgroup of H, then $\langle N^G\rangle$ is finite.*

Proof Suppose first that $s = 1$ and write a in place of a_1. Let M be the subgroup generated by all commutators of the form $[x,_j a]$, where $x \in N$ and j is a nonnegative integer. Since both N and $\mathcal{E}(a)$ are finite, it follows that there exists an integer k such that M is contained in the product $\prod_{i=0}^{k} N^{a^i}$. It is clear that the product $\prod_{i=0}^{k} N^{a^i}$ is normal in H and a normalizes M. Therefore $\langle M^H\rangle$ is normal in G and is contained in $\prod_{i=0}^{k} N^{a^i}$. Moreover, $\langle N^G\rangle = \langle M^H\rangle$ so in the case where $s = 1$ the lemma follows.

Therefore we will assume that $s \ge 2$ and use induction on s. Assume additionally that G/H is abelian. Set $H_0 = H$ and $H_i = H_{i-1}\langle a_i\rangle$ for $i = 1, \ldots, s$. The subgroups H_i are normal in G and $H_s = G$. By induction, $K = \langle N^{H_{s-1}}\rangle$ is finite. Since $G = H_{s-1}\langle a_s\rangle$, the above paragraph shows that $\langle K^G\rangle$ is finite. Obviously, $\langle K^G\rangle = \langle N^G\rangle$ and so in the case where G/H is abelian the lemma follows.

We will now allow G/H to be nonabelian, say of nilpotency class c. We will use induction on c. Set $B = \langle a_s^G \rangle$ and $G_1 = HB$. Since G/H is a finitely generated nilpotent group, it follows that each subgroup of G/H is finitely generated and so B has finitely many conjugates of a_s, say $a_s^{g_1} \dots, a_s^{g_r}$ such that $G_1 = H\langle a_s^{g_1} \dots, a_s^{g_r} \rangle$. Since G_1/H has nilpotency class at most $c - 1$, by induction $\langle N^{G_1} \rangle$ is finite. We now note that $G = G_1\langle a_1, \dots, a_{s-1} \rangle$ so the induction on s completes the proof. □

Lemma 3.2 *Let $G = H\langle a \rangle$, where H is a virtually abelian normal subgroup and a is an almost Engel element. Then $\langle a^G \rangle$ is finite-by-(locally nilpotent).*

Proof Assume that G is a counter-example with $|\mathcal{E}(a)|$ as small as possible. In view of Lemma 2.1 we can choose a maximal characteristic abelian subgroup V in H. Since V is abelian, we have $[v_1, a][v_2, a] = [v_1 v_2, a]$ for any $v_1, v_2 \in V$. In other words, a product of two commutators of the form $[v, a]$, where $v \in V$, again has the same form. Therefore $\mathcal{E}(a, V)$ is a finite subgroup. Obviously, the normalizer in G of $\mathcal{E}(a, V)$ has finite index. It follows that $\mathcal{E}(a, V)$ is contained in a finite normal subgroup N. If $\mathcal{E}(a, V) \neq 1$, we pass to the quotient G/N and use induction on $|\mathcal{E}(a)|$. Therefore without loss of generality we will assume that $\mathcal{E}(a, V) = 1$, that is, a is Engel in $V\langle a \rangle$. Since $\mathcal{E}(a)$ consists of commutators of the form $[x, a]$ with $x \in \mathcal{E}(a)$, it follows that $\mathcal{E}(a) \cap V = \{1\}$. Let $C_0 = 1$ and

$$C_i = \{v \in V \mid [v, a] \in C_{i-1}\}$$

for $i = 1, 2, \dots$. Since a is Engel in V, we have $V = \cup_i C_i$.

Let $T = \langle \mathcal{E}(a), a \rangle$ and $U = V \cap T$. We observe that U is a finitely generated abelian subgroup. In view of the fact that V is the union of the C_i we deduce that there exists a positive integer n such that $U = C_n \cap U$.

For $i = 0, \dots, n$ set $U_i = C_i \cap U$. Thus, $U = U_n$. Observe that U_1 centralizes a and therefore U_1 normalizes the set $\mathcal{E}(a)$. Denote by W_1 the intersection $U_1 \cap C_G(\mathcal{E}(a))$. Since $\mathcal{E}(a)$ is finite, it follows that W_1 has finite index in U_1. Further, it is clear that W_1 is contained in the center $Z(T)$.

The finiteness of the index $[U_1 : W_1]$ implies that U_2 contains a normal in T subgroup W_2 such that the index $[U_2 : W_2]$ is finite, and $[W_2, T] \leq W_1$. Thus, W_2 is contained in $Z_2(T)$, the second term of the upper central series of T.

Next, in a similar way we conclude that $U_3 \cap Z_3(T)$ has finite index in U_3 and so on. Eventually, we deduce that $U \cap Z_n(T)$ has finite index in U. Thus, $T/Z_n(T)$ is finite-by-cyclic and therefore there exists a positive integer k such that $a^k \in Z_{n+1}(T)$. Hence, $T/Z_{n+1}(T)$ is finite and so, in view of Baer's theorem, we deduce that T is finite-by-nilpotent. In particular, for some positive integer r the subgroup $\gamma_r(T)$ is finite. The observation that for each $x \in \mathcal{E}(a)$ there exists $y \in \mathcal{E}(a)$ such that $x = [y, g]$ guarantees that $\mathcal{E}(a)$ is contained in $\gamma_r(T)$. In particular, we proved that the subgroup $\langle \mathcal{E}(a) \rangle$ is finite. Because V is abelian, it is obvious that V normalizes $V \cap \langle \mathcal{E}(a) \rangle$. Thus, $V \cap \langle \mathcal{E}(a) \rangle$ is a finite subgroup with normalizer of finite index. It follows that $V \cap \langle \mathcal{E}(a) \rangle$ is contained in a finite normal subgroup of G. We can factor out the latter and without loss of generality assume that $V \cap \langle \mathcal{E}(a) \rangle = 1$.

Recall that $C_1 = C_V(a)$. Therefore C_1 normalizes $\langle \mathcal{E}(a) \rangle$ and in view of the fact that $V \cap \langle \mathcal{E}(a) \rangle = 1$ we conclude that C_1 centralizes $\langle \mathcal{E}(a) \rangle$. So $C_1 \leq Z(VT)$.

Same argument shows that $C_2/C_1 \leq Z(VT/C_1)$ and, more generally, $C_{i+1}/C_i \leq Z(VT/C_i)$ for $i = 0, 1, 2 \ldots$. Thus, $V \leq Z_\infty(VT)$ where $Z_\infty(VT)$ stands for the hypercenter of T. Of course, it follows that there exists a positive integer k such that $a^k \in Z_\infty(VT)$. We deduce that $Z_\infty(VT)$ has finite index in VT. Theorem 2.2 now tells us that VT has a finite normal subgroup N such that the quotient group $(VT)/N$ is hypercentral. The hypercentral groups are locally nilpotent and so VT is finite-by-(locally nilpotent). The observation that for each $x \in \mathcal{E}(a)$ there exists $y \in \mathcal{E}(a)$ such that $x = [y, g]$ guarantees that $\mathcal{E}(a)$ is contained in N.

Since VT has finite index in G, Dicman's lemma tells us that G contains a finite normal subgroup R such that $\mathcal{E}(a) \subseteq N \leq R$. The image of a in G/R is Engel and the required result follows from Lemma 2.4. □

Theorem 3.3 *A virtually soluble group generated by finitely many almost Engel elements is finite-by-nilpotent.*

Proof Let G be a virtually soluble group generated by finitely many almost Engel elements $a_1, \ldots, a_s$ and let S be a normal soluble subgroup of finite index in G. We assume that $S \neq 1$ and let V be the last nontrivial term of the derived series of S. By induction on the derived length of S we assume that G/V is finite-by-nilpotent. Therefore G contains a normal subgroup H such that V has finite index in H and the quotient G/H is nilpotent. For $i = 1, \ldots, s$ set $G_i = H\langle a_i \rangle$. By Lemma 3.2 each subgroup $\langle a_i^{G_i} \rangle$ has a finite normal subgroup N_i such that $\langle a_i^{G_i} \rangle / N_i$ is locally nilpotent. Since G_i/H are abelian, it is clear that all quotients $G_i/H \cap N_i$ are locally nilpotent and so, replacing if necessary N_i by $H \cap N_i$, without loss of generality we can assume that all subgroups N_i are normal subgroups of H. Therefore the product of the subgroups N_i is finite. By Lemma 3.1 the product of $N_1 \cdots N_s$ is contained in a finite subgroup N which is normal in G. Obviously the images in G/N of the generators $a_1, \ldots, a_s$ are Engel. Thus, G/N is a virtually soluble group generated by finitely many Engel elements. It follows from Lemma 2.4 that G/N is nilpotent. The proof is complete. □

The next lemma is well-known. For the reader's convenience we provide the proof.

Lemma 3.4 *Let $G = H\langle a \rangle$, where H is a nilpotent normal subgroup and a is a nil element. Then G is nilpotent.*

Proof Suppose that a is n-Engel. Let $K = Z(H)$ and set $K_0 = K$ and $K_{i+1} = [K_i, a]$ for $i = 0, 1, \ldots$. Then $K_{n-1} \leq K \cap C_K(a)$ and so $K_{n-1} \leq Z(G)$. Moreover we observe that $[K_{i-1}, G] \leq K_i$ and it follows that $K_{n-i} \leq Z_i(G)$ for $i = 1, 2, \ldots, n$. Therefore $K \leq Z_n(G)$. Passing to the quotient $G/Z_n(G)$ and using induction on the nilpotency class of H we deduce that if H is nilpotent with class c, then G is nilpotent with class at most cn. □

Lemma 3.5 *Let $G = H\langle a \rangle$, where H is a hypercentral normal subgroup and a is an Engel element. Then G is hypercentral.*

Proof It is sufficient to show that $Z(G) \neq 1$. Let $Z = Z(H)$. Since a is an Engel element, $C_Z(a) \neq 1$. Obviously, $C_Z(a) \leq Z(G)$. The proof is complete. □

Lemma 3.6 *Let a be an almost Engel element in a group G and assume that $\mathcal{E}(a)$ is contained in a locally nilpotent subgroup. Then the subgroup $\langle \mathcal{E}(a) \rangle$ is finite.*

Proof Set $D = \langle \mathcal{E}(a) \rangle$. Without loss of generality we can assume that $G = D\langle a \rangle$. Since $\mathcal{E}(a)$ is finite, D is nilpotent and we can use induction on the nilpotency class of D. Thus, by induction assume that the quotient of D over its center is finite. By Schur's theorem the derived group D' is finite as well. Factoring out D' we can assume that D is abelian. So now D is abelian and $D = [D, a]$. By [6, Lemma 2.3], $D = \mathcal{E}(a)$ and hence D is finite. □

Lemma 3.7 *Let $G = H\langle a \rangle$, where H is a hypercentral normal subgroup.*

1. *If a is almost Engel, then G is finite-by-hypercentral.*
2. *If H is nilpotent and a is almost nil, then G is finite-by-nilpotent.*

Proof We will prove Claim 1 first. Assume that a is almost Engel. Let N be the product of all normal subgroups of G whose intersection with $\mathcal{E}(a)$ is $\{1\}$. It is easy to see that $N \cap \mathcal{E}(a) = \{1\}$ and N is the unique maximal normal subgroup with that property. Therefore $K \cap \mathcal{E}(a) \neq \{1\}$ whenever K is a normal subgroup containing N as a proper subgroup. Since $\mathcal{E}(a)$ is finite, the group G contains a minimal normal subgroup M such that $N < M$. Taking into account that H is hypercentral, we observe that M/N is central in H/N.

Let $D = \langle \mathcal{E}(a) \rangle \cap M$. It follows that $M = ND$. Suppose that D is not normal in M and set $L = N_M(N_M(D))$. Since M is hypercentral, it satisfies the normalizer condition and so $L \neq N_M(D)$. Obviously a normalizes both L and $N_M(D)$. Since a acts on $L/N_M(D)$ as an Engel element, the centralizer of a in $L/N_M(D)$ is nontrivial. Thus, L has a subgroup C such that $N_M(D) < C$ and C normalizes $N_M(D)\langle a \rangle$. Of course, D is normal in $N_M(D)\langle a \rangle$. By Lemma 3.5 the quotient of $N_M(D)\langle a \rangle$ by D is hypercentral. It is easy to see that D is a unique minimal normal subgroup of $N_M(D)\langle a \rangle$ whose quotient is hypercentral. Therefore D is characteristic in $N_M(D)\langle a \rangle$ and so C normalizes D. This is a contradiction since $N_M(D) < C$.

Hence, D is normal in M. Again, it is easy to see that D is a unique minimal normal subgroup of $M\langle a \rangle$ whose quotient is hypercentral. Therefore D is characteristic in M and so it is normal in G. We pass to the quotient G/D and Claim 1 now follows by straightforward induction on $|\mathcal{E}(a)|$.

We now assume that H is nilpotent and a is almost nil. We already know that G is finite-by-hypercentral. Factoring out a finite normal subgroup we can assume that G is hypercentral. In that case a is actually nil and so by Lemma 3.4 G is nilpotent. The proof of the lemma is complete. □

4 Linear groups

Lemma 4.1 *A virtually soluble almost Engel linear group is finite-by-hypercentral.*

Proof Suppose that G is a virtually soluble almost Engel linear group. Let S be a normal soluble subgroup of finite index in G. By induction on the derived length of S we assume that S' is finite-by-hypercentral. Passing to the quotient over a normal finite subgroup without loss of generality we can assume that S' is hypercentral. By Lemma 3.7 the subgroup $\langle S', x\rangle$ is finite-by-hypercentral for each $x \in G$. Thus, for each $x \in G$ there exists a finite characteristic subgroup $R_x \leq \langle S', x\rangle$ such that $\langle S', x\rangle/R_x$ is hypercentral. Since $\langle S', x\rangle$ is normal in S, it follows that each element in R_x has centralizer of finite index in S, hence centralizer of finite index in G. Therefore G^0 centralizes R_x and it follows that $\langle S', x\rangle$ is hypercentral for each $x \in G^0$. The subgroup $\prod\langle S', x\rangle$, where x ranges over $S \cap G^0$, is locally nilpotent and therefore hypercentral. In particular $N = S \cap G^0$ is hypercentral and so G is virtually hypercentral. By Lemma 3.7 the subgroup $\langle N, x\rangle$ is finite-by-hypercentral for each $x \in G$. In other words, for each $x \in G$ there exists a finite characteristic subgroup $Q_x \leq \langle N, x\rangle$ such that the quotient $\langle N, x\rangle/Q_x$ is hypercentral. Since N has finite index in G, it follows that G contains only finitely many subgroups of the form $\langle N, x\rangle$. Set $N_0 = \prod_{x\in G} Q_x$. We see that N_0 is a finite normal subgroup. Pass to the quotient G/N_0. Now the subgroup $\langle N, x\rangle$ is hypercentral for each $x \in G$. It follows that N consists of right Engel elements and so, by the result of Gruenberg, N is contained in the hypercenter of G. It follows from Theorem 2.2 that G is finite-by-hypercentral, as required. □

We are now ready to prove Theorem 1.1 in its full generality. For the reader's convenience we restate it here.

Theorem 4.2 *Let G be a linear group. If G is almost Engel, then G is finite-by-hypercentral. If G is almost nil, then G is finite-by-nilpotent.*

Proof Assume that G is almost Engel. In view of Lemma 4.1 it is sufficient to show that G is virtually soluble. By the Zassenhaus theorem a linear group is soluble if and only if it is locally soluble. Therefore it is sufficient to show that G is virtually locally soluble. It is clear that G does not contain a subgroup isomorphic to a nonabelian free group. Hence, by Tits alternative, any finitely generated subgroup of G is virtually soluble. Therefore, by Theorem 3.3, any finitely generated subgroup of G is finite-by-nilpotent. It becomes obvious that elements of finite order in G generate a periodic subgroup. Moreover, the quotient of G over the subgroup generated by all elements of finite order is locally nilpotent. Hence, G is virtually locally soluble if and only if so is the subgroup generated by elements of finite order. Therefore without loss of generality we can assume that G is an infinite periodic (and locally finite) group.

Let R be the soluble radical of G. We can pass to the quotient and without loss of generality assume that $R = 1$. So in particular G has no nontrivial Engel elements. By the theorem of Hall–Kulatilaka G contains an infinite abelian subgroup [4]. We conclude that some centralizers in G are infinite. Since G satisfies the minimal condition on centralizers, it follows that G has a subgroup $D \neq 1$ such that the centralizer $C = C_G(D)$ is infinite while $C_G(\langle D, x\rangle)$ is finite for each $x \in G \setminus D$. Using that C is infinite we deduce from the Hall–Kulatilaka theorem that C contains an infinite abelian subgroup A. Obviously $A \leq C_G(\langle D, A\rangle)$ and it follows that $A \leq D$. Thus, $A \leq Z(C)$.

Now choose $1 \neq a \in A$. The centralizer C normalizes the finite set $\mathcal{E}(a)$ because $a \in Z(C)$. Hence, C contains a subgroup of finite index which centralizes $\mathcal{E}(a)$. It follows that $C_G(\langle D, \mathcal{E}(a)\rangle)$ is infinite and we conclude that $\mathcal{E}(a)$ is contained in D

and C centralizes $\mathcal{E}(a)$. In particular, a centralizes $\mathcal{E}(a)$ and so $\mathcal{E}(a) = \{1\}$. Thus, a is an Engel element, a contradiction. This completes the proof of Claim 1.

Suppose now that G is almost nil. We already know that G is finite-by-hypercentral. Passing to a quotient over a finite normal subgroup we can assume that G is hypercentral. Then obviously G, being both hypercentral and almost nil, must be nil. By the result of Gruenberg, G is nilpotent. □

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