

Spectral radius and k -connectedness of a graph

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Abstract A connected graph G is said to be k -connected if it has more than k vertices and remains connected whenever fewer than k vertices are deleted. In this paper, we present a tight sufficient condition for a connected graph with fixed minimum degree to be k -connected based on its spectral radius, for sufficiently large order.

Keywords Spectral radius · Closure · Degree sequence · Minimum degree · k -Connected

Mathematics Subject Classification 05C50

1 Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V| = n$ and $|E| = m$. We denote by d_v the degree of a vertex v in G and δ the minimum degree. Let K_n denote a complete graph on n vertices. For two vertex-disjoint graphs G and H , we use $G \vee H$ to denote the join of G and H . A graph G is said to be k -connected

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if it has more than k vertices and remains connected whenever fewer than k vertices are deleted. Similarly, G is k -edge-connected if it has at least two vertices and remains connected whenever fewer than k edges are deleted.

The *adjacency matrix* of G is defined as $A(G) = (a_{ij})_{n \times n}$, where $a_{ij} = 1$ if two vertices i and j are adjacent in G , and $a_{ij} = 0$ otherwise. The largest eigenvalue of $A(G)$, denoted by $\lambda(G)$, is called the *spectral radius* of G .

In spectral graph theory, one of the most attractive problems is the Brualdi–Solheid problem [5]: Given a set \mathcal{G} of graphs, find an upper bound for the spectral radius in this set and characterize the graphs in which the maximal spectral radius is attained. In recent decades, many researchers [2, 13, 31] intended to investigate this problem for various graph classes with certain constraints. For more details, one may refer to the recent comprehensive monograph by Stevanović [30].

Analogous to the Brualdi–Solheid problem, Nikiforov [23] proposed the following so-called Brualdi–Solheid–Turán type problem.

Problem 1 *For a given graph F , what is the maximum spectral radius of a graph G on n vertices without subgraph isomorphic to F ?*

Regarding Problem 1 and its variations, Nikiforov and many other mathematicians discovered wide recognized results. Fiedler and Nikiforov [16] obtained tight sufficient conditions for graphs to be hamiltonian or traceable, this paper became the motivation of many related results [19, 21, 22, 27, 28]. Cioabă and his collaborators [10–12] studied the matchings and the eigenvalues in regular graphs extensively. When we talk about the connectivity and the eigenvalues, perhaps the most famous one is due to Fiedler [15] which states that the second smallest Laplacian eigenvalue is at most the connectivity for any non-complete graph. For adjacency eigenvalues, the relation between the connectivity, edge-connectivity and the eigenvalues was reported in [8, 9, 17, 20, 29]. Extending and improving the result in [6], Cioabă [8] obtained that

Theorem 1.1 [8] *Let $d \geq k \geq 2$. If the second largest eigenvalue λ_2 of a d -regular graph satisfies*

$$\lambda_2 < d - \frac{(k-1)n}{(d+1)(n-d-1)},$$

then the edge-connectivity of G is at least k .

Let G be a simple graph of order $n \geq k + 1$. It is known [3, Page 4] that, if $\delta \geq \frac{1}{2}(n + k - 2)$, then G is k -connected. From the spectral point of view, we naturally ask the following question: can one find a sufficient spectral condition for a connected graph to be k -connected? Along this line, the authors obtained such a sufficient condition for a general graph without any minimum degree restriction [14]. Borrowing ideas from [25], in this paper, by utilizing the degree sequence and the closure concept, we shall present a tight sufficient spectral condition for a graph with fixed minimum degree to be k -connected, for sufficiently large order (and therefore for relatively small δ). Such results may be of independent interest.

2 Preliminaries

We first present several graph notations.

An integer sequence $\pi = (d_1 \leq d_2 \leq \dots \leq d_n)$ is called *graphical* if there exists a graph G having π as its vertex degree sequence; in that case, G is called a *realization* of π . If P is a graph property, such as hamiltonian or k -connected, we call a graphical sequence π *forcibly P* if every realization of π has property P . More results in this subject can be found in the survey paper [1].

Bondy and Chvátal introduced the concept of stability in [4], which plays a prominent role in many structural graph theory problems. Let P be a property defined on all graphs of order n . Let k be a nonnegative integer. Then P is said to be *k -stable* if whenever $G + uv$ has property P and

$$d_G(u) + d_G(v) \geq k,$$

where $uv \notin E$, then G itself has property P . The most famous properties such as the hamiltonicity and traceability are respectively n -stable and $(n - 1)$ -stable. Among all the graphs H of order n such that $G \subset H$ and

$$d_G(u) + d_G(v) < k$$

for all $uv \notin E(H)$, there is a unique smallest one, we shall call this graph the *k -closure* of G , denoted $cl_k(G)$. Obviously, $cl_k(G)$ can be obtained from G by iteratively joining two nonadjacent vertices such that their degree sum is at least k .

The following lemmas will be used in this paper.

Lemma 2.1 [7] *Let G be a connected graph and $\pi = (d_1 \leq d_2 \leq \dots \leq d_n)$ be a graphical sequence. Suppose $n \geq 2$, and $1 \leq k \leq n - 1$. If*

$$d_i \leq i + k - 2 \Rightarrow d_{n-k+1} \geq n - i,$$

for $1 \leq i \leq \frac{1}{2}(n - k + 1)$, then π is forcibly k -connected.

Lemma 2.2 [4] *The property that “ G is k -connected” is $(n + k - 2)$ -stable.*

Lemma 2.3 [4] *Let P be a property of graph G of order $n \geq 4$. If P is k -stable and $cl_k(G)$ has property P , then G itself has property P .*

Lemma 2.4 [18, 24] *If G is a graph of order n , size m , and minimum degree t , then*

$$\lambda(G) \leq \frac{1}{2} \left(t - 1 + \sqrt{8m - 4nt + (t + 1)^2} \right).$$

Lemma 2.5 [18] *If $2m \leq n(n - 1)$, the function*

$$f(x) = \frac{x - 1}{2} + \sqrt{2m - nx + \frac{(x + 1)^2}{4}}$$

is decreasing in x for $x \leq n - 1$.

3 Main results

For convenience, in the rest of this paper, we denote

$$A(n, k, \delta) = K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1}),$$

$$F_0(k, \delta) = (\delta - k + 2)(k^2 - 2k + 4) + 3.$$

We first describe our strategy here. We will prove Theorem 3.4 by contradiction. Assume that $\lambda(G) \geq n - \delta + k - 3$, but G is not k -connected. In order to get our result, we need to show $G = A(n, k, \delta)$, and $A(n, k, \delta)$ is exactly the unique closure. To prove the uniqueness of $A(n, k, \delta)$, we need to demonstrate that the spectral radius of the subgraph of $A(n, k, \delta)$ is less than $n - \delta + k - 3$, which is stated in Theorem 3.1. Thus Theorem 3.1 is the basis in the sense of closure in the proof of our main result.

In the proof the Theorem 3.1, it is required that the K_{k-1} part of $A(n, k, \delta)$ has at least 2 vertices, so we assume $k \geq 3$ in Theorem 3.1.

Theorem 3.1 *Let $\delta \geq k \geq 3$, and let G be a connected graph of order $n \geq F_0(k, \delta)$ and minimum degree δ . If G is a subgraph of $A(n, k, \delta)$, then*

$$\lambda(G) < n - \delta + k - 3,$$

unless $G = A(n, k, \delta)$.

Proof We write $\lambda := \lambda(G)$ for short. Let $x = (x_1, x_2, \dots, x_n)^T$ be the unique positive unit eigenvector corresponding to λ . According to the Rayleigh’s principle, we have

$$\lambda = \langle A(G)x, x \rangle = x^T A(G)x = 2 \sum_{ij \in E(G)} x_i x_j.$$

Let G be a proper subgraph of $A(n, k, \delta)$. Without loss of generality, we may assume that G is obtained by omitting just one edge $\{u, v\}$ of $A(n, k, \delta)$. Now we denote by X for the set of vertices in $A(n, k, \delta)$ of degree δ , Y the set of their neighbors, and Z the set of the remaining $n - \delta - 1$ vertices. Since G is a connected graph with minimum degree δ , G must contain all the edges incident with X . Therefore $\{u, v\} \subset Y \cup Z$, and this implies three possibilities: (a) $\{u, v\} \subset Y$; (b) $u \in Y, v \in Z$; (c) $\{u, v\} \subset Z$. We denote the corresponding graph in each of these three cases by G_1, G_2 and G_3 , respectively. We shall show that $\lambda(G_1) \leq \lambda(G_2) \leq \lambda(G_3)$.

Firstly, suppose that case (a) holds, that is, $\{u, v\} \subset Y$ and $G = G_1$. We will show that $\lambda(G_1) \leq \lambda(G_2)$. We choose a vertex $w \in Z$, then remove the edge $\{u, w\}$ and add the edge $\{u, v\}$, we obtain one graph G' , which is of the form G_2 in case (b). If $x_w \leq x_v = x_u$, then $x^T A(G')x - x^T A(G_1)x = 2x_u(x_v - x_w) \geq 0$, by the Rayleigh principle, we have $\lambda(G') \geq \lambda(G_1)$. If $x_w > x_v$, we construct the vector x' of G' from x by swap the entries x_v and x_w . In this case, we note that $G_1 \setminus \{v, w\}$ and $G' \setminus \{v, w\}$ are isomorphic, then we get that

$$\begin{aligned}
& x'^T A(G')x' - x^T A(G_1)x \\
&= 2x_w \left(\sum_{i \in X} x_i + \sum_{i \in Y \setminus \{u, v\}} x_i + \sum_{i \in Z \setminus \{w\}} x_i + x_u + x_v \right) + 2x_v \left(\sum_{i \in Y \setminus \{u, v\}} x_i + \sum_{i \in Z \setminus \{w\}} x_i \right) \\
&\quad - 2x_v \left(\sum_{i \in X} x_i + \sum_{i \in Y \setminus \{u, v\}} x_i + \sum_{i \in Z \setminus \{w\}} x_i + x_w \right) - 2x_w \left(\sum_{i \in Y \setminus \{u, v\}} x_i + \sum_{i \in Z \setminus \{w\}} x_i + x_u \right) \\
&= 2(x_w - x_v) \sum_{i \in X} x_i > 0,
\end{aligned}$$

so by the Rayleigh principle, we have $\lambda(G') > \lambda(G_1)$.

Secondly, suppose that case (b) holds, that is, $u \in Y$, $v \in Z$, and $G = G_2$. We will show that $\lambda(G_2) \leq \lambda(G_3)$. We choose a vertex $w \in Z$, then remove the edge $\{v, w\}$ and add the edge $\{u, v\}$, we obtain one graph G'' , which is of the form G_3 in case (c). If $x_u \geq x_w$, then $x^T A(G'')x - x^T A(G_2)x = 2x_v(x_u - x_w) \geq 0$, by the Rayleigh principle, we have $\lambda(G'') \geq \lambda(G_2)$. If $x_u < x_w$, we construct the vector x' from x by swap the entries x_w and x_u . Since $G_2 \setminus \{u, w\}$ and $G'' \setminus \{u, w\}$ are isomorphic, then we get that

$$\begin{aligned}
& x'^T A(G'')x' - x^T A(G_2)x \\
&= 2x_w \left(\sum_{i \in X} x_i + \sum_{i \in Y \setminus \{u\}} x_i + \sum_{i \in Z \setminus \{v, w\}} x_i + x_u + x_v \right) + 2x_u \left(\sum_{i \in Y \setminus \{u\}} x_i + \sum_{i \in Z \setminus \{v, w\}} x_i \right) \\
&\quad - 2x_u \left(\sum_{i \in X} x_i + \sum_{i \in Y \setminus \{u\}} x_i + \sum_{i \in Z \setminus \{v, w\}} x_i + x_w \right) - 2x_w \left(\sum_{i \in Y \setminus \{u\}} x_i + \sum_{i \in Z \setminus \{v, w\}} x_i + x_v \right) \\
&= 2(x_w - x_u) \sum_{i \in X} x_i > 0,
\end{aligned}$$

again by the Rayleigh principle, we have $\lambda(G'') > \lambda(G_2)$.

Therefore, we may assume $\{u, v\} \subset Z$ and $G = G_3$ in the rest of this theorem. From symmetry, we obtain $x_i = x_j := x$ for any $i, j \in X$; $x_i = x_j := y$ for any $i, j \in Y$; and $x_i = x_j := z$ for any $i, j \in Z \setminus \{u, v\}$. As the vertices u and v are symmetric, we may write $x_u = x_v := t$. From $\lambda x_i = \sum_{j \in E(G)} x_j$, we have

$$\begin{aligned}
\lambda x &= (\delta - k + 1)x + (k - 1)y, \\
\lambda y &= (\delta - k + 2)x + (k - 2)y + (n - \delta - 3)z + 2t, \\
\lambda z &= (k - 1)y + (n - \delta - 4)z + 2t, \\
\lambda t &= (k - 1)y + (n - \delta - 3)z.
\end{aligned}$$

From the above system, we obtain

$$x = \left(\frac{k - 1}{\lambda - \delta + k - 1} \right) y, \quad (1)$$

$$\begin{aligned}
 z &= \left(1 - \frac{(\delta - k + 2)(k - 1)}{(\lambda + 1)(\lambda - \delta + k - 1)}\right)y, \\
 t &= \left(\frac{\lambda + 1}{\lambda + 2}\right)\left(1 - \frac{(\delta - k + 2)(k - 1)}{(\lambda + 1)(\lambda - \delta + k - 1)}\right)y.
 \end{aligned}
 \tag{2}$$

Note that if we delete all edges incident to X , and add the edge $\{u, v\}$ to G , we obtain the graph $\overline{K_{\delta-k+2}} \cup K_{n-\delta+k-2}$. Let x'' be the restriction of x to $K_{n-\delta+k-2}$. We see that

$$\begin{aligned}
 &\langle A(K_{n-\delta+k-2})x'', x'' \rangle \\
 &= \langle A(G)x, x \rangle + 2t^2 - 2(k - 1)(\delta - k + 2)xy - (\delta - k + 2)(\delta - k + 1)x^2 \\
 &= \lambda + 2t^2 - 2(k - 1)(\delta - k + 2)xy - (\delta - k + 2)(\delta - k + 1)x^2.
 \end{aligned}$$

Observe that

$$\langle A(K_{n-\delta+k-2})x'', x'' \rangle \leq \lambda(K_{n-\delta+k-2}) = n - \delta + k - 3,$$

from the above, we have

$$\lambda + 2t^2 - 2(k - 1)(\delta - k + 2)xy - (\delta - k + 2)(\delta - k + 1)x^2 \leq n - \delta + k - 3.
 \tag{3}$$

Suppose on the contrary that $\lambda \geq n - \delta + k - 3$, then (3) implies

$$2(k - 1)(\delta - k + 2)xy + (\delta - k + 2)(\delta - k + 1)x^2 \geq 2t^2.$$

By applying (1) and (2), we obtain

$$\begin{aligned}
 &2\frac{(k - 1)^2(\delta - k + 2)}{\lambda - \delta + k - 1}y^2 + \frac{(\delta - k + 2)(\delta - k + 1)(k - 1)^2}{(\lambda - \delta + k - 1)^2}y^2 \\
 &> 2\left(1 - \frac{1}{\lambda + 2}\right)^2\left(1 - \frac{(\delta - k + 2)(k - 1)}{(\lambda + 1)(\lambda - \delta + k - 1)}\right)^2y^2.
 \end{aligned}$$

Recall that the Bernoulli inequality states that: for any nonzero real number $x > -1$, and an integer $n > 1$, we have

$$(1 + x)^n > 1 + nx.
 \tag{4}$$

Eliminating y^2 and applying (4) to the previous inequality, we get

$$\begin{aligned}
 &2\frac{(k - 1)^2(\delta - k + 2)}{\lambda - \delta + k - 1} + \frac{(\delta - k + 2)(\delta - k + 1)(k - 1)^2}{(\lambda - \delta + k - 1)^2} \\
 &> 2\left(1 - \frac{2}{\lambda + 2} - \frac{2(\delta - k + 2)(k - 1)}{(\lambda + 1)(\lambda - \delta + k - 1)}\right).
 \end{aligned}$$

Rewriting this inequality, we have

$$\begin{aligned} & \frac{(\delta - k + 2)(\delta - k + 1)(k - 1)^2}{(\lambda - \delta + k - 1)} + 2(k - 1)^2(\delta - k + 2) \\ & > 2 \left(\lambda - \delta + k - 1 - \frac{2(\lambda - \delta + k - 1)}{\lambda + 2} - \frac{2(\delta - k + 2)(k - 1)}{\lambda + 1} \right) \\ & > 2(\lambda - \delta + k - 1 - 4). \end{aligned}$$

Note that

$$\begin{aligned} \lambda - \delta + k - 1 & > (n - \delta + k - 3) - \delta + k - 1 \quad \text{as } n \geq F_0(k, \delta) \\ & \geq (\delta - k + 2)(k^2 - 2k + 4) + 3 - 2(\delta - k + 2) \\ & = (\delta - k + 2)(k^2 - 2k + 2) + 3, \end{aligned}$$

the previous inequality in turn becomes

$$\begin{aligned} & (\delta - k + 2)(\delta - k + 1)(k - 1)^2 \\ & > 2(\lambda - \delta + k - 1) \left(\lambda - \delta + k - 1 - 4 - (k - 1)^2(\delta - k + 2) \right) \\ & > 2 \left((\delta - k + 2)(k^2 - 2k + 2) + 3 \right) \left((\delta - k + 2)(k^2 - 2k + 2) + 3 \right. \\ & \quad \left. - 4 - (k - 1)^2(\delta - k + 2) \right) \\ & = 2 \left((\delta - k + 2)(k^2 - 2k + 2) + 3 \right) (\delta - k + 1), \end{aligned}$$

obviously, this is a contradiction and the proof is complete. \square

The following result is of independent interest, and its ideas will be used in the proof of Theorem 3.4.

Theorem 3.2 *Let G be a connected graph of order n with minimum degree δ . If*

$$2m \geq n^2 - (2\delta - 2k + 5)n + 2(\delta + 1)(\delta - k + 2),$$

then G is k -connected unless $G = A(n, k, \delta)$.

Proof Suppose G is not k -connected. From Lemma 2.1, there exists an integer $1 \leq i \leq \frac{n-k+1}{2}$ such that $d_i \leq i + k - 2$ and $d_{n-k+1} \leq n - i - 1$. Thus we have

$$\begin{aligned} 2m & \leq i(i + k - 2) + (n - i - k + 1)(n - i - 1) + (k - 1)(n - 1) \\ & = 2i^2 - (2n - 2k + 2)i + n(n - 1). \end{aligned} \tag{5}$$

Since $\delta \leq i + k - 2$, we have $\delta - k + 2 \leq i \leq \frac{n-k+1}{2}$. We assume that $f(x) = 2x^2 - (2n - 2k + 2)x$ with $\delta - k + 2 \leq x \leq \frac{1}{2}(n - k + 1)$. As the symmetric axis of $f(x)$ is $x_0 = \frac{1}{2}(n - k + 1)$, we find that

$$f_{\max}(x) = f(\delta - k + 2) = 2(\delta + 1 - n)(\delta - k + 2).$$

Therefore $2m \leq n^2 - (2\delta - 2k + 5)n + 2(\delta + 1)(\delta - k + 2)$.

If $2m = n^2 - (2\delta - 2k + 5)n + 2(\delta + 1)(\delta - k + 2)$, then $i = \delta - k + 2$, $d_1 = d_2 = \dots = d_{\delta-k+2} = \delta$, $d_{\delta-k+3} = \dots = d_{n-k+1} = n - \delta + k - 3$, $d_{n-k+2} = \dots = d_n = n - 1$. Thus $G = K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$, which is not k -connected from the definition. □

Nikiforov [25] recently obtained a sufficient spectral condition for a graph to be hamiltonian. Soon he [26] obtained the following refined result.

Theorem 3.3 *Let $t \geq 1, n \geq t^3/2 + t + 4$, and let G be a graph of order n . If $\delta(G) \geq t$ and*

$$\lambda(G) \geq n - t - 1,$$

then G has a hamiltonian cycle unless $G = K_t \vee (K_{n-2t} \cup \overline{K_t})$ or $G = K_1 \vee (K_{n-t-1} \cup K_t)$.

Thus, except for at most two graphs, when $\lambda(G) \geq n - t - 1$, the graph G is 2-connected. So in the next main theorem, we only consider the case the connectivity is at least 3.

Theorem 3.4 *Let $\delta \geq k \geq 3, n \geq F_0(k, \delta)$. Let G be a connected graph of order n and minimum degree $\delta(G) \geq \delta$. If*

$$\lambda(G) \geq n - \delta + k - 3,$$

then G is k -connected unless $G = A(n, k, \delta)$.

Proof Suppose that $\lambda(G) \geq n - \delta + k - 3$ and G is not k -connected. From Lemma 2.2, we consider the closure $H := cl_{n+k-2}(G)$ of G . By Lemma 2.3, H is not k -connected, and $\delta(H) \geq \delta(G) \geq \delta$ and $\lambda(H) \geq \lambda(G) \geq n - \delta + k - 3$. Obviously, for any two nonadjacent vertices $i, j \in V(H)$, $d_i + d_j \leq n + k - 3$. From Theorem 3.1, we need only to prove $H = A(n, k, \delta)$.

According to the assumptions of the theorem and Lemmas 2.4, 2.5, we have

$$n - \delta + k - 3 \leq \lambda(H) \leq \frac{1}{2} \left(\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - n\delta)} \right),$$

this implies

$$2m \geq n^2 + (2k - 2\delta - 5)n + (2\delta - k + 3)(\delta - k + 2) := g_1(n). \tag{6}$$

We next prove that $i = \delta - k + 2$. Suppose on the contrary $i \geq \delta - k + 3$. Since G is not k -connected, from (5) of Theorem 3.2, we consider $f(x) = 2x^2 - (2n - 2k + 2)x$ with $\delta - k + 3 \leq x \leq \frac{1}{2}(n - k + 1)$. Obviously $f_{\max}(x) = f(\delta - k + 3)$, we have

$$2m \leq n^2 + (2k - 2\delta - 7)n + 2(\delta - k + 3)(\delta + 2) := g_2(n). \tag{7}$$

From (6) and (7), bearing in mind that $n \geq 2k\delta - 2k^2 + 5k := F_1(k, \delta)$, $F_1(k, \delta) \leq F_0(k, \delta)$, we obtain

$$\begin{aligned} g_1(n) - g_2(n) &= -6 - k + 2n - k\delta - 3\delta + k^2 \\ &\geq -6 - k + 2(2k\delta - 2k^2 + 5k) - k\delta - 3\delta + k^2 \\ &= 3(k - 1)(\delta - k + 2) > 0, \end{aligned}$$

hence inequality (7) cannot hold. Therefore, we have $i = \delta - k + 2$, and thus

$$d_1 = d_2 = \dots = d_{\delta-k+2} = \delta.$$

Next, we shall show that $d_{\delta-k+3} \geq n - \delta + k - 3 - k\delta + \delta - 3k + 2 + k^2$. Since $n \geq F_0(k, \delta)$, $n - \delta + k - 3 - k\delta + \delta - 3k + 2 + k^2 > 0$. Suppose on the contrary that $d_{\delta-k+3} < n - \delta + k - 3 - k\delta + \delta - 3k + 2 + k^2$, we have

$$\begin{aligned} 2m &< (\delta - k + 2)\delta + (n - \delta + k - 3 - k\delta + \delta - 3k + 2 + k^2) \\ &\quad + (n - \delta - 2)(n - \delta + k - 3) + (k - 1)(n - 1) \\ &= n^2 + (2k - 2\delta - 5)n + (2\delta - k + 3)(\delta - k + 2) \\ &\leq 2m. \end{aligned}$$

This is a contradiction. Hence $d_i \geq n - \delta + k - 3 - k\delta + \delta - 3k + 2 + k^2$ for any $i \in \{\delta - k + 3, \dots, n\}$.

Now, we conclude that $\{\delta - k + 3, \dots, n\}$ induces a complete graph. Suppose $i, j \in \{\delta - k + 3, \dots, n\}$ and i, j are not adjacent, we have

$$\begin{aligned} d_i + d_j &\geq 2n - 2\delta + 2k - 6 - 2k\delta + 2\delta - 6k + 4 + 2k^2 \\ &\geq n + (2k\delta - 2k^2 + 5k) - 2\delta + 2k - 6 - 2k\delta + 2\delta - 6k + 4 + 2k^2 \\ &= n + k - 2 > n + k - 3, \end{aligned}$$

the second inequality holds as $n \geq 2k\delta - 2k^2 + 5k$. This contradicts the property of H . Hence we can get $\{\delta - k + 3, \dots, n\}$ induces a complete graph.

Let $X = \{1, 2, \dots, \delta - k + 2\}$, Y be the set of vertices in $\{\delta - k + 3, \dots, n\}$ having neighbors in X . In fact, every vertex from Y is adjacent to every vertex in X . Indeed, suppose that is not the case, and let $w \in Y$, $u, v \in X$ be such that w is adjacent to u , but not to v . We see that $d_w + d_v \geq (n - \delta + k - 2) + \delta = n + k - 2$, which contradicts the fact that H is the closure of G .

Next, let $\ell := |Y|$. As the degree of the vertices in X is δ , we have $k - 1 \leq \ell \leq \delta$.

If $\ell = k - 1$, then $H = A(n, k, \delta)$. If we delete $k - 1$ vertices in K_{k-1} , we obtain a graph that is not connected. Thus in this case, H is not k -connected.

If $\ell = k$, then H is k -connected from definition.

If $\ell = \delta$, then $H = K_\delta \vee (K_{n-2\delta+k-2} \cup \overline{K_{\delta-k+2}})$. Since every vertex from Y is adjacent to every vertex in X , and $\delta \geq k$, it follows that if we delete any $k - 1$ vertices, the resulting graph is connected. Therefore in this case, H is k -connected.

If $k \leq \ell < \delta$, obviously H is k -connected.

The proof is completed. \square

Since $\lambda(A(n, k, \delta)) \geq n - \delta + k - 3$, we immediately have

Corollary 3.5 *Let $\delta \geq k \geq 3$, $n \geq F_0(k, \delta)$. Let G be a connected graph of order n and minimum degree $\delta(G) \geq \delta$. If*

$$\lambda(G) \geq \lambda(A(n, k, \delta)),$$

then G is k -connected unless $G = A(n, k, \delta)$.

Since $A(n, k, \delta)$ is k -edge-connected, we have the following corollary.

Corollary 3.6 *Let $\delta \geq k \geq 3$, $n \geq F_0(k, \delta)$. Let G be a connected graph of order n and minimum degree $\delta(G) \geq \delta$. If*

$$\lambda(G) \geq n - \delta + k - 3,$$

then G is k -edge-connected.

At last, we would like to mention that we do not make too much efforts to optimize the lower bound for n . For smaller n , we can not find the way to work out. We leave it for further study.

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References

1. Bauer, D., Broersma, H.J., van den Heuvel, J., Kahl, N., Nevo, A., Schmeichel, E., Woodall, D.R., Yatauro, M.: Best monotone degree conditions for graph properties: a survey. *Graphs Comb.* **31**, 1–22 (2015)
2. Berman, A., Zhang, X.D.: On the spectral radius of graphs with cut vertices. *J. Comb. Theory Ser. B* **83**, 233–240 (2001)
3. Bollobás, B.: *Extremal Graph Theory*. Academic Press, New York (1978)
4. Bondy, J.A., Chvátal, V.: A method in graph theory. *Discrete Math.* **15**, 111–135 (1976)
5. Brualdi, R.A., Solheid, E.S.: On the spectral radius of complementary acyclic matrices of zeros and ones. *SIAM J. Algebr. Discrete Method* **7**, 265–272 (1986)
6. Chandran, S.L.: Minimum cuts, girth and spectral threshold. *Inf. Process. Lett.* **89**(3), 105–110 (2004)
7. Chvátal, V.: On Hamiltons ideals. *J. Comb. Theory Ser. B* **12**, 163–168 (1972)
8. Cioabă, S.M.: Eigenvalues and edge-connectivity of regular graphs. *Linear Algebra Appl.* **432**, 458–470 (2010)
9. Cioabă, S.M., Gu, X.: Connectivity, toughness, spanning trees of bounded degree, and the spectrum of regular graphs. *Czechoslov. Math. J.* **66**(141), 913–924 (2016)
10. Cioabă, S.M., Gregory, D.: Large matchings from eigenvalues. *Linear Algebra Appl.* **422**, 308–317 (2007)
11. Cioabă, S.M.: Perfect matchings eigenvalues and expansion. *C.R. Math. Acad. Sci. Soc. R. Can.* **27**(4), 101–104 (2005)
12. Cioabă, S.M., Gregory, D., Haemers, W.: Matchings in regular graphs from eigenvalues. *J. Comb. Theory Ser. B* **99**, 287–297 (2009)
13. Feng, L.H., Cao, J., Liu, W., Ding, S., Liu, H.: The spectral radius of edge chromatic critical graphs. *Linear Algebra Appl.* **492**, 78–88 (2016)

14. Feng, L.H., Zhang, P.L., Liu, H., Liu, W., Liu, M., Hu, Y.: Spectral conditions for some graphical properties. *Linear Algebra Appl.* **524**, 182–198 (2017)
15. Fiedler, M.: Algebraic connectivity of graphs. *Czechoslov. Math. J.* **23**, 298–305 (1973)
16. Fiedler, M., Nikiforov, V.: Spectral radius and Hamiltonicity of graphs. *Linear Algebra Appl.* **432**(9), 2170–2173 (2010)
17. Gu, X., Lai, H.-J., Li, P., Yao, S.: Edge-disjoint spanning trees, edge connectivity and eigenvalues in graphs. *J. Graph Theory* **81**, 16–29 (2016)
18. Hong, Y., Shu, J.L., Fang, K.: A sharp upper bound of the spectral radius of graphs. *J. Comb. Theory Ser. B* **81**, 177–183 (2001)
19. Li, B.-L., Ning, B.: Spectral analogues of Moon–Moser’s theorem on Hamiltonian paths in bipartite graphs. *Linear Algebra Appl.* **515**, 180–195 (2017)
20. Liu, H., Lu, M., Tian, F.: Edge-connectivity and (signless) Laplacian eigenvalue of graphs. *Linear Algebra Appl.* **439**, 3777–3784 (2013)
21. Liu, R., Shiu, W.C., Xue, J.: Sufficient spectral conditions on Hamiltonian and traceable graphs. *Linear Algebra Appl.* **467**, 254–266 (2015)
22. Lu, M., Liu, H., Tian, F.: Spectral radius and Hamiltonian graphs. *Linear Algebra Appl.* **437**(7), 1670–1674 (2012)
23. Nikiforov, V.: The spectral radius of graphs without paths and cycles of specified length. *Linear Algebra Appl.* **432**, 2243–2256 (2010)
24. Nikiforov, V.: Some inequalities for the largest eigenvalue of a graph. *Comb. Probab. Comput.* **11**, 179–189 (2002)
25. Nikiforov, V.: Spectral radius and Hamiltonicity of graphs with large minimum degree. *Czechoslov. Math. J.* **66**(141), 925–940 (2016)
26. Nikiforov, V.: Spectral radius and Hamiltonicity of graphs with large minimum degree. [arXiv:1602.01033v3](https://arxiv.org/abs/1602.01033v3)
27. Ning, B., Ge, J.: Spectral radius and Hamiltonian properties of graphs. *Linear Multilinear Algebra* **63**(8), 1520–1530 (2015)
28. Ning, B., Li, B.: Spectral radius and traceability of connected claw-free graphs. *Filomat* **30**(9), 2445–2452 (2016)
29. Suil, O., Cioabă, S.M.: Edge-connectivity, eigenvalues, and matchings in regular graphs. *SIAM J. Discrete Math.* **24**, 1470–1481 (2010)
30. Stevanović, D.: *Spectral Radius of Graphs*. Academic Press, Amsterdam (2015)
31. Stevanović, D., Aouchiche, M., Hansen, P.: On the spectral radius of graphs with a given domination number. *Linear Algebra Appl.* **428**, 1854–1864 (2008)