Monatsh Math (2018) 185:489–501
https://doi.org/10.1007/s00605-017-1024-3

Logarithmic coefficients and a coefficient conjecture for univalent functions

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Received: 30 September 2016 / Accepted: 20 January 2017 / Published online: 3 February 2017

Abstract Let $\mathcal{U}(\lambda)$ denote the family of analytic functions $f(z)$, $f(0) = 0 = f'(0) - 1$, in the unit disk $\mathbb{D}$, which satisfy the condition $\left|\left(z/f(z)\right)^2 f'(z) - 1\right| < \lambda$ for some $0 < \lambda \leq 1$. The logarithmic coefficients γ_n of f are defined by the formula $\log(f(z)/z) = 2\sum_{n=1}^{\infty} \gamma_n z^n$. In a recent paper, the present authors proposed a conjecture that if $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq 1$, then $|a_n| \leq \sum_{k=0}^{n-1} \lambda^k$ for $n \geq 2$ and provided a new proof for the case $n = 2$. One of the aims of this article is to present a proof of this conjecture for $n = 3, 4$ and an elegant proof of the inequality for $n = 2$, with equality for $f(z) = z/[(1+z)(1+\lambda z)]$. In addition, the authors prove the following sharp inequality for $f \in \mathcal{U}(\lambda)$:

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4}\left(\frac{\pi^2}{6} + 2\mathrm{Li}_2(\lambda) + \mathrm{Li}_2(\lambda^2)\right),$$

Communicated by A. Constantin.

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where Li_2 denotes the dilogarithm function. Furthermore, the authors prove two such new inequalities satisfied by the corresponding logarithmic coefficients of some other subfamilies of $\mathcal{S}$.

Keywords Univalent · Starlike · Convex and close-to-convex functions · Subordination · Logarithmic coefficients and coefficient estimates

Mathematics Subject Classification 30C45

1 Introduction

Let $\mathcal{A}$ be the class of functions f analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ denote the class of functions f from $\mathcal{A}$ that are univalent in $\mathbb{D}$. Then the logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined by the formula

$$\frac{1}{2}\log\left(\frac{f(z)}{z}\right) = \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

These coefficients play an important role for various estimates in the theory of univalent functions. When we require a distinction, we use the notation $\gamma_n(f)$ instead of γ_n. For example, the Koebe function $k(z) = z(1 - e^{i\theta}z)^{-2}$ for each θ has logarithmic coefficients $\gamma_n(k) = e^{in\theta}/n$, $n \geq 1$. If $f \in \mathcal{S}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then by (1) it follows that $2\gamma_1 = a_2$ and hence, by the Bieberbach inequality, $|\gamma_1| \leq 1$. Let $\mathcal{S}^\star$ denote the class of functions $f \in \mathcal{S}$ such that $f(\mathbb{D})$ is starlike with respect to the origin. Functions $f \in \mathcal{S}^\star$ are characterized by the condition $\mathrm{Re}\,(zf'(z)/f(z)) > 0$ in $\mathbb{D}$. The inequality $|\gamma_n| \leq 1/n$ holds for starlike functions $f \in \mathcal{S}$, but is false for the full class $\mathcal{S}$, even in order of magnitude. See [4, Theorem 8.4 on page 242]. In [6], Girela pointed out that this bound is actually false for the class of close-to-convex functions in $\mathbb{D}$ which is defined as follows: A function $f \in \mathcal{A}$ is called close-to-convex, denoted by $f \in \mathcal{K}$, if there exists a real α and a $g \in \mathcal{S}^\star$ such that

$$\mathrm{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}.$$

For $0 \leq \beta < 1$, a function $f \in \mathcal{S}$ is said to belong to the class of starlike functions of order β, denoted by $f \in \mathcal{S}^\star(\beta)$, if $\mathrm{Re}\left(zf'(z)/f(z)\right) > \beta$ for $z \in \mathbb{D}$. Note that $\mathcal{S}(0) =: \mathcal{S}^\star$. The class of all convex functions of order β, denoted by $\mathcal{C}(\beta)$, is then defined by $\mathcal{C}(\beta) = \{f \in \mathcal{S} : zf' \in \mathcal{S}^\star(\beta)\}$. The class $\mathcal{C}(0) =: \mathcal{C}$ is usually referred to as the class of convex functions in $\mathbb{D}$. With the class $\mathcal{S}$ being of the first priority, its subclasses such as $\mathcal{S}^\star$, $\mathcal{K}$, and $\mathcal{C}$, respectively, have been extensively studied in the literature and they appear in different contexts. We refer to [4,7,10,12] for a general reference related to the present study. In [5, Theorem 4], it was shown that the logarithmic coefficients γ_n of every function $f \in \mathcal{S}$ satisfy

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6} \tag{2}$$

and the equality is attained for the Koebe function. The proof uses ideas from the work of Baernstein [3] on integral means. However, this result is easy to prove (see Theorem 1) in the case of functions in the class $\mathcal{U} := \mathcal{U}(1)$ which is defined as follows:

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, \quad z \in \mathbb{D} \right\},$$

where $\lambda \in (0, 1]$. It is known that [1,2,11] every $f \in \mathcal{U}$ is univalent in $\mathbb{D}$ and hence, $\mathcal{U}(\lambda) \subset \mathcal{U} \subset \mathcal{S}$ for $\lambda \in (0, 1]$. The present authors have established many interesting properties of the family $\mathcal{U}(\lambda)$. See [10] and the references therein. For example, if $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq 1$ and $a_2 = f''(0)/2$, then we have the subordination relations

$$\frac{f(z)}{z} \prec \frac{1}{1 + (1 + \lambda)z + \lambda z^2} = \frac{1}{(1 + z)(1 + \lambda z)}, \ z \in \mathbb{D}, \tag{3}$$

and

$$\frac{z}{f(z)} + a_2 z \prec 1 + 2\lambda z + \lambda z^2, \ z \in \mathbb{D}.$$

Here $\prec$ denotes the usual subordination [4,7,12]. In addition, the following conjecture was proposed in [10].

Conjecture 1 *Suppose that* $f \in \mathcal{U}(\lambda)$ *for some* $0 < \lambda \leq 1$. *Then* $|a_n| \leq \sum_{k=0}^{n-1} \lambda^k$ *for* $n \geq 2$.

In Theorem 1, we present a direct proof of an inequality analogous to (2) for functions in $\mathcal{U}(\lambda)$ and in Corollary 1, we obtain the inequality (2) as a special case for $\mathcal{U}$. At the end of Sect. 2, we also consider estimates of the type (2) for some interesting subclasses of univalent functions. However, Conjecture 1 remains open for $n \geq 5$. On the other hand, the proof for the case $n = 2$ of this conjecture is due to [17] and an alternate proof was obtained recently by the present authors in [10, Theorem 1]. In this paper, we show that Conjecture 1 is true for $n = 3, 4$, and our proof includes an elegant proof of the case $n = 2$. The main results and their proofs are presented in Sects. 2 and 3.

2 Logarithmic coefficients of functions in $\mathcal{U}(\lambda)$

Theorem 1 *For* $0 < \lambda \leq 1$, *the logarithmic coefficients of* $f \in \mathcal{U}(\lambda)$ *satisfy the inequality*

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \left(\frac{\pi^2}{6} + 2\mathrm{Li}_2(\lambda) + \mathrm{Li}_2(\lambda^2) \right), \tag{4}$$

where Li_2 *denotes the dilogarithm function given by*

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = z \int_0^1 \frac{\log(1/t)}{1 - tz}\, dt.$$

The inequality (4) *is sharp. Further, there exists a function* $f \in \mathcal{U}$ *such that* $|\gamma_n| > (1 + \lambda^n)/(2n)$ *for some* n.

Proof Let $f \in \mathcal{U}(\lambda)$. Then, by (3), we have

$$\frac{z}{f(z)} \prec (1 - z)(1 - \lambda z)$$

which clearly gives

$$\sum_{n=1}^{\infty} \gamma_n z^n = \log \sqrt{\frac{f(z)}{z}} \prec \frac{-\log(1 - z) - \log(1 - \lambda z)}{2} = \sum_{n=1}^{\infty} \frac{1}{2n}(1 + \lambda^n) z^n. \quad (5)$$

Again, by Rogosinski's theorem (see [4, 6.2]), we obtain

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{1}{4n^2}(1 + \lambda^n)^2 = \frac{1}{4}\left(\sum_{n=1}^{\infty} \frac{1}{n^2} + 2\sum_{n=1}^{\infty} \frac{\lambda^n}{n^2} + \sum_{n=1}^{\infty} \frac{\lambda^{2n}}{n^2}\right)$$

and the desired inequality (4) follows. For the function

$$g_\lambda(z) = \frac{z}{(1 - z)(1 - \lambda z)},$$

we find that $\gamma_n(g_\lambda) = (1 + \lambda^n)/(2n)$ for $n \geq 1$ and therefore, we have the equality in (4). Note that $g_1(z)$ is the Koebe function $z/(1 - z)^2$.

From the relation (5), we cannot conclude that

$$|\gamma_n(f)| \leq |\gamma_n(g_\lambda)| = \frac{1 + \lambda^n}{2n} \quad \text{for } f \in \mathcal{U}(\lambda).$$

Indeed for the function f_λ defined by

$$f_\lambda(z) = \frac{z}{(1 - z)(1 - \lambda z)(1 + (\lambda/(1 + \lambda))z)} \quad (6)$$

we find that

$$\frac{z}{f_\lambda(z)} = 1 + \frac{\lambda - (1 + \lambda)^2}{1 + \lambda} z + \frac{\lambda^2}{1 + \lambda} z^3$$

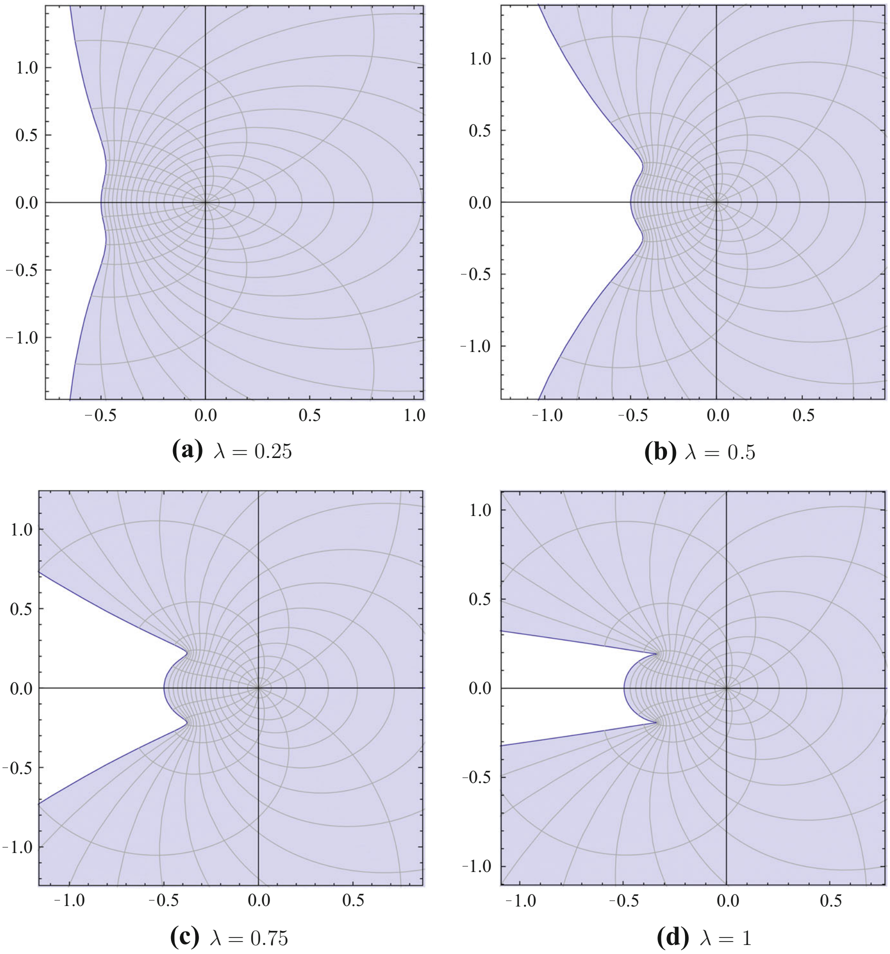

Fig. 1 The image of $f_\lambda(z) = \dfrac{z}{(1-z)(1-\lambda z)(1+(\lambda/(1+\lambda))z)}$ under $\mathbb{D}$ for certain values of λ

and

$$\left(\frac{z}{f_\lambda(z)}\right)^2 f_\lambda'(z) - 1 = -\frac{2\lambda^2}{1+\lambda} z^3 = -\left(1 - \frac{(1+2\lambda)(1-\lambda)}{1+\lambda}\right) z^3$$

which clearly shows that $f_\lambda \in \mathcal{U}(\lambda)$. The images of $\mathbb{D}$ under $f_\lambda(z)$ for certain values of λ are shown in Fig. 1a–d. Moreover, for this function, we have

$$\begin{aligned}\log\left(\frac{f_\lambda(z)}{z}\right) &= -\log(1-z) - \log(1-\lambda z) - \log\left(1 + \frac{\lambda}{1+\lambda} z\right)\\ &= 2\sum_{n=1}^{\infty} \gamma_n(f_\lambda) z^n,\end{aligned}$$

where

$$\gamma_n(f_\lambda) = \frac{1}{2}\left(\frac{1+\lambda^n}{n} + (-1)^n \frac{\lambda^n}{(1+\lambda)^n}\right).$$

This contradicts the above inequality at least for even integer values of $n \geq 2$. Moreover, with these $\gamma_n(f_\lambda)$ for $n \geq 1$, we obtain

$$\sum_{n=1}^{\infty} |\gamma_n(f_\lambda)|^2 = \frac{1}{4}\sum_{n=1}^{\infty}\left\{\frac{(1+\lambda^n)^2}{n^2} + 2\frac{(-1)^n}{n}\left(\left(\frac{\lambda^2}{1+\lambda}\right)^n + \left(\frac{\lambda}{1+\lambda}\right)^n\right) + \left(\frac{\lambda}{1+\lambda}\right)^{2n}\right\}$$

and by a computation, it follows easily that

$$\begin{aligned}\sum_{n=1}^{\infty} |\gamma_n(f_\lambda)|^2 &= \frac{1}{4}\left(\frac{\pi^2}{6} + 2\mathrm{Li}_2(\lambda) + \mathrm{Li}_2(\lambda^2)\right) \\ &\quad -\frac{1}{2}\log\left[\left(1+\frac{\lambda^2}{1+\lambda}\right)\left(1+\frac{\lambda}{1+\lambda}\right)\right] + \frac{\lambda^2}{4(1+2\lambda)} \\ &< \frac{1}{4}\left(\frac{\pi^2}{6} + 2\mathrm{Li}_2(\lambda) + \mathrm{Li}_2(\lambda^2)\right) \quad \text{for } 0 < \lambda \leq 1,\end{aligned}$$

and we complete the proof. □

Corollary 1 *The logarithmic coefficients of $f \in \mathcal{U}$ satisfy the inequality*

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{7}$$

We have equality in the last inequality for the Koebe function $k(z) = z(1-e^{i\theta}z)^{-2}$. Further there exists a function $f \in \mathcal{U}$ such that $|\gamma_n| > 1/n$ for some n.

Remark 1 From the analytic characterization of starlike functions, it is easy to see that for $f \in \mathcal{S}^\star$,

$$\frac{zf'(z)}{f(z)} - 1 = z\left(\log\left(\frac{f(z)}{z}\right)\right)' = 2\sum_{n=1}^{\infty} n\gamma_n z^n \prec \frac{2z}{1-z}$$

and thus, by Rogosinski's result, we obtain that $|\gamma_n| \leq 1/n$ for $n \geq 1$. In fact for starlike functions of order α, $\alpha \in [0, 1)$, the corresponding logarithmic coefficients satisfy the inequality $|\gamma_n| \leq (1-\alpha)/n$ for $n \geq 1$. Moreover, one can quickly obtain that

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq (1-\alpha)^2 \frac{\pi^2}{6}$$

if $f \in \mathcal{S}^\star(\alpha)$, $\alpha \in [0, 1)$ (See also the proof of Theorem 2 and Remark 3). As remarked in the proof of Theorem 1, from the relation (7), we cannot conclude the same fact, namely, $|\gamma_n| \leq 1/n$ for $n \geq 1$, for the class $\mathcal{U}$ although the Koebe function $k(z) = z/(1-z)^2$ belongs to $\mathcal{U} \cap \mathcal{S}^\star$. For example, if we set $\lambda = 1$ in (6), then we have

$$\frac{z}{f_1(z)} = (1-z)^2 \left(1 + \frac{z}{2}\right) = 1 - \frac{3}{2} z + \frac{z^3}{2},$$

where $f_1 \in \mathcal{U}$ and for this function, we obtain

$$\sum_{n=1}^{\infty} |\gamma_n(f_1)|^2 = \sum_{n=1}^{\infty} \left(\frac{1}{n} + (-1)^n \frac{1}{2^{n+1}} \right)^2 = \frac{\pi^2}{6} + \frac{1}{12} \\ - \log \frac{3}{2} < \frac{\pi^2}{6}.$$

On the other hand, it is a simple exercise to verify that $f_1 \notin \mathcal{S}^\star$. The graph of this function is shown in Fig. 1d.

Let $\mathcal{G}(\alpha)$ denote the class of locally univalent normalized analytic functions f in the unit disk $|z| < 1$ satisfying the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2} \quad \text{for } |z| < 1,$$

and for some $0 < \alpha \leq 1$. Set $\mathcal{G}(1) =: \mathcal{G}$. It is known (see [13, Equation (16)]) that $\mathcal{G} \subset \mathcal{S}^\star$ and thus, functions in $\mathcal{G}(\alpha)$ are starlike. This class has been studied extensively in the recent past, see for instance [9] and the references therein. We now consider the estimate of the type (2) for the subclass $\mathcal{G}(\alpha)$.

Theorem 2 *Let $0 < \alpha \leq 1$ and $\mathcal{G}(\alpha)$ be defined as above. Then the logarithmic coefficients γ_n of $f \in \mathcal{G}(\alpha)$ satisfy the inequalities*

$$\sum_{n=1}^{\infty} n^2 |\gamma_n|^2 \leq \frac{\alpha}{4(\alpha+2)} \tag{8}$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\alpha^2}{4} \mathrm{Li}_2\left(\frac{1}{(1+\alpha)^2}\right). \tag{9}$$

Also we have

$$|\gamma_n| \leq \frac{\alpha}{2(\alpha+1)n} \quad \textit{for } n \geq 1. \tag{10}$$

Proof If $f \in \mathcal{G}(\alpha)$, then we have (see eg. [8, Theorem 1] and [13])

$$\frac{zf'(z)}{f(z)} - 1 \prec \frac{(1+\alpha)(1-z)}{1+\alpha-z} - 1 = -\alpha\left(\frac{z/(1+\alpha)}{1-(z/(1+\alpha))}\right), \quad z \in \mathbb{D}, \tag{11}$$

which, in terms of the logarithmic coefficients γ_n of f defined by (1), is equivalent to

$$\sum_{n=1}^{\infty}(-2n\gamma_n)z^n \prec \alpha \sum_{n=1}^{\infty} \frac{z^n}{(1+\alpha)^n}. \tag{12}$$

Again, by Rogosinski's result, we obtain that

$$\sum_{n=1}^{\infty} 4n^2|\gamma_n|^2 \leq \alpha^2 \sum_{n=1}^{\infty} \frac{1}{(1+\alpha)^{2n}} = \frac{\alpha}{\alpha+2}$$

which is (8).

Now, since the sequence $A_n = \frac{1}{(1+\alpha)^n}$ is convex decreasing, we obtain from (12) and [15, Theorem VII, p.64] that

$$|-2n\gamma_n| \leq A_1 = \frac{1}{1+\alpha},$$

which implies the desired inequality (10). As an alternate approach to prove this inequality, we may rewrite (11) as

$$\sum_{n=1}^{\infty}(2n\gamma_n)z^n = z\left(\log\left(\frac{f(z)}{z}\right)\right)' \prec \phi(z) = -\alpha\left(\frac{z/(1+\alpha)}{1-(z/(1+\alpha))}\right)$$

and, since $\phi(z)$ is convex in $\mathbb{D}$ with $\phi'(0) = -\alpha/(1+\alpha)$, it follows from Rogosinski's result (see also [4, Theorem 6.4(i), p. 195]) that $|2n\gamma_n| \leq \alpha/(1+\alpha)$. Again, this proves the inequality (10).

Finally, we prove the inequality (9). From the formula (12) and the result of Rogosinski (see also [12, Theorem 2.2] and [4, Theorem 6.2]), it follows that for $k \in \mathbb{N}$ the inequalities

$$\sum_{n=1}^{k} n^2 |\gamma_n|^2 \leq \frac{\alpha^2}{4} \sum_{n=1}^{k} \frac{1}{(1+\alpha)^{2n}}$$

are valid. Clearly, this implies the inequality (8) as well. On the other hand, consider these inequalities for $k = 1, \ldots, N$, and multiply the k-th inequality by the factor $\frac{1}{k^2} - \frac{1}{(k+1)^2}$, if $k = 1, \ldots, N-1$ and by $\frac{1}{N^2}$ for $k = N$. Then the summation of the multiplied inequalities yields

$$
\begin{aligned}
\sum_{k=1}^{N} |\gamma_k|^2 &\le \frac{\alpha^2}{4} \sum_{k=1}^{N} \frac{1}{k^2(1+\alpha)^{2k}} \\
&\le \frac{\alpha^2}{4} \sum_{k=1}^{\infty} \frac{1}{k^2(1+\alpha)^{2k}} \\
&= \frac{\alpha^2}{4} \mathrm{Li}_2\left(\frac{1}{(1+\alpha)^2}\right) \quad \text{for } N = 1, 2, \ldots,
\end{aligned}
$$

which proves the desired assertion (9) if we allow $N \to \infty$. ◻

Corollary 2 *The logarithmic coefficients γ_n of $f \in \mathcal{G} := \mathcal{G}(1)$ satisfy the inequalities*

$$
\sum_{n=1}^{\infty} n^2 |\gamma_n|^2 \le \frac{1}{12} \quad \textit{and} \quad \sum_{n=1}^{\infty} |\gamma_n|^2 \le \frac{1}{4} \mathrm{Li}_2\left(\frac{1}{4}\right).
$$

The results are the best possible as the function $f_0(z) = z - \frac{1}{2}z^2$ shows. Also we have $|\gamma_n| \le 1/(4n)$ for $n \ge 1$.

Remark 2 For the function $f_0(z) = z - \frac{1}{2}z^2$, we have that $\gamma_n(f_0) = -\frac{1}{n2^{n+1}}$ for $n = 1, 2, \ldots$ and thus, it is reasonable to expect that the inequality $|\gamma_n| \le \frac{1}{n2^{n+1}}$ is valid for the logarithmic coefficients γ_n of each $f \in \mathcal{G}$. But that is not the case as the function f_n defined by $f_n'(z) = (1 - z^n)^{\frac{1}{n}}$ shows. Indeed for this function we have

$$
1 + \frac{z f_n''(z)}{f_n'(z)} = \frac{1 - 2z^n}{1 - z^n}
$$

showing that $f_n \in \mathcal{G}$. Moreover,

$$
\log \frac{f_n(z)}{z} = -\frac{1}{n(n+1)} z^n + \cdots,
$$

which implies that $|\gamma_n(f_n)| = \frac{1}{2n(n+1)}$ for $n = 1, 2, \ldots$, and observe that $\frac{1}{2n(n+1)} > \frac{1}{n2^{n+1}}$ for $n = 2, 3, \ldots$. Thus, we conjecture that the logarithmic coefficients γ_n of each $f \in \mathcal{G}$ satisfy the inequality $|\gamma_n| \le \frac{1}{2n(n+1)}$ for $n = 1, 2, \ldots$. Clearly, Corollary 2 shows that the conjecture is true for $n = 1$.

Remark 3 Let $f \in \mathcal{C}(\alpha)$, where $0 \le \alpha < 1$. Then we have [18]

$$
\frac{z f'(z)}{f(z)} - 1 \prec G_\alpha(z) - 1 = \sum_{n=1}^{\infty} \delta_n z^n, \tag{13}
$$

where δ_n is real for each n,

$$G_\alpha(z) = \begin{cases} \dfrac{(2\alpha-1)z}{(1-z)[(1-z)^{1-2\alpha}-1]} & \text{if } \alpha \neq 1/2, \\ \dfrac{-z}{(1-z)\log(1-z)} & \text{if } \alpha = 1/2, \end{cases}$$

and

$$\beta(\alpha) = G_\alpha(-1) = \inf_{|z|<1} G_\alpha(z) = \begin{cases} \dfrac{1-2\alpha}{2[2^{1-2\alpha}-1]} & \text{if } 0 \leq \alpha \neq 1/2 < 1, \\ \dfrac{1}{2\log 2} & \text{if } \alpha = 1/2 \end{cases}$$

so that $f \in \mathcal{S}^\star(\beta(\alpha))$. Also, we have [16]

$$\frac{f(z)}{z} \prec \frac{K_\alpha(z)}{z} = \begin{cases} \dfrac{(1-z)^{2\alpha-1}-1}{1-2\alpha} & \text{if } 0 \leq \alpha \neq 1/2 < 1, \\ -\dfrac{\log(1-z)}{z} & \text{if } \alpha = 1/2, \end{cases}$$

and $K_\alpha(z)/z$ is univalent and convex (not normalized in the usual sense) in $\mathbb{D}$.

Now, the subordination relation (13), in terms of the logarithmic coefficients γ_n of f defined by (1), is equivalent to

$$2\sum_{n=1}^{\infty} n\gamma_n z^n \prec G_\alpha(z) - 1 = \sum_{n=1}^{\infty} \delta_n z^n, \quad z \in \mathbb{D},$$

and thus,

$$\sum_{n=1}^{k} n^2 |\gamma_n|^2 \leq \frac{1}{4}\sum_{n=1}^{k} \delta_n^2 \quad \text{for each } k \in \mathbb{N}. \tag{14}$$

Since f is starlike of order $\beta(\alpha)$, it follows that

$$\frac{zK'_\alpha(z)}{K_\alpha(z)} - 1 = G_\alpha(z) - 1 \prec 2(1-\beta(\alpha))\frac{z}{1-z}$$

and therefore, $|\delta_n| \leq 2(1-\beta(\alpha))$ for each $n \geq 1$. Again, the relation (14) by the previous approach gives

$$\sum_{k=1}^{N} |\gamma_k|^2 \leq \frac{1}{4}\sum_{k=1}^{N} \frac{\delta_k^2}{k^2} \leq (1-\beta(\alpha))^2 \sum_{k=1}^{N} \frac{1}{k^2}$$

for $N = 1, 2, \ldots,$ and hence, we have

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4}\sum_{n=1}^{\infty} \frac{\delta_n^2}{n^2} \leq (1-\beta(\alpha))^2 \sum_{n=1}^{\infty} \frac{1}{n^2} = (1-\beta(\alpha))^2 \frac{\pi^2}{6}$$

and equality holds in the first inequality for $K_\alpha(z)$. In particular, if f is convex then $\beta(0) = 1/2$ and hence, the last inequality reduces to

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{24}$$

which is sharp as the convex function $z/(1-z)$ shows.

3 Proof of Conjecture 1 for $n = 2, 3, 4$

Theorem 3 *Let* $f \in \mathcal{U}(\lambda)$ *for* $0 < \lambda \leq 1$ *and let* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. *Then*

$$|a_n| \leq \frac{1-\lambda^n}{1-\lambda} \text{ for } 0 < \lambda < 1 \text{ and } n = 2, 3, 4, \tag{15}$$

and $|a_n| \leq n$ *for* $\lambda = 1$ *and* $n \geq 2$. *The results are the best possible.*

Proof The case $\lambda = 1$ is well-known because $\mathcal{U} = \mathcal{U}(1) \subset \mathcal{S}$ and hence, by the de Branges theorem, we have $|a_n| \leq n$ for $f \in \mathcal{U}$ and $n \geq 2$. Here is an alternate proof without using the de Branges theorem. From the subordination result (3) with $\lambda = 1$, one has

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}$$

and thus, by Rogosinski's theorem [4, Theorem 6.4(ii), p. 195], it follows that $|a_n| \leq n$ for $n \geq 2$.

So, we may consider $f \in \mathcal{U}(\lambda)$ with $0 < \lambda < 1$. The result for $n = 2$, namely, $|a_2| \leq 1 + \lambda$ is proved in [10,17] and thus, it suffices to prove (15) for $n = 3, 4$ although our proof below is elegant and simple for the case $n = 2$ as well. To do this, we begin to recall from (3) that

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)(1-\lambda z)} = 1 + \sum_{n=1}^{\infty} \frac{1-\lambda^{n+1}}{1-\lambda} z^n$$

and thus

$$\frac{f(z)}{z} = \frac{1}{(1-z\omega(z))(1-\lambda z\omega(z))},$$

where ω is analytic in $\mathbb{D}$ and $|\omega(z)| \leq 1$ for $z \in \mathbb{D}$. In terms of series formulation, we have

$$\sum_{n=1}^{\infty} a_{n+1} z^n = \sum_{n=1}^{\infty} \frac{1-\lambda^{n+1}}{1-\lambda} \omega^n(z) z^n.$$

We now set $\omega(z) = c_1 + c_2 z + \cdots$ and rewrite the last relation as

$$\sum_{n=1}^{\infty}(1-\lambda)a_{n+1}z^n = \sum_{n=1}^{\infty}(1-\lambda^{n+1})(c_1 + c_2 z + \cdots)^n z^n. \tag{16}$$

By comparing the coefficients of z^n for $n = 1, 2, 3$ on both sides of (16), we obtain

$$\begin{cases} (1-\lambda)a_2 = (1-\lambda^2)c_1 \\ (1-\lambda)a_3 = (1-\lambda^2)c_2 + (1-\lambda^3)c_1^2 \\ (1-\lambda)a_4 = (1-\lambda^2)\left(c_3 + \mu c_1 c_2 + \nu c_1^3\right), \end{cases} \tag{17}$$

where

$$\mu = 2\frac{1-\lambda^3}{1-\lambda^2} \text{ and } \nu = \frac{1-\lambda^4}{1-\lambda^2}.$$

It is well-known that $|c_1| \leq 1$ and $|c_2| \leq 1 - |c_1|^2$. From the first relation in (17) and the fact that $|c_1| \leq 1$, we obtain

$$(1-\lambda)|a_2| = (1-\lambda^2)|c_1| \leq 1 - \lambda^2,$$

which gives a new proof for the inequality $|a_2| \leq 1 + \lambda$.

Next we present a proof of (15) for $n = 3$. Using the second relation in (17), $|c_1| \leq 1$ and the inequality $|c_2| \leq 1 - |c_1|^2$, we get

$$\begin{aligned} (1-\lambda)|a_3| &\leq (1-\lambda^2)|c_2| + (1-\lambda^3)|c_1|^2 \\ &\leq (1-\lambda^2)(1-|c_1|^2) + (1-\lambda^3)|c_1|^2 \\ &= 1 - \lambda^2 + (\lambda^2 - \lambda^3)|c_1|^2 \\ &\leq 1 - \lambda^3, \end{aligned}$$

which implies $|a_3| \leq 1 + \lambda + \lambda^2$.

Finally, we present a proof of (15) for $n = 4$. To do this, we recall the sharp upper bounds for the functionals $\left|c_3 + \mu c_1 c_2 + \nu c_1^3\right|$ when μ and ν are real. In [14], Prokhorov and Szynal proved among other results that

$$\left|c_3 + \mu c_1 c_2 + \nu c_1^3\right| \leq |\nu|$$

if $2 \leq |\mu| \leq 4$ and $\nu \geq (1/12)(\mu^2 + 8)$. From the third relation in (17), this condition is fulfilled and thus, we find that

$$(1-\lambda)|a_4| = (1-\lambda^2)\left|c_3 + \mu c_1 c_2 + \nu c_1^3\right| \leq (1-\lambda^2)\left(\frac{1-\lambda^4}{1-\lambda^2}\right) = 1 - \lambda^4$$

which proves the desired inequality $|a_4| \leq 1 + \lambda + \lambda^2 + \lambda^3$. $\square$

Acknowledgements The work of the first author was supported by MNZZS Grant, No. ON174017, Serbia. The second author is on leave from the IIT Madras.

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