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Partial group actions and partial Galois extensions

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Abstract Let α be a partial action of a group G on a ring S which has an enveloping action. Suppose that (S, α) is a partial Galois extension. We study partial Galois extensions inside (S, α). In particular, we derive some results on partial orbits and partial stabilizers and apply them to associate to each subgroup K of G certain partial Galois extensions inside (S, α) with partial actions of α restricted to K.

Keywords Partial action of a group · Partial Galois extension · Partial orbit · Partial stabilizer

Mathematics Subject Classification 13B05 · 16W22

1 Introduction

A partial action of a group, a generalization of a group action, has been studied and applied in various areas of mathematics since it firstly appeared in the theory of operator algebras as a powerful tool (see [8–10,14,16]). The formal definition of this concept was firstly given by Exel [9], and later Abadie in his PhD thesis (see also [1]) and independently Kellendonk and Lawson [11] showed that every partial action of a group on a set possesses an enveloping action.

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The definition of partial action of a group in a purely algebraic context was formulated in [6]. We give this definition in the next section and use it throughout the paper. Briefly speaking, a partial action α of a group G on a set S is a collection of subsets $\{S_g \mid g \in G\}$ together with bijections $\alpha_g : S_{g^{-1}} \to S_g$ satisfying certain conditions. In the case where S is an algebra, S_g are required to be ideals of S and α_g are isomorphisms of (not necessarily unital) algebras; furthermore, the definition of an enveloping action needs a few modifications (see [3] for more details and discussion). It was shown in [6] that a partial group action on a unital algebra S possesses an enveloping action if and only if every ideal S_g is unital. Throughout this paper, whenever S is a ring, we assume that it is unital with $1 \neq 0$ and each ideal S_g is generated by a central idempotent 1_g of S.

Dokuchaev et al. [7] introduced the notion of a partial Galois extension, and generalized the results on Galois theory of commutative rings by Chase, Harrison and Rosenberg [4] in the context of partial group actions, assuming the existence of an enveloping action. We have studied the structure of a partial Galois extension in [12, 13], and in this paper we will continue the investigation.

Suppose now that (S, α) is a partial Galois extension. The main goal of this paper is to study partial Galois extensions inside (S, α), especially those generated by central idempotents of S. In Sect. 4, we in particular show that for any subgroup K of G, there exists a nonzero central idempotent e of S such that Se with the partial action of α restricted to K, denoted α_K, is a partial Galois extension. More specifically, the existence of e is constructed via taking the Boolean sum of all element in $\mathscr{M}_K$, the set of minimal elements of the Boolean ring generated by $\{1_k \mid k \in K\}$. It turns out that the subset $\mathscr{M}_K$ is α_K-invariant; hence α induces a partial action of K on $\mathscr{M}_K$. In Sect. 6, we show, among other things, how to construct partial Galois extensions in (S, α) via partial orbits in $(\mathscr{M}_K, \alpha_K)$. Results derived in Sect. 6 require some properties of partial orbits and partial stabilizers, which will be presented in Sect. 5.

For each central idempotent e of S, let $N(e) = \{g \in G \mid e1_g = e\}$ and $G(e) = \{g \in G \mid e1_g \neq 0\}$. The discussion in Sects. 4 and 6 is closely related to these two subsets of G. We will present some properties of $N(\cdot)$ and $G(\cdot)$ in Sect. 3 so that they can easily be applied whenever needed. In the next section, we will recall the notions of a partial group action and a partial Galois extension and two examples in details which will be used very often later. Throughout this paper, G is assumed to be a finite group unless mentioned otherwise.

2 Preliminaries

We firstly recall the definition of a partial action of a group on a set. A partial action α of a group G on a set S is a collection of subsets $\{S_g \mid g \in G\}$ together with bijections $\alpha_g : S_{g^{-1}} \to S_g$ satisfying the following conditions:

(P1) $S_1 = S$ and α_1 is the identity map of S;
(P2) $\alpha_g(S_{g^{-1}} \cap S_h) = S_g \cap S_{gh}$ for all $g, h \in G$;
(P3) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$ for all $x \in S_{h^{-1}} \cap S_{(gh)^{-1}}$ and $g, h \in G$.

Notice that $\alpha_{g^{-1}} = \alpha_g^{-1}$ for each $g \in G$, and in particular, if $S_g = S$ for every $g \in G$, then α is a usual global action of G on S. In the case where S is a ring, S_g are required to be ideals of S and α_g are isomorphisms of (non-necessarily unital) algebras. As mentioned in the introduction, throughout the rest of the paper, whenever S is a ring, we assume that S is unital with $1 \neq 0$ and $S_g = S1_g$ for each $g \in G$, where 1_g is a central idempotent of S. We remark that the following identity holds

$$\alpha_g(1_h 1_{g^{-1}}) = 1_{gh} 1_g \quad \text{for all } g, h \in G.$$

As in [13], we use $(\mathscr{B}, \bullet)$ to denote the Boolean semigroup generated by $\{1_g \mid g \in G\}$ under the multiplication of S.

Next, we recall the definition of a partial Galois extension. Let (S, α) be a ring with a partial action of a group G. As defined in [7], the subring of the invariant elements of S under α is

$$S^\alpha = \{x \in S \mid \alpha_g(x 1_{g^{-1}}) = x 1_g \quad \text{for all } g \in G\}.$$

If there exist elements x_i and y_i in S, $i = 1, 2, \ldots, m$ for some positive integer m, such that

$$\sum_{i=1}^{m} x_i \alpha_g(y_i 1_{g^{-1}}) = \delta_{1,g} 1_S \quad \text{for each } g \in G,$$

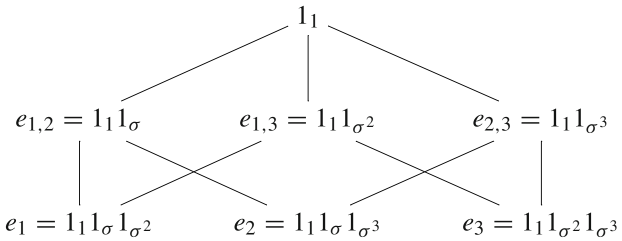
then (S, α) is called a partial Galois extension of S^α (we often simply say (S, α) is a partial Galois extension), and the set $V = \{x_i, y_i \mid i = 1, 2, \ldots, m\}$ is called an α-partial Galois system for S. This is obviously a generalization of Galois extension since under this definition, if $S_g = S$ for every $g \in G$, then S^α is the usual invariant subring S^G of S under the global action of G and V is a G-Galois system for S such that S is a Galois extension of S^G with Galois group G.

Examples of partial Galois extensions can be found in [7] and [15]. The authors in [12] also presented an easy way of constructing partial Galois extensions, which shows that any direct sum of a finite number of Galois extensions is a partial Galois extension. Example 6.1 in [7] and Example 4.2 in [15] will be used very often throughout this paper, so for readers' convenience, we provide details of these two examples below.

Example 2.1 (see [7, Example 6.1]) Let R be a commutative ring and $S = Re_1 \oplus Re_2 \oplus Re_3$, where $\{e_1, e_2, e_3\}$ is a set of nonzero orthogonal idempotents whose sum is one. Let G be a cyclic group of order 4 generated by σ. The partial action of G on S is defined as follows: taking $S_1 = S$, $S_\sigma = Re_1 \oplus Re_2$, $S_{\sigma^2} = Re_1 \oplus Re_3$ and $S_{\sigma^3} = Re_2 \oplus Re_3$, and defining $\alpha_1 = id_S$,

$$\begin{aligned}
&\alpha_\sigma : S_{\sigma^3} \to S_\sigma && \text{by } \alpha_\sigma(e_2) = e_1 \text{ and } \alpha_\sigma(e_3) = e_2,\\
&\alpha_{\sigma^2} : S_{\sigma^2} \to S_{\sigma^2} && \text{by } \alpha_{\sigma^2}(e_1) = e_3 \text{ and } \alpha_{\sigma^2}(e_3) = e_1,\\
&\alpha_{\sigma^3} : S_\sigma \to S_{\sigma^3} && \text{by } \alpha_{\sigma^3}(e_1) = e_2 \text{ and } \alpha_{\sigma^3}(e_2) = e_3.
\end{aligned}$$

It is easy to check that S is an α-partial Galois extension of R. The induced tree for the nonzero elements of the Boolean semigroup $(\mathscr{B}, \bullet)$ associated to the partial Galois extension (S, α) is given in [12, Example 1(1)] as follows:

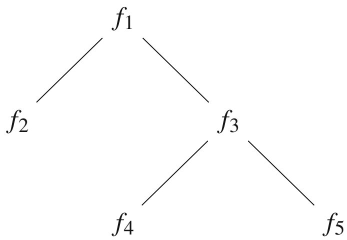


Example 2.2 (see [15, Example 4.2]) Let A be a ring and $S = \Pi_{i=1}^{6} A_i$, where A_i is A if $i = 1, 3, 6$ and the zero ring otherwise. Then $S = Ae_1 \oplus Ae_3 \oplus Ae_6$, where each e_i is the six-tuple whose jth-coordinate is 1_A if $j = i$ and 0_A otherwise. Let G be a cyclic group of order 6 generated by σ. The partial action α of G on S is defined in the following way:

$$1_1 = 1_S = e_1 + e_3 + e_6,\ 1_\sigma = 1_{\sigma^4} = e_1,\ 1_{\sigma^2} = e_3,\ 1_{\sigma^3} = e_3 + e_6,\ 1_{\sigma^5} = e_6,$$

$$\begin{aligned} &\alpha_1 = \mathrm{id}_S, && \alpha_\sigma\ : S_{\sigma^5} \to S_\sigma, ae_6 \mapsto ae_1,\\ &\alpha_{\sigma^2} : S_{\sigma^4} \to S_{\sigma^2}, ae_1 \mapsto ae_3, && \alpha_{\sigma^3}\ : S_{\sigma^3} \to S_{\sigma^3}, ae_3 + be_6 \mapsto be_3 + ae_6,\\ &\alpha_{\sigma^4} : S_{\sigma^2} \to S_{\sigma^4}, ae_3 \mapsto ae_1, && \alpha_{\sigma^5}\ : S_{\sigma^1} \to S_{\sigma^5}, ae_1 \mapsto ae_6. \end{aligned}$$

It is clear that S is an α-partial Galois extension of $S^\alpha = A(e_1 + e_3 + e_6) \cong A$ with α-partial Galois system $\{x_1 = y_1 = e_1, x_2 = y_2 = e_3, x_3 = y_3 = e_6\}$. The induced tree for the nonzero elements of the Boolean semigroup $(\mathscr{B}, \bullet)$ associated to (S, α) is given in [12, Example 1(3)] as follows:



where $f_1 = 1_1$, $f_2 = 1_1 1_\sigma = 1_1 1_{\sigma^4} = 1_1 1_\sigma 1_{\sigma^4}$, $f_3 = 1_1 1_{\sigma^3}$, $f_4 = 1_1 1_{\sigma^2} = 1_1 1_{\sigma^2} 1_{\sigma^3}$ and $f_5 = 1_1 1_{\sigma^5} = 1_1 1_{\sigma^3} 1_{\sigma^5}$ (Here we provide all possible expressions for each nonzero element of $(\mathscr{B}, \bullet)$).

3 Properties of $N(\cdot)$ and $G(\cdot)$

Throughout this section, let (S, α) be a ring with a partial action of a group G. Let $\mathscr{I}(S)$ denote the set of all central idempotents in S. The Boolean sum on $\mathscr{I}(S)$, denoted $\vee$, is defined by $e \vee e' = e + e' - ee'$, and the Boolean multiplication on $\mathscr{I}(S)$, denoted $\wedge$, is just the multiplication of S reduced to $\mathscr{I}(S)$. We say $e, e' \in \mathscr{I}(S)$ are orthogonal if $e \wedge e' = 0$. Let $\preceq$ denote the canonical partial order on $\mathscr{I}(S)$ defined as follows:

$e \preceq e'$ if and only if $e \wedge e' = e$. Clearly, minimal elements of $\mathscr{I}(S)^\times = \mathscr{I}(S)\backslash\{0\}$ are mutually orthogonal.

For any $e \in \mathscr{I}(S)$, define $G(e) = \{g \in G \mid e1_g \neq 0\}$ and $N(e) = \{g \in G \mid e1_g = e\}$. We shall firstly present some properties of $N(\cdot)$ and $G(\cdot)$, some of which will be used in the next section. At the end, we shall apply them to show the main result of this section: Let $\mathscr{B}$ denote the Boolean subring of $\mathscr{I}(S)$ generated by $\{1_g \mid g \in G\}$. Then there do not exist two distinct minimal elements e, e' of $\mathscr{B}^\times = \mathscr{B}\backslash\{0\}$ such that $G(e)$, $G(e')$ and $G(e \vee e')$ are all subgroups of G. We begin with the reaction of $G(\cdot)$ and $N(\cdot)$ under the operations of $\vee$ and $\wedge$ on $\mathscr{I}(S)$.

Lemma 3.1 *For any $e, e' \in \mathscr{I}(S)$, $G(e \wedge e') \subseteq G(e) \cap G(e') \subseteq G(e) \cup G(e') = G(e \vee e')$.*

Proof Suppose $g \in G(e \wedge e')$; that is, $ee'1_g \neq 0$. Then in particular $e1_g \neq 0 \neq e'1_g$ and so $g \in G(e) \cap G(e')$. For any $g \in G(e \vee e')$, we have $(e \vee e')1_g \neq 0$; that is, $(e + e' - ee')1_g \neq 0$, which, if $g \notin G(e)$, becomes $e'1_g \neq 0$; that is, $g \in G(e')$. Conversely, let $g \in G(e)$. Since $e(e \vee e') = e$, it follows that $g \in G(e \vee e')$. Similarly, $G(e') \subseteq G(e \vee e')$. We conclude that $G(e \vee e') = G(e) \cup G(e')$. □

Example 3.2 In Example 2.1, we see that $e_{1,2} = e_1 \vee e_2$. Clearly, $G(e_1) = \{1, \sigma, \sigma^2\}$, $G(e_2) = \{1, \sigma, \sigma^3\}$ and $G(e_{1,2}) = G$, and so $G(e_1 \vee e_2) = G(e_1) \cup G(e_2)$. On the other hand, it could be the case where $G(e) \cap G(e') \nsubseteq G(e \wedge e')$. Consider Example 2.1 again. We have $\sigma \in G(e_{1,3}) \cap G(e_{2,3})$, but $\sigma \notin G(e_{1,3} \wedge e_{2,3}) = G(e_3)$.

Lemma 3.3 *For any $e, e' \in \mathscr{I}(S)$, $N(e \vee e') = N(e) \cap N(e') \subseteq N(e) \cup N(e') \subseteq N(e \wedge e')$.*

Proof For any $g \in N(e) \cap N(e')$, $e1_g = e$ and $e'1_g = e'$. Hence $(e + e' - ee')1_g = e + e' - ee'$; that is, $g \in N(e \vee e')$. Conversely, take any $g \in N(e \vee e')$. Since $e(e \vee e') = e$, $e1_g = e(e \vee e')1_g = e(e \vee e') = e$. Similarly, we can get $e'1_g = e'$. Hence $g \in N(e) \cap N(e')$. We have shown that $N(e \vee e') = N(e) \cap N(e')$. Now, if $g \in N(e)$, then $(e \wedge e')1_g = e1_g e' = e \wedge e'$, and similarly, if $g \in N(e')$, then $g \in N(e \wedge e')$. Hence $N(e) \cup N(e') \subseteq N(e \wedge e')$. □

Example 3.4 It is possible that $N(e \wedge e')$ is not equal to $N(e) \cup N(e')$. Consider Example 2.2. We have $N(f_2) = \{1, \sigma, \sigma^4\}$, $N(f_3) = \{1, \sigma^3\}$, but $N(f_2 \wedge f_3) = N(0) = G \neq N(f_2) \cup N(f_3)$. Also, if taking $e = f_2 \vee f_4$ and $e' = f_3$, then $e \wedge e' = f_4$, $N(e) = \{1\}$ and $N(e') = \{1, \sigma^3\}$; hence $N(e) \cup N(e') \neq N(e \wedge e') = N(f_4) = \{1, \sigma^2, \sigma^3\}$.

With the former one of the preceding example in mind, one might wonder if it is true that $N(e \wedge e') \subseteq N(e) \cup N(e')$ whenever $e, e' \in (\mathscr{B}, \bullet)$ such that $e \wedge e' \neq 0$. This is, however, too naive from the following example.

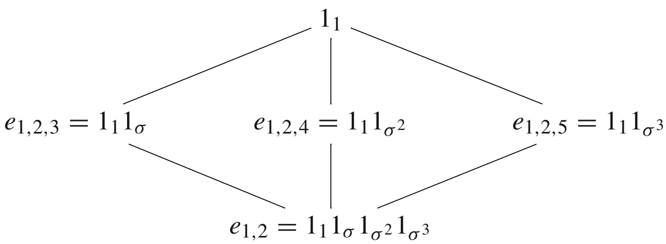
Example 3.5 Let R be a commutative ring and $S = \oplus_{i=1}^{5} Re_i$, where $\{e_i \mid 1 \leq i \leq 5\}$ is a set of nonzero orthogonal idempotents whose sum is one. Let G be a cyclic group of order 4 generated by σ. We define a partial action of G on S as follows: taking

$S_1 = S$, $S_\sigma = \oplus_{i\in\{1,2,3\}} Re_i$, $S_{\sigma^2} = \oplus_{i\in\{1,2,4\}} Re_i$ and $S_{\sigma^3} = \oplus_{i\in\{1,2,5\}} Re_i$, and defining $\alpha_1 = id_S$,

$$\begin{aligned}
\alpha_\sigma &: S_{\sigma^3} \to S_\sigma, & ae_1 + be_2 + ce_5 &\mapsto be_1 + ae_2 + ce_3 \\
\alpha_{\sigma^2} &: S_{\sigma^2} \to S_{\sigma^2}, & ae_1 + be_2 + ce_4 &\mapsto ae_1 + be_2 + ce_4 \\
\alpha_{\sigma^3} &: S_\sigma \to S_{\sigma^3}, & ae_1 + be_2 + ce_3 &\mapsto be_1 + ae_2 + ce_5.
\end{aligned}$$

One should check that conditions (P1)–(P3) listed in the definition of partial group action are satisfied.

The induced tree for the associated Boolean semigroup $(\mathscr{B}, \bullet)$ is given below:



Here $e_{1,2,4} \wedge e_{1,2,5} = e_{1,2} \neq 0$, and $N(e_{1,2,4} \wedge e_{1,2,5}) = G \nsubseteq N(e_{1,2,4}) \cup N(e_{1,2,5}) = \{1, \sigma^2, \sigma^3\}$.

Proposition 3.6 *If $e_1, e_2, \ldots, e_k$ are elements of $\mathscr{I}(S)$, then $G(\vee_{i=1}^k e_i) = \cup_{i=1}^k G(e_i)$ and $N(\vee_{i=1}^k e_i) = \cap_{i=1}^k N(e_i)$.*

Proof The statement holds by Lemmas 3.1, 3.3 and induction. □

The following result follows easily from a well-known result in group theory.

Proposition 3.7 *For any $e, e' \in \mathscr{I}(S)$, if $G(e)$, $G(e')$ and $G(e \vee e')$ are all subgroups of G, then either $G(e) \subseteq G(e')$ or $G(e') \subseteq G(e)$.*

Proof By Proposition 3.6, $G(e \vee e') = G(e) \cup G(e')$. Thus by hypothesis, $G(e)$ and $G(e')$ are subgroups of G such that their union is also a subgroup, so they are comparable. □

Corollary 3.8 *If $e_1, e_2, \ldots, e_k$ are elements of $\mathscr{I}(S)$ such that each $G(e_i \vee e_j)$, where $i, j \in \{1, 2, \ldots, k\}$, is a subgroup of G, then $G(e_1), G(e_2), \ldots, G(e_k)$ form a chain of groups under the inclusion of sets.*

Proof Notice firstly that for each $i = 1, 2, \ldots, k$, $G(e_i) = G(e_i \vee e_i)$ is a subgroup of G. Now assume to the contrary that these subgroups $G(e_1), G(e_2), \ldots, G(e_k)$ do not form a chain. Then there exist some $i, j \in \{1, 2, \ldots, k\}$ such that $G(e_i)$ and $G(e_j)$ are not comparable. But by the assumption, $G(e_i)$, $G(e_j)$ and $G(e_i \vee e_j)$ are all groups, so either $G(e_i) \subseteq G(e_j)$ or $G(e_j) \subseteq G(e_i)$ by Proposition 3.7, a contradiction. □

We next show that $G(\cdot)$ respects the canonical partial order $\preceq$ on $\mathscr{I}(S)$ and so does $N(\cdot)$ but in the reversing way.

Lemma 3.9 *For any $e, e' \in \mathscr{I}(S)$, if $e \preceq e'$, then $G(e) \subseteq G(e')$ and $N(e') \subseteq N(e)$.*

Proof Suppose e, e' are elements of $\mathscr{I}(S)$ such that $e \preceq e'$. For any $g \in G(e)$, $ee'1_g = e1_g \neq 0$, so $e'1_g \neq 0$. If $g \in N(e')$, then $e1_g = ee'1_g = ee' = e$; hence $g \in N(e)$. □

The converse of the preceding lemma for $N(\cdot)$ is true if $e, e' \in (\mathscr{B}, \bullet)^\times = (\mathscr{B}, \bullet)\backslash\{0\}$. Recall that for any $e \in (\mathscr{B}, \bullet)^\times$, as observed in [13, Lemma 1], $e = \Pi_{g \in N(e)} 1_g$, and furthermore as observed in [13, Lemma 9], e is minimal if and only if $G(e) = N(e)$.

Lemma 3.10 *If $e, e' \in (\mathscr{B}, \bullet)^\times$ such that $N(e') \subseteq N(e)$, then $e \preceq e'$.*

Proof Suppose that $e, e' \in (\mathscr{B}, \bullet)^\times$. Then as mentioned above, $e = \Pi_{g \in N(e)} 1_g$ and $e' = \Pi_{g \in N(e')} 1_g$. Thus if $N(e') \subseteq N(e)$, then $e \wedge e' = e$; that is, $e \preceq e'$. □

Remark 3.11 The corresponding statement does not hold for $G(\cdot)$; that is, even when $e, e' \in (\mathscr{B}, \bullet)^\times$ such that $G(e) \subseteq G(e')$, it is not necessarily true that $e \preceq e'$. In Example 2.1, $G(e_{1,2}) = G(e_{1,3}) = G$, but $e_{1,2} \npreceq e_{1,3} \npreceq e_{1,2}$. The corresponding statement also does not hold if $(\mathscr{B}, \bullet)^\times$ is replaced by $\mathscr{B}^\times$. In Example 2.2, $N(f_2 \vee f_4) = \{1\} \subseteq N(f_3)$ but $f_3 \npreceq f_2 \vee f_4$.

To show the main result of this section, we need two more observations.

Proposition 3.12 *The minimal elements of $\mathscr{B}^\times$ are exactly the same as those of $(\mathscr{B}, \bullet)^\times$.*

Proof Let f be a minimal element of $(\mathscr{B}, \bullet)^\times$. To show that f is also minimal in $\mathscr{B}^\times$, assume that e is an element of $\mathscr{B}^\times$ such that $e \preceq f$ but $e \neq f$. By definition of $\mathscr{B}$, $e = \vee_{i=1}^l e_i$ for some $e_i \in (\mathscr{B}, \bullet)$. Since f is minimal in $(\mathscr{B}, \bullet)^\times$, $f \wedge e_i$ is either 0 or f for each $i = 1, 2, \ldots, l$. Now from $e \preceq f$, we then have $e = (\vee_{i=1}^l e_i) \wedge f = \vee_{i=1}^l (e_i \wedge f)$ equals f or 0, a contradiction. This shows that f is minimal in $\mathscr{B}^\times$. Conversely, let f be a minimal element of $\mathscr{B}^\times$. We claim that actually $f \in (\mathscr{B}, \bullet)^\times$. This follows easily from the fact that for any $e, e' \in \mathscr{B}$, $e, e' \preceq e \vee e'$. We then conclude that f is a minimal element of $(\mathscr{B}, \bullet)^\times$. □

An immediate application of the preceding proposition is the following generalization of [13, Lemma 9].

Proposition 3.13 *For any $e \in \mathscr{B}^\times$, e is minimal in $\mathscr{B}^\times$ if and only if $G(e) = N(e)$.*

Proof Since e is nonzero, clearly $N(e) \subseteq G(e)$. If e is minimal in $\mathscr{B}^\times$, then for each $g \in G$, $e1_g$ equals either 0 or e; thus $G(e) \subseteq N(e)$. Suppose now that $e \in \mathscr{B}^\times$ such that $G(e) = N(e)$. We claim that e actually belongs to $(\mathscr{B}, \bullet)^\times$. Let $e = \vee_{i=1}^l e_i$, where each $e_i \in (\mathscr{B}, \bullet)^\times$. Then by Proposition 3.6,

$$\cup_{i=1}^l N(e_i) \subseteq \cup_{i=1}^l G(e_i) = G(\vee_{i=1}^l e_i) = G(e) = N(e) = N(\vee_{i=1}^l e_i) = \cap_{i=1}^l N(e_i),$$

which forces that for any $i, j \in \{1, 2, \ldots, l\}$, $N(e_i) = N(e_j)$ and so $e_i = e_j$ by Lemma 3.10. Thus $e = e_1 \in (\mathscr{B}, \bullet)^\times$. Therefore, e is minimal in $(\mathscr{B}, \bullet)^\times$ by [13, Lemma 9], and hence e is minimal in $\mathscr{B}^\times$ by Proposition 3.12. □

One can check easily that similarly for any $e \in \mathscr{I}(S)^\times$, if e is minimal in $\mathscr{I}(S)^\times$, then $G(e) = N(e)$; the converse, however, is false. In Example 3.5, $N(e_{1,2}) = G(e_{1,2}) = G$, but $e_{1,2}$ is not minimal in $\mathscr{I}(S)^\times$.

We are now ready to show the main result of this section.

Theorem 3.14 *There do not exist two distinct minimal elements e, e' of $\mathscr{B}^\times$ such that $G(e)$, $G(e')$ and $G(e \vee e')$ are all subgroups of G.*

Proof Suppose e and e' are two distinct minimal elements of $\mathscr{B}^\times$ such that $G(e)$, $G(e')$ and $G(e \vee e')$ are all subgroups of G. Then in particular by Proposition 3.7, $G(e) \subseteq G(e')$ or $G(e') \subseteq G(e)$. Since e, e' are minimal elements of $\mathscr{B}^\times$, they are also minimal in $(\mathscr{B}, \bullet)^\times$ by Proposition 3.12, and $G(e) = N(e)$ and $G(e') = N(e')$ by Proposition 3.13. But then by Lemma 3.10, we conclude that $e' \preceq e$ or $e \preceq e'$, either of which implies that $e = e'$, a contradiction. ◻

Example 3.15 In [13, Example 19], e_1 and e_2 are the only two minimal elements of $\mathscr{B}^\times$. We see that $G(e_1) = H \times \{1\}$ and $G(e_2) = \{1\} \times K$ are both subgroups of G, but $G(e_1 \vee e_2) = G(1_1) = H \times \{1\} \cup \{1\} \times K$ is not a subgroup of G.

4 Partial Galois extensions in (S, α)

Throughout this section, let (S, α) be a ring with a partial action of a group G and K be a subgroup of G. We will sometimes assume in addition that (S, α) is a partial Galois extension. Suppose that A is a nonzero ring contained in S with identity denoted 1_A. Recall in [13] we say that α induces a partial action of K on A if $1_A 1_k \in A$ for each $k \in K$ and $\{A1_k \mid k \in K\}$ forms the collection of associated ideals of A for the partial action; that is, for each $k \in K$, α_k restricted to $A1_{k^{-1}}$ is an isomorphism of rings onto $A1_k$. We shall discuss when (A, α_K) forms a partial Galois extension with a special attention to the case where A is an ideal of S generated by a central idempotent. At the end of this section, we will in particular present a way of associating to every subgroup K of G a partial Galois extension (Se, α_K), where $e \in \mathscr{I}(S)^\times$. We recall that for any subset H of G, $S^{\alpha_H} = \{x \in S \mid \alpha_h(x1_{h^{-1}}) = x1_h \text{ for all } h \in H\}$.

Lemma 4.1 *If α induces a partial action of K on A, then $1_A \in S^{\alpha_K}$.*

Proof By definition, α_k, for each $k \in K$, restricted to $A1_{k^{-1}}$ is an isomorphism of rings onto $A1_k$. In particular, $\alpha_k(1_A 1_{k^{-1}}) = 1_A 1_k$ for all $k \in K$. Thus $1_A \in S^{\alpha_K}$. ◻

Theorem 4.2 *If α induces a partial action of K on A, then α also induces a partial action of K on the maximum ring extension of A in S, namely $1_A S 1_A$. Furthermore, if (A, α_K) is a partial Galois extension, then so is $(1_A S 1_A, \alpha_K)$.*

Proof Clearly, $1_A S 1_A$ is the maximum ring extension of A in S. Since α induces a partial action of K on A, $1_A 1_k \in A \subseteq 1_A S 1_A$ for each $k \in K$ and $1_A \in S^{\alpha_K}$ by Lemma 4.1; hence for each $k \in K$, $\alpha_k(1_A S 1_A 1_{k^{-1}}) = \alpha_k(1_A 1_{k^{-1}})\alpha_k(S1_{k^{-1}})\alpha_k(1_A 1_{k^{-1}}) = 1_A S 1_A 1_k$. Thus α induces a partial action of K on $1_A S 1_A$. Furthermore, it is clear that every α_K-partial Galois system for A is also an α_K-partial Galois system for $1_A S 1_A$. ◻

Applying Theorem 4.2, the set of all partial Galois extensions in (S, α) can basically be determined as follows in four steps. Step 1. For each subgroup K of G, determine the set I_K of nonzero idempotents e such that $e \in S^{\alpha_K}$. Step 2. Determine the subset $J_K = \{e \in I_K \mid (eSe, \alpha_K)$ is a partial Galois extension$\}$. Step 3. For each $e \in J_K$, determine the set $\mathscr{S}_K^e = \{A$ is a subring of $eSe \mid (A, \alpha_K)$ is a partial Galois extension$\}$. Then $\mathscr{S}_K := \cup_{e \in J_K} \mathscr{S}_K^e$ is the set of all partial Galois extensions in (S, α) with a partial action of K. Step 4. The union of $\mathscr{S}_K$ for all subgroups K is then the set of all partial Galois extensions contained in (S, α).

We next study the case where A is an ideal of S generated by an element of $\mathscr{I}(S)$. The converse of Lemma 4.1 holds here.

Lemma 4.3 *For any $e \in \mathscr{I}(S)$, $e \in S^{\alpha_K}$ if and only if α induces a partial action of K on Se.*

Proof Obviously, $e1_k \in Se$ for each $k \in K$. Now, if $e \in S^{\alpha_K}$, then for each $k \in K$, $\alpha_k(Se1_{k^{-1}}) = \alpha_k(S1_{k^{-1}})\alpha_k(e1_{k^{-1}}) = Se1_k$. Hence α induces a partial action of K on Se. Thus we are done by Lemma 4.1 □

Theorem 4.4 *Let K be a subgroup of G and e an element of $\mathscr{I}(S)^\times$. Suppose that (S, α) is a partial Galois extension. Then (Se, α_K) is a partial Galois extension in (S, α) if and only if $e \in S^{\alpha_K}$. Furthermore, Se is a Galois extension in (S, α) with Galois group K if and only if $e \in S^{\alpha_K}$ and $K \subseteq N(e)$.*

Proof By Lemma 4.3, it suffices to show that under the assumption that (S, α) is a partial Galois extension, if $e \in S^{\alpha_K}$, then Se with the induced partial action α_K forms a partial Galois extension. Indeed, if $\{x_i, y_i\}_{i=1}^l$ is an α-partial Galois system for S, then $\{x_i e, y_i e\}_{i=1}^l$ is an α_K-partial Galois system for Se: for each $k \in K$,

$$\begin{aligned}\sum_{i=1}^{l} x_i e\alpha_k(y_i e1_{k^{-1}}) &= e\sum_{i=1}^{l} x_i\alpha_k(y_i 1_{k^{-1}})\alpha_k(e1_{k^{-1}}) \\ &= e1_k\sum_{i=1}^{l} x_i\alpha_k(y_i 1_{k^{-1}}) = e1_k\delta_{1,k}1_S = \delta_{1,k}e.\end{aligned}$$

Finally, the partial Galois extension (Se, α_K) is Galois if and only if $e1_k = e$ for all $k \in K$, or equivalently, $K \subseteq N(e)$. □

Let $\mathscr{C}_K$ ($\mathscr{D}_K$, resp.) denote the set of elements e in $\mathscr{I}(S)$ such that (Se, α_K) is a partial Galois extension (Se is a Galois extension with Galois group K, resp.). We now show that both $\mathscr{C}_K$ and $\mathscr{D}_K$ are closed under the Boolean sum $\vee$ and the Boolean multiplication $\wedge$ on $\mathscr{I}(S)$.

Proposition 4.5 *Suppose that (S, α) is a partial Galois extension. If e and e' are elements of $\mathscr{C}_K$ ($\mathscr{D}_K$, resp.), then $e \vee e'$ and $e \wedge e'$ are also elements of $\mathscr{C}_K$ ($\mathscr{D}_K$, resp.).*

Proof By Theorem 4.4, we need only show that $e \vee e', e \wedge e' \in S^{\alpha_K}$ if $e, e' \in S^{\alpha_K}$, and $K \subseteq N(e \vee e') \cap N(e \wedge e')$ if $K \subseteq N(e) \cap N(e')$. The latter one is obvious since $N(e \vee e') \cap N(e \wedge e') = N(e) \cap N(e')$ by Lemma 3.3. Now for each $k \in K$, if $\alpha_k(e1_{k^{-1}}) = e1_k$ and $\alpha_k(e'1_{k^{-1}}) = e'1_k$, then we have $\alpha_k((e \vee e')1_{k^{-1}}) = \alpha_k(e1_{k^{-1}}) \vee \alpha_k(e'1_{k^{-1}}) = e1_k \vee e'1_k = (e \vee e')1_k$, and $\alpha_k((e \wedge e')1_{k^{-1}}) = \alpha_k(e1_{k^{-1}}) \wedge \alpha_k(e'1_{k^{-1}}) = e1_k \wedge e'1_k = (e \wedge e')1_k$. □

Notice that the latter part of Theorem 4.4 generalizes [13, Theorem 2], where the ideal Se under consideration is generated by an element in the Boolean semigroup $(\mathscr{B}, \bullet)^\times$. It was further proved in [13, Proposition 5] that $N(e)$ is a subgroup of G if and only if $e \in S^{\alpha_{N(e)}}$. We do not know that whether this result is true if e is any element in $\mathscr{B}^\times$ or even in $\mathscr{I}(S)^\times$. It turns out that the argument used to prove the if-direction of [13, Proposition 5] can be applied here to prove the following result, and so we skip the proof.

Proposition 4.6 *If e is an element of $\mathscr{I}(S)^\times$ such that $e \in S^{\alpha_{N(e)}}$, then $N(e)$ is a subgroup of G.*

Remark 4.7 We point out that it is not necessarily true that $e \in S^{\alpha_{N(e)}}$ whenever e is an element of $\mathscr{I}(S) \backslash \mathscr{B}$ such that $N(e)$ is a subgroup of G. In Example 3.5, we see that $e_2 \in \mathscr{I}(S) \backslash \mathscr{B}$ with $N(e_2) = G$, but $\alpha_\sigma(e_2 1_{\sigma^3}) = e_1 \neq e_2 1_\sigma$. It remains open whether $e \in S^{\alpha_{N(e)}}$ if e is any element in $\mathscr{B} \backslash (\mathscr{B}, \bullet)$ such that $N(e)$ is a subgroup of G.

Corollary 4.8 *Suppose that (S, α) is a partial Galois extension. If e is an element of $\mathscr{I}(S)^\times$ such that $e \in S^{\alpha_{N(e)}}$, then Se is a Galois extension in (S, α) with Galois group $N(e)$. Furthermore, if e is minimal in $\mathscr{I}(S)^\times$ or in $\mathscr{B}^\times$, then $(Se)^{N(e)} = S^\alpha e$.*

Proof The first result follows immediately from Proposition 4.6 and Theorem 4.4. Since $e \in S^{\alpha_{N(e)}}$, it is clear that $S^\alpha e \subseteq (Se)^{N(e)}$. Suppose in addition that e is minimal in $\mathscr{I}(S)^\times$ or in $\mathscr{B}^\times$. Then $N(e) = G(e)$ by Proposition 3.13 and the statement right after its proof. Now if $x \in (Se)^{N(e)}$, then for each $g \in N(e)$, $\alpha_g(x1_{g^{-1}}) = x1_g$; as for any $g \notin N(e)$, we have $\alpha_g(x1_{g^{-1}}) = \alpha_g(xe1_{g^{-1}}) = \alpha_g(0) = xe1_g = x1_g$. Therefore, $x = xe \in S^\alpha e$. □

Example 4.9 Let S be a Galois extension of S^G with Galois group G. Let H be a subgroup of G. We shall define a partial action α of G on S such that S is an α-partial Galois extension of S^H. For $g \in G$, let

$$1_g = \begin{cases} 1_S & \text{if } g \in H \\ 0 & \text{otherwise} \end{cases}$$

and define $\alpha_g \colon S_{g^{-1}} \to S_g$ by $x1_{g^{-1}} \mapsto g(x)1_g$. Obviously, each α_g is a ring isomorphism. Actually, α is a partial action of G on S. Clearly, $S^\alpha = S^H$. Now, one can check easily that every Galois system for S is also an α-partial Galois system for S. Therefore, S is an α-partial Galois extension of S^H.

For this partial Galois extension (S, α), $1_S = 1_1 = \Pi_{g \in H} 1_g$ is the only nonzero element in $\mathscr{B}$, which is of course invariant under $\alpha_{N(1_1)}$, where $N(1_1) = H$ is a subgroup of G as promised by Proposition 4.6. Furthermore, by Corollary 4.8, we obtain that $S = S1_1$ is a Galois extension of $S^\alpha 1_1 = S^H$ with Galois group H. This conclusion should not be surprising according to classical Galois theory.

Example 4.10 Here we present another example to interpret Proposition 4.6 and Corollary 4.8. This example is constructed in a way similar to Example 2.1. Let R be a commutative ring and $S = \oplus_{i=1}^{5} Re_i$, where $\{e_i \mid 1 \leq i \leq 5\}$ is a set of nonzero orthogonal idempotents whose sum is one. Let G be a cyclic group of order 6 generated by σ. We define a partial action of G on S as follows: taking $S_1 = S$, $S_\sigma = \oplus_{i\in\{1,2,3,4\}} Re_i$, $S_{\sigma^2} = \oplus_{i\in\{1,2,3,5\}} Re_i$, $S_{\sigma^3} = \oplus_{i\in\{1,2,4,5\}} Re_i$, $S_{\sigma^4} = \oplus_{i\in\{1,3,4,5\}} Re_i$ and $S_{\sigma^5} = \oplus_{i\in\{2,3,4,5\}} Re_i$, and defining $\alpha_1 = id_S$,

$$\begin{aligned}
&\alpha_\sigma : S_{\sigma^5} \to S_\sigma && \text{by } e_2 \mapsto e_1, e_3 \mapsto e_2, e_4 \mapsto e_3 \text{ and } e_5 \mapsto e_4 \\
&\alpha_{\sigma^2} : S_{\sigma^4} \to S_{\sigma^2} && \text{by } e_1 \mapsto e_5, e_3 \mapsto e_1, e_4 \mapsto e_2 \text{ and } e_5 \mapsto e_3, \\
&\alpha_{\sigma^3} : S_{\sigma^3} \to S_{\sigma^3} && \text{by } e_1 \mapsto e_4, e_2 \mapsto e_5, e_4 \mapsto e_1 \text{ and } e_5 \mapsto e_2, \\
&\alpha_{\sigma^4} : S_{\sigma^2} \to S_{\sigma^4} && \text{by } \alpha_{\sigma^4} = \alpha_{\sigma^2}^{-1}, \\
&\alpha_{\sigma^5} : S_\sigma \to S_{\sigma^5} && \text{by } \alpha_{\sigma^5} = \alpha_\sigma^{-1}.
\end{aligned}$$

One can check easily that S is an α-partial Galois extension of R with α-partial Galois system $\{x_i = y_i = e_i \mid 1 \leq i \leq 5\}$. Consider $e_{1,3} := 1_1 1_\sigma 1_{\sigma^2} 1_{\sigma^4}$, $e_{1,5} := 1_1 1_{\sigma^2} 1_{\sigma^3} 1_{\sigma^4}$ and $e_{3,5} := 1_1 1_{\sigma^2} 1_{\sigma^4} 1_{\sigma^5}$, and take $e = e_{1,3} \vee e_{1,5}$. Then $N(e) = \{1, \sigma^2, \sigma^4\}$ is a subgroup of G and $e \in S^{\alpha_{N(e)}}$: $\alpha_{\sigma^2}(e1_{\sigma^4}) = \alpha_{\sigma^2}(e_{1,3}1_{\sigma^4}) \vee \alpha_{\sigma^2}(e_{1,5}1_{\sigma^4}) = e_{1,5} \vee e_{3,5} = e_{1,3} \vee e_{1,5} = e1_{\sigma^2}$. By Corollary 4.8, Se is a Galois extension in (S, α) with Galois group $N(e)$.

We next show that if (S, α) is a partial Galois extension, then for any subgroup K of G, there exists an element e in $\mathscr{I}(S)^\times$ such that (Se, α_K) is a partial Galois extension in (S, α).

Let $\mathscr{B}_K$ be the Boolean ring generated by $\{1_k \mid k \in K\}$. Clearly, $\mathscr{B}_K \subseteq \mathscr{B}$. Let $\mathscr{M}$ and $\mathscr{M}_K$ denote the set of all minimal elements in $\mathscr{B}^\times$ and $\mathscr{B}_K^\times = \mathscr{B}_K \backslash \{0\}$, respectively. It is not necessary that $\mathscr{M}_K \subseteq \mathscr{M}$. In Example 2.2, if letting $K = \{1, \sigma^3\}$, then $\mathscr{M}_K = \{f_3\} \nsubseteq \mathscr{M}$.

Lemma 4.11 *For each $k \in K$, $\alpha_k(\mathscr{B}_K 1_{k^{-1}}) \subseteq \mathscr{B}_K$. In particular, $\alpha_k(\mathscr{B}_K 1_{k^{-1}}) = \mathscr{B}_K 1_k$ for each $k \in K$.*

Proof By definition of $\mathscr{B}_K$, every element of $\mathscr{B}_K$ is a Boolean sum of some Boolean multiplications of elements of the form 1_h, where $h \in K$. Thus it suffices to show that for each pair of $k, h \in K$, $\alpha_k(1_h 1_{k^{-1}}) \in \mathscr{B}_K$, since $\alpha_k((e \wedge e')1_{k^{-1}}) = \alpha_k(e1_{k^{-1}}) \wedge \alpha_k(e'1_{k^{-1}})$ and $\alpha_k((e \vee e')1_{k^{-1}}) = \alpha_k(e1_{k^{-1}}) \vee \alpha_k(e'1_{k^{-1}})$ for any $e, e' \in \mathscr{B}_K$. But $\alpha_k(1_h 1_{k^{-1}}) = 1_{kh} 1_k$ and K is a subgroup. Hence we are done. □

Proposition 4.12 *If f is an element of $\mathscr{M}_K$, then so is $\alpha_k(f1_{k^{-1}})$ for each $k \in K$ such that $f1_{k^{-1}} \neq 0$.*

Proof Let $f \in \mathscr{M}_K$ and $k \in K$ such that $f1_{k^{-1}} \neq 0$; equivalently, $f1_{k^{-1}} = f$. Then $f' := \alpha_k(f1_{k^{-1}})$ is nonzero and belongs to $\mathscr{B}_K$ by Lemma 4.11. Assume e is an element of $\mathscr{B}_K^\times$ smaller than but not equal to f'; that is, $ef' = e \neq f'$. In particular, $e1_k = ef'1_k = ef' = e$. Let $e' = \alpha_{k^{-1}}(e1_k)$. Then e' is nonzero and belongs to $\mathscr{B}_K$ by Lemma 4.11 again. Now, $e' \neq \alpha_{k^{-1}}(f'1_k) = f1_{k^{-1}} = f$ and $e' = \alpha_{k^{-1}}(ef'1_k) = \alpha_{k^{-1}}(e1_k)\alpha_{k^{-1}}(f'1_k) = e'f$, showing that $e' \in \mathscr{B}_K^\times$ is smaller than but not equal to f, a contradiction to $f \in \mathscr{M}_K$. Thus we conclude that $f' \in \mathscr{M}_K$. □

Corollary 4.13 *$\alpha_k(\mathscr{M}_K 1_{k^{-1}}\backslash\{0\}) = \mathscr{M}_K 1_k\backslash\{0\}$ for each $k \in K$.*

Example 4.14 In Example 2.2, if letting $K = \{1, \sigma^2, \sigma^4\}$, then $\mathscr{M}_K = \{f_2, f_4\}$. Here $f_4 1_{\sigma^2} \neq 0$ and $\alpha_{\sigma^4}(f_4 1_{\sigma^2}) = f_2 \in \mathscr{M}_K$.

Notice that for each $f \in \mathscr{M}_K$ and $k \in K$, $f 1_k$ is either f or 0. Let $\mathscr{M}_K = \{f_i \mid i = 1, 2, \ldots, m\}$. For each $k \in K$, consider the following two subsets of $\{1, 2, \ldots, m\}$: $I_k^+ = \{i \mid f_i 1_k = f_i\} = \{i \mid f_i 1_k \neq 0\}$ and $I_k^- = \{i \mid f_i 1_{k^{-1}} = f_i\} = \{i \mid f_i 1_{k^{-1}} \neq 0\}$.

Lemma 4.15 *For each $k \in K$, $|I_k^+| = |I_k^-|$.*

Proof By Corollary 4.13, $\alpha_k(\mathscr{M}_K 1_{k^{-1}}\backslash\{0\}) = \mathscr{M}_K 1_k\backslash\{0\}$ for each $k \in K$. Hence the number of nonzero elements in $\mathscr{M}_K 1_k$ is equal to that of nonzero elements in $\mathscr{M}_K 1_{k^{-1}}$. □

Theorem 4.16 *Let $f_K = \vee_{i=1}^m f_i$, the Boolean sum of all elements in $\mathscr{M}_K$. Then $f_K \in S^{\alpha_K}$.*

Proof Let $k \in K$. We have that $f_K 1_k = \vee_{i=1}^m f_i 1_k = \vee_{i \in I_k^+} f_i$, the Boolean sum of all elements f in $\mathscr{M}_K$ such that $f 1_k = f$, and that $\alpha_k(f_K 1_{k^{-1}}) = \vee_{i=1}^m \alpha_k(f_i 1_{k^{-1}}) = \vee_{i \in I_k^-} \alpha_k(f_i 1_{k^{-1}})$, which by Proposition 4.12 is a Boolean sum of $|I_k^-|$ distinct elements f in $\mathscr{M}_K$ such that $f 1_k = f$. Therefore it follows from Lemma 4.15 that $\alpha_k(f_K 1_{k^{-1}}) = f_K 1_k$. □

Corollary 4.17 *Let $f_K = \vee_{i=1}^m f_i$. Suppose that (S, α) is a partial Galois extension. Then $(S f_K, \alpha_K)$ is a partial Galois extension. Furthermore, $S f_K$ is a Galois extension in (S, α) with Galois group K if and only if $K \subseteq N(f_K)$.*

Proof This follows immediately from the preceding theorem and Theorem 4.4. □

Example 4.18 Under the notations in Example 4.14, $(S(f_2 \vee f_4), \alpha_K)$ is a partial Galois extension. Indeed, $f_2 \vee f_4 \in S^{\alpha_K}$: $\alpha_{\sigma^2}((f_2 \vee f_4)1_{\sigma^4}) = \alpha_{\sigma^2}(f_2 1_{\sigma^4}) = f_4 = (f_2 \vee f_4)1_{\sigma^2}$. However, $(S(f_2 \vee f_4), \alpha_K)$ is not a Galois extension since K is not contained in $N(f_2 \vee f_4) = N(f_2) \cap N(f_4) = \{1\}$.

Example 4.19 In Example 4.10, if letting $K = \{1, \sigma^2, \sigma^4\}$, then $\mathscr{M}_K = \{e_{1,3,5} := 1_1 1_{\sigma^2} 1_{\sigma^4}\}$. By Corollary 4.17, $(S e_{1,3,5}, \alpha_K)$ is a partial Galois extension and actually is a Galois extension since $K = N(e_{1,3,5})$. Notice that $e_{1,3,5}$ equals the element e given in Example 4.10. Therefore the result here coincides with the conclusion of Example 4.10.

Let $\mathscr{M} = \{e_1, e_2, \ldots, e_n\}$. In the remaining of this section, assume that the Boolean sum of all elements in $\mathscr{M}$ is 1_S. Under this assumption, every element of $\mathscr{B}^\times$ is a Boolean sum of elements in $\mathscr{M}$, which we call its Boolean components. We have seen earlier that the inclusion $\mathscr{M}_K \subseteq \mathscr{M}$ does not hold in general. We close this section by showing that this must be true if $K = G(e)$ for some $e \in \mathscr{B}^\times$.

Proposition 4.20 *If $K = G(e)$ for some $e \in \mathscr{B}^\times$, then each Boolean component of e belongs to $\mathscr{M}_K$; in fact, $\mathscr{M}_K$ consists of the Boolean components of e. In particular, $\mathscr{M}_K \subseteq \mathscr{M}$.*

Proof Write $e = \vee_{i\in I} e_i$, where $I \subseteq \{1, 2, \ldots, n\}$. Then $G(e) = \cup_{i\in I} G(e_i)$ by Proposition 3.6. Thus $K = \cup_{i\in I} N(e_i)$ by Proposition 3.13, where $e_i = \Pi_{g\in N(e_i)} 1_g$ for each $i \in I$ by Proposition 3.12 and the canonical form of elements in $(\mathscr{B}, \bullet)^{\times}$ (see [13, Lemma 1]). Note that this implies that $\mathscr{M}_K = \{e_i \mid i \in I\}$. Hence $\mathscr{M}_K \subseteq \mathscr{M}$. □

We present an example below to show that it is possible that $\mathscr{M}_K \subseteq \mathscr{M}$ but $K \neq G(e)$ for any $e \in \mathscr{B}^{\times}$.

Example 4.21 In Example 2.2, we see that $1_S = f_2 \vee f_4 \vee f_5$, the Boolean sum of all elements in $\mathscr{M}$. As seen in Example 4.14, if letting $K = \{1, \sigma^2, \sigma^4\}$, then $\mathscr{M}_K = \{f_2, f_4\} \subseteq \mathscr{M}$. As $\mathscr{B}^{\times} = \{f_i \mid 1 \leq i \leq 5\} \cup \{f_2 \vee f_4, f_2 \vee f_5\}$, one can check that $K \neq G(e)$ for any $e \in \mathscr{B}^{\times}$.

5 Partial orbits and partial stabilizers

Throughout this section, let (S, α) be simply a set with a partial action of a group G (not necessarily finite). The concept of partial stabilizer was firstly introduced implicitly in [5]. Later in [2], the concept of partial orbit was also defined and the relationship of partial stabilizers and partial orbits with global notions arising from the associated enveloping action was studied. It was proved that the partial stabilizer coincides with the associated global stabilizer, and each partial orbit is the intersection of S with the associated global orbit. In this section we will derive some results on partial stabilizers and partial orbits, some of which will be applied in the next section to construct partial Galois extensions inside a fixed partial Galois extension. We will in particular generalize the orbit-stabilizer theorem and Burnside's lemma in the context of partial actions of groups.

We begin with showing that, as in the theory of group action, the definition of partial group action gives rise to an equivalence relation, and then defining partial orbits as equivalence classes. Define a relation on S by $x \sim y$ if and only if there exists some $g \in G$ such that $x \in S_{g^{-1}}$ and $y = \alpha_g(x)$. This is in fact an equivalence relation.

Lemma 5.1 *The relation $\sim$ on S is an equivalence relation.*

Proof Obviously, for any $x \in S$, $x \sim x$ by simply taking $g = 1$. If $x \sim y$, say $y = \alpha_g(x)$ for some $g \in G$ such that $x \in S_{g^{-1}}$, then $y \in S_g$ and $x = \alpha_{g^{-1}}(y)$; hence $y \sim x$. Suppose that $x \sim y$ and $y \sim z$, say $y = \alpha_g(x)$ and $z = \alpha_h(y)$ for some $g, h \in G$ such that $x \in S_{g^{-1}}$ and $y \in S_{h^{-1}}$. Noticing that $z \in S_h \cap S_{hg}$ by condition (P2) in the definition of partial group action, we then see from condition (P3) that $\alpha_{(hg)^{-1}}(z) = \alpha_{g^{-1}} \circ \alpha_{h^{-1}}(z) = \alpha_{g^{-1}}(y) = x$. Hence $z \sim x$ and so $x \sim z$. □

The equivalence class containing x, denoted by $\mathscr{O}_x$, is

$$\mathscr{O}_x = \{\alpha_g(x) \mid g \in G \text{ such that } x \in S_{g^{-1}}\},$$

and is called the α-partial orbit of x. The set S is partitioned into α-partial orbits. Let S/α denote the set of all α-partial orbits of S. The partial action α of G on S is said to be transitive if $|S/\alpha| = 1$; that is, there is only one α-partial orbit. This is equivalent to saying that for each pair of elements $x, y \in S$, there exists some $g \in G$ such that $x \in S_{g^{-1}}$ and $y = \alpha_g(x)$.

As in [2,5], the α-partial stabilizer of $x \in S$, denoted by $\mathscr{G}_x$, is defined to be

$$\mathscr{G}_x = \{g \in G \mid x \in S_{g^{-1}} \quad \text{and} \quad \alpha_g(x) = x\}.$$

This set has been shown to be a subgroup of G ([5, Proposition 2.5], [2, Corollary 10]). The proof in [5] used the definition of partial group action given by Exel [9] and the authors in [2] presented a proof via globalization, showing that the partial stabilizer coincides with the associated global stabilizer. We record a proof here using the definition of partial group action presented in Sect. 2, as given in [6], without passing through its enveloping action.

Proposition 5.2 *The α-partial stabilizer $\mathscr{G}_x$, for each $x \in S$, is a subgroup of G.*

Proof Clearly, $1 \in \mathscr{G}_x$. Let $g, h \in \mathscr{G}_x$. Then $x \in S_{g^{-1}} \cap S_{h^{-1}}$ and $\alpha_g(x) = x = \alpha_h(x)$. Hence $x \in S_{g^{-1}} \cap S_g \cap S_{h^{-1}} \cap S_h$ and $\alpha_{h^{-1}}(x) = x$. In particular, $h^{-1} \in \mathscr{G}_x$. Furthermore, x belongs to $S_{(gh)^{-1}}$ by condition (P2). Now by condition (P3), we then have $\alpha_{gh}(x) = \alpha_g(\alpha_h(x)) = x$, so $gh \in \mathscr{G}_x$. □

The following result is a generalization of the latter statement of [5, Proposition 2.5].

Proposition 5.3 *If $y \in \mathscr{O}_x$, then $\mathscr{G}_y$ and $\mathscr{G}_x$ are conjugate. More specifically, if $y = \alpha_g(x)$ for some $g \in G$ such that $x \in S_{g^{-1}}$, then $\mathscr{G}_y = g\mathscr{G}_x g^{-1}$.*

Proof Suppose $y = \alpha_g(x)$ for some $g \in G$ such that $x \in S_{g^{-1}}$. Let $h \in \mathscr{G}_x$. Then $x \in S_{h^{-1}}$ and $\alpha_h(x) = x$; hence $x \in S_h$ and $\alpha_{h^{-1}}(x) = x$. In particular, $x \in S_{h^{-1}} \cap S_{g^{-1}} \cap S_{(gh)^{-1}}$ and $y \in S_g \cap S_{gh^{-1}g^{-1}}$. Hence $\alpha_{ghg^{-1}}(y) = \alpha_{gh} \circ \alpha_{g^{-1}}(y) = \alpha_{gh}(x) = \alpha_g \circ \alpha_h(x) = \alpha_g(x) = y$, so $ghg^{-1} \in \mathscr{G}_y$. We have shown that $g\mathscr{G}_x g^{-1} \subseteq \mathscr{G}_y$. Since $x = \alpha_{g^{-1}}(y)$, a similar argument shows that $g^{-1}\mathscr{G}_y g \subseteq \mathscr{G}_x$. Therefore, $\mathscr{G}_y = g\mathscr{G}_x g^{-1}$. □

As a generalization of the orbit-stabilizer theorem in the theory of group action, the following result could be called the partial orbit-partial stabilizer theorem.

Theorem 5.4 *For each $x \in S$, there is a one-to-one correspondence between the elements of $\mathscr{O}_x$ and the left cosets $g\mathscr{G}_x$ in G with $x \in S_{g^{-1}}$.*

Proof Define a mapping from $\mathscr{O}_x$ to the set $\{g\mathscr{G}_x \mid g \in G \text{ such that } x \in S_{g^{-1}}\}$ by sending $\alpha_g(x)$ to $g\mathscr{G}_x$ for each $g \in G$ such that $x \in S_{g^{-1}}$. Suppose $g, h \in G$ such that $x \in S_{g^{-1}} \cap S_{h^{-1}}$ and $\alpha_g(x) = \alpha_h(x)$, which we denote by y. Then $y \in S_g \cap S_h$ and $x = \alpha_{g^{-1}}(y) \in S_{g^{-1}} \cap S_{g^{-1}h}$; hence $x = \alpha_{h^{-1}}(y) = \alpha_{h^{-1}}(\alpha_g(x)) = \alpha_{h^{-1}g}(x)$. Hence $h^{-1}g \in \mathscr{G}_x$, which means $g\mathscr{G}_x = h\mathscr{G}_x$. This shows that the mapping is well-defined. Suppose now that $h^{-1}g \in \mathscr{G}_x$, where $g, h \in G$ such that $x \in S_{g^{-1}} \cap S_{h^{-1}}$. Then $x \in S_{g^{-1}h}$ and $\alpha_{h^{-1}g}(x) = x$. Thus $\alpha_g(x) = \alpha_h(\alpha_{h^{-1}g}(x)) = \alpha_h(x)$. Therefore, the mapping is one-to-one. Obviously, it is surjective. □

Corollary 5.5 *Let $x \in S$ and suppose $\{g_1, g_2, \ldots, g_l\}$ be a system of representatives for the left cosets $g\mathscr{G}_x$ in G with $x \in S_{g^{-1}}$. Then $\mathscr{O}_x = \{\alpha_{g_1}(x), \ldots, \alpha_{g_l}(x)\}$. Furthermore, if $h \in G$ such that $x \in S_{(hg_i)^{-1}}$ for each $1 \leq i \leq l$, then $\mathscr{O}_x = \{\alpha_{hg_1}(x), \ldots, \alpha_{hg_l}(x)\}$*

Proof The first result follows immediately from the one-to-one correspondence in Theorem 5.4. Since $\{g_1, g_2, \ldots, g_l\}$ is a system of representatives for the left cosets $g\mathscr{G}_x$ in G with $x \in S_{g^{-1}}$, so is $\{hg_1, hg_2, \ldots, hg_l\}$ for any $h \in G$ with $x \in S_{(hg_i)^{-1}}$ for each $1 \leq i \leq l$. Thus by Theorem 5.4 again, $\mathscr{O}_x = \{\alpha_{hg_1}(x), \ldots, \alpha_{hg_l}(x)\}$. □

We will next derive a generalization of Burnside's lemma in the theory of group action. To do so, for each $x \in S$, let $G^x = \{g \in G \mid x \in S_{g^{-1}}\}$. Then $\mathscr{O}_x = \{\alpha_g(x) \mid g \in G^x\}$ and $\mathscr{G}_x = \{g \in G^x \mid \alpha_g(x) = x\}$. Notice that G^x might not be a subgroup of G. We show that elements in G^x and elements in $G \backslash G^x$ do not give rise to the same left cosets of the subgroup $\mathscr{G}_x$.

Proposition 5.6 *For any $g, h \in G$, if $g\mathscr{G}_x = h\mathscr{G}_x$, then either $g, h \in G^x$ or $g, h \in G \backslash G^x$.*

Proof Suppose $g\mathscr{G}_x = h\mathscr{G}_x$. Then $h^{-1}g \in \mathscr{G}_x$; equivalently, $x \in S_{g^{-1}h}$ and $\alpha_{h^{-1}g}(x) = x$. If $g \in G^x$; that is, $x \in S_{g^{-1}}$, then by (P3), $x = \alpha_{h^{-1}} \circ \alpha_g(x)$, belonging to $S_{h^{-1}}$, and hence $h \in G^x$. Since $g^{-1}h \in \mathscr{G}_x$, a similar argument shows that if $h \in G^x$, then $g \in G^x$. Therefore, we are done. □

Corollary 5.7 *Suppose G is a finite group. Then for each $x \in S$, $|\mathscr{G}_x|$ divides $|G^x|$ and $|\mathscr{O}_x| = |G^x|/|\mathscr{G}_x|$.*

Proof By Proposition 5.6, we see that the left cosets of $\mathscr{G}_x$ partition G in a way such that $G^x = \cup_{g \in G^x} g\mathscr{G}_x$ and $G \backslash G^x = \cup_{g \in G \backslash G^x} g\mathscr{G}_x$. Thus if G is finite, then $|\mathscr{G}_x|$ divides $|G^x|$. Furthermore, by Theorem 5.4, we then get $|\mathscr{O}_x| = |G^x|/|\mathscr{G}_x|$. □

The following result shows that if $y, z \in S$ belong to the same orbit $\mathscr{O}_x$, then there is a one-to-one correspondence between G^y and G^z. When G is finite, we denote this common number of elements by $|G^{\mathscr{O}_x}|$.

Lemma 5.8 *For any $x \in S$, if $y = \alpha_g(x)$ for some $g \in G^x$, then $G^y = G^x g^{-1}$.*

Proof Suppose $y = \alpha_g(x)$ for some $g \in G$ such that $x \in S_{g^{-1}}$. If $h \in G^x$; equivalently, $x \in S_{h^{-1}}$, then $y \in S_{gh^{-1}}$; hence $hg^{-1} \in G^y$. Hence $G^x g^{-1} \subseteq G^y$. Since $x = \alpha_{g^{-1}}(y)$, where $g^{-1} \in G^y$, a similar argument shows that $G^y g \subseteq G^x$. Therefore, $G^y = G^x g^{-1}$. □

We are now ready to prove the generalization of Burnside's lemma in the context of partial group action. For each $g \in G$, let $S^g = \{x \in S_{g^{-1}} \mid \alpha_g(x) = x\}$.

Theorem 5.9 *Suppose G is a finite group. Then $\sum_{g \in G} |S^g| = \sum_{\mathscr{O} \in S/\alpha} |G^{\mathscr{O}}|$.*

Proof The sum

$$\begin{aligned}\textstyle\sum_{g\in G}|S^g| &= \textstyle\sum_{g\in G}|\{x\in S_{g^{-1}} \mid \alpha_g(x)=x\}|\\ &= |\{(g,x)\in G\times S \mid x\in S_{g^{-1}} \quad \text{and} \quad \alpha_g(x)=x\}|\\ &= |\{(g,x)\in G\times S \mid g\in \mathcal{G}_x\}|,\end{aligned}$$

when expressed as a sum over the elements of S, is $\sum_{x\in S}|\mathcal{G}_x|$, which by Corollary 5.7, equals $\sum_{x\in S}|G^x|/|\mathcal{O}_x|$. Now notice that S is partitioned into partial orbits by the partial action α of G, so the sum over S may be broken up into separate sums over S/α, grouping the elements of each partial orbit together:

$$\sum_{\mathcal{O}\in S/\alpha}\sum_{x\in\mathcal{O}}|G^x|/|\mathcal{O}_x|.$$

Furthermore, Lemma 5.8 shows that $|G^y|=|G^z|$ whenever y,z belong to the same orbit $\mathcal{O}$, and this common number of elements is denoted by $|G^{\mathcal{O}}|$. Therefore, we conclude that

$$\sum_{g\in G}|S^g| = \sum_{\mathcal{O}\in S/\alpha}\left(|G^{\mathcal{O}}|\sum_{x\in\mathcal{O}}1/|\mathcal{O}|\right) = \sum_{\mathcal{O}\in S/\alpha}|G^{\mathcal{O}}|.$$

□

To give an example, where the set under the partial action is finite, to interpret the preceding theorem, we introduce the concept of invariant subset. Let R be a subset of S and K a subgroup of G. We say R is an α_K-invariant subset of S if $\alpha_k(R\cap S_{k^{-1}}) = R\cap S_k$ for each $k\in K$. In the case where S is a ring, $\mathcal{B}_K$ and $\mathcal{M}_K$ are both α_K-invariant by Lemma 4.11 and Corollary 4.13 since $\mathcal{B}_K\cap S_k=\mathcal{B}_K 1_k$ and $\mathcal{M}_K\cap S_k = \mathcal{M}_K 1_k\backslash\{0\}$ for each $k\in K$. Note that if R is an α_K-invariant subset of S, then α induces a partial action of K on R with $\{R\cap S_k \mid k\in K\}$ as the collection of associated subsets.

Example 5.10 Notice that in Example 2.1, $\mathcal{B}=(\mathcal{B},\bullet)$ with 8 elements, which, as pointed above, is α_G-invariant. Let α also denote the induced partial action of G on $\mathcal{B}$. One can check easily that for $(\mathcal{B},\alpha)$ there are five partial orbits, namely,

$$\begin{aligned}&\mathcal{O}_0=\{0\},\ \mathcal{O}_1=\{1_1\},\ \mathcal{O}_2=\{e_{1,2}, e_{2,3}=\alpha_{\sigma^3}(e_{1,2})\},\ \mathcal{O}_3=\{e_{1,3}\},\\ &\mathcal{O}_4=\{e_1, e_2=\alpha_{\sigma^3}(e_1), e_3=\alpha_{\sigma^2}(e_1)\}\end{aligned}$$

Also,

$$\begin{aligned}&G^0=G,\ G^{1_1}=\{1\},\ G^{e_{1,2}}=\{1,\sigma^3\},\ G^{e_{1,3}}=\{1,\sigma^2\},\ G^{e_{2,3}}=\{1,\sigma\},\\ &G^{e_1}=\{1,\sigma^2,\sigma^3\},\ G^{e_2}=\{1,\sigma,\sigma^3\},\ G^{e_3}=\{1,\sigma,\sigma^2\}.\end{aligned}$$

Lemma 5.8 indeed holds here, and $|G^{\mathcal{O}_0}|=4$, $|G^{\mathcal{O}_1}|=1$, $|G^{\mathcal{O}_2}|=2$, $|G^{\mathcal{O}_3}|=2$ and $|G^{\mathcal{O}_4}|=3$, so $\sum_{\mathcal{O}\in\mathcal{B}/\alpha}|G^{\mathcal{O}}|=12$. On the other hand, $\mathcal{B}^1=\mathcal{B}$, $\mathcal{B}^\sigma=\{0\}$,

$\mathscr{B}^{\sigma^2} = \{0, e_{1,3}\}$ and $\mathscr{B}^{\sigma^3} = \{0\}$. Hence $\sum_{g\in G} |\mathscr{B}^g| = 12 = \sum_{\mathscr{O}\in\mathscr{B}/\alpha} |G^{\mathscr{O}}|$, as expected.

We now close this section by showing that each α-partial orbit of S is an α_G-invariant subset as expected.

Proposition 5.11 *The α-partial orbit $\mathscr{O}_x$, for each $x \in S$, is an α_G-invariant subset of S.*

Proof It suffices to show that $\alpha_g(\mathscr{O}_x \cap S_{g^{-1}}) \subseteq \mathscr{O}_x \cap S_g$ for each $g \in G$. Let $g \in G$ and suppose $y \in \mathscr{O}_x \cap S_{g^{-1}}$. Then $y = \alpha_h(x)$ for some $h \in G$ such that $x \in S_{h^{-1}} \cap S_{h^{-1}g^{-1}}$. Hence $\alpha_g(y) = \alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$ belongs to $\mathscr{O}_x \cap S_g$. □

Clearly, the induced partial action of G on any α-partial orbit of S is transitive.

6 Special invariant subsets

Throughout this section, let (S, α) be a ring with a partial action of a group G and K be a subgroup of G. The definition of α_K-invariant subset of S is given in the end of the last section; as mentioned there, for any α_K-invariant subset R, α induces a partial action of K on R with $\{R \cap S_k \mid k \in K\}$ as the collection of associated subsets. Recall that in Sect. 4 and in [13], we say that α induces a partial action of K on the unital ring A contained in S if $1_A 1_k \in A$ for each $k \in K$ and $\{A1_k \mid k \in K\}$ forms the collection of associated ideals of A. We remark that such A is an α_K-invariant subset of S since $A1_k = A \cap S1_k$ whenever $1_A 1_k \in A$.

In this section, we will study certain α_K-invariant subsets of S, including $\mathscr{B}_K$ and $\mathscr{M}_K$, and show how to constructing more partial Galois extensions inside (S, α) via these special invariant subsets. We begin with showing that $\mathscr{I}(S)$ and its subset consisting of minimal elements of $\mathscr{I}(S)^\times$, denoted by $\mathscr{M}\mathscr{I}(S)$, are also α_K-invariant subsets of S.

Proposition 6.1 *The sets $\mathscr{I}(S)$ and $\mathscr{M}\mathscr{I}(S)$ are both α_K-invariant subsets of S.*

Proof It suffices to show that for each $k \in K$, $\alpha_k(R \cap S_{k^{-1}}) \subseteq R \cap S_k$ whenever R is $\mathscr{I}(S)$ or $\mathscr{M}\mathscr{I}(S)$. Let $k \in K$ and suppose $x \in \mathscr{I}(S) \cap S_{k^{-1}}$. It is clear that $\alpha_k(x)$ is a central idempotent of S contained in S_k. Thus $\mathscr{I}(S)$ is an α_K-invariant subset of S. Assume furthermore that x is minimal in $\mathscr{I}(S)^\times$. Let y be any element in $\mathscr{I}(S)^\times$ such that $\alpha_k(x)y \neq 0$. Since $y1_k \in \mathscr{I}(S) \cap S_k$, $y1_k = \alpha_k(u)$ for some $u \in \mathscr{I}(S) \cap S_{k^{-1}}$. Hence $\alpha_k(x)y = \alpha_k(x)y1_k = \alpha_k(xu)$, where $xu \neq 0$ and thus $xu = x$ since $x \in \mathscr{M}\mathscr{I}(S)$. Hence $\alpha_k(x) \preceq y$. This shows that $\alpha_k(x)$ is contained in $\mathscr{M}\mathscr{I}(S) \cap S_k$. Therefore, $\mathscr{M}\mathscr{I}(S)$ is also an α_K-invariant subset of S. □

We next focus on these four α_K-invariant subsets of S: $\mathscr{B}_K$, $\mathscr{M}_K$, $\mathscr{I}(S)$, $\mathscr{M}\mathscr{I}(S)$ with the induced partial action of K. For any subset R of S, $R \cap S1_k \subseteq R1_k$ for each $k \in K$. Notice that the equality holds whenever R is $\mathscr{B}_K$ or $\mathscr{I}(S)$. On the other hand, it is obvious that $R \cap S1_k = R1_k \backslash \{0\}$ for each $k \in K$ whenever R is $\mathscr{M}_K$ or $\mathscr{M}\mathscr{I}(S)$. As shown below, the preceding equality holds too whenever R is any α_K-partial orbit of $\mathscr{M}_K$ or of $\mathscr{M}\mathscr{I}(S)$.

Lemma 6.2 *Let R be $\mathscr{M}_K$ or $\mathscr{M}\mathscr{I}(S)$. Let $x \in R$ and $\mathscr{O}_x$ denote the α_K-partial orbit of x in R. Then $\mathscr{O}_x \cap S1_k = \mathscr{O}_x 1_k \backslash \{0\}$ for each $k \in K$; in particular, $\alpha_k(\mathscr{O}_x 1_{k^{-1}} \backslash \{0\}) = \mathscr{O}_x 1_k \backslash \{0\}$.*

Proof For any $k \in K$, clearly $\mathscr{O}_x \cap S1_k \subseteq \mathscr{O}_x 1_k \backslash \{0\}$. Let $y \in \mathscr{O}_x$ such that $y1_k \neq 0$. Then $y = \alpha_h(x)$ for some $h \in K$ such that $x \in S_{h^{-1}}$. Hence $y1_k = \alpha_h(x)1_k = \alpha_h(x1_{h^{-1}k})$, where $x1_{h^{-1}k} \neq 0$, with $h^{-1}k \in K$, and so equals x since x is minimal in $\mathscr{B}_K^{\times}$ ($\mathscr{I}(S)^{\times}$ resp.). Hence $y1_k = \alpha_h(x) = y \in \mathscr{O}_x \cap S1_k$, as desired. The last statement follows from the previous result and the fact that each partial orbit under the partial action of K is α_K-invariant (see Proposition 5.11). □

We remark that the preceding lemma is not true if R is replaced by $\mathscr{B}_K$. Consider $\mathscr{B}_K$ with $K = G$ in Example 2.1. We have seen in Example 5.10 that $\mathscr{O}_3 = \{e_{1,3}\}$ is a partial orbit, but $e_{1,3}1_{\sigma^3} = e_3 \notin \mathscr{O}_3$.

With the preceding lemma, we now can apply the idea of the proof of Theorem 4.16 to prove the following stronger result than Theorem 4.16.

Theorem 6.3 *Let R be $\mathscr{M}_K$ or $\mathscr{M}\mathscr{I}(S)$. For each $x \in R$, the Boolean sum of all elements in the α_K-partial orbit $\mathscr{O}_x$ of x in R belongs to S^{α_K}.*

Proof The key of the proof is that for each $x \in R$ and $k \in K$, $x1_k$ is either x or 0. Let $\mathscr{O}_x = \{x_1, x_2, \ldots, x_m\}$ and $\tilde{x} = \vee_{i=1}^m x_i$. For each $k \in K$, consider the following two subsets of $\{1, 2, \ldots, m\}$: $I_k^+ = \{i \mid x_i 1_k = x_i\} = \{i \mid x_i 1_k \neq 0\}$ and $I_k^- = \{i \mid x_i 1_{k^{-1}} = x_i\} = \{i \mid x_i 1_{k^{-1}} \neq 0\}$. Since $\alpha_k(\mathscr{O}_x 1_{k^{-1}} \backslash \{0\}) = \mathscr{O}_x 1_k \backslash \{0\}$ by Lemma 6.2, it follows that the number of nonzero elements in $\mathscr{O}_x 1_k$ is equal to that of nonzero elements in $\mathscr{O}_x 1_{k^{-1}}$; equivalently, $|I_k^+| = |I_k^-|$. Observe that $\tilde{x}1_k = \vee_{i=1}^m x_i 1_k = \vee_{i \in I_k^+} x_i$ is the Boolean sum of all elements y in $\mathscr{O}_x$ such that $y1_k = y$. On the other hand, $\alpha_k(\tilde{x}1_{k^{-1}}) = \vee_{i=1}^m \alpha_k(x_i 1_{k^{-1}}) = \vee_{i \in I_k^-} \alpha_k(x_i)$, which by Proposition 5.11 or Lemma 6.2 is a Boolean sum of $|I_k^-|$ distinct elements y in $\mathscr{O}_x$ such that $y1_k = y$. We therefore conclude from $|I_k^+| = |I_k^-|$ that $\alpha_k(\tilde{x}1_{k^{-1}}) = \tilde{x}1_k$. Hence $\tilde{x} \in S^{\alpha_K}$. □

Corollary 6.4 *Let R be $\mathscr{M}_K$ or $\mathscr{M}\mathscr{I}(S)$. Let $x \in R$ and $\tilde{x}$ denote the Boolean sum of all elements in the α_K-partial orbit $\mathscr{O}_x$ of x in R. Suppose that (S, α) is a partial Galois extension. Then $(S\tilde{x}, \alpha_K)$ is a partial Galois extension. Furthermore, $S\tilde{x}$ is a Galois extension in (S, α) with Galois group K if and only if $K \subseteq N(\tilde{x})$.*

Proof This follows immediately from the preceding theorem and Theorem 4.4. □

Example 6.5 In Example 4.10, take $e_2 \in \mathscr{M}\mathscr{I}(S)$ and $K = \{1, \sigma^2, \sigma^4\}$. The α_K-partial orbit of e_2 in $\mathscr{M}\mathscr{I}(S)$ consists of e_2, e_4. Observe that $N(\tilde{e_2}) = N(e_2 \vee e_4) = G\backslash\{\sigma^2, \sigma^4\}$ does not contain K. Hence by the preceding Corollary, $(S(e_2 \vee e_4), \alpha_K)$ is a partial Galois extension, but not a Galois extension.

Remark 6.6 We point out that Theorem 6.3 does not hold if R is replaced by $\mathscr{B}_K$. Consider $\mathscr{B}_K$ with $K = G$ in Example 2.2. Let $x = f_2 \vee f_4$, which belongs to $S_{k^{-1}}$ only when $k = 1$ and so the α_K-partial orbit of x in $\mathscr{B}_x$ consists of x only. However $x \notin S^{\alpha_K}$ since $x1_{\sigma^5} = 0$ while $\alpha_{\sigma^5}(x1_\sigma) = \alpha_{\sigma^5}(f_2) = f_5$.

In the following let R be $\mathscr{I}(S)$ or $\mathscr{B}_K$. For $x \in R^\times = R\backslash\{0\}$, let $N_K(x) = N(x) \cap K$. If $N_K(x)$ is a subgroup of K, then α induces a partial action of $N_K(x)$ on R. We show that in this case the Boolean sum of all elements in the $\alpha_{N_K(x)}$-partial orbit of x in R gives rise to a Galois extension in (S, α) with Galois group $N_K(x)$.

Theorem 6.7 *Let R be $\mathscr{I}(S)$ or $\mathscr{B}_K$ and $x \in R^\times$ such that $N_K(x)$ is a subgroup of K. Let $\tilde{x}$ denote the Boolean sum of all elements in the $\alpha_{N_K(x)}$-partial orbit of x in R. Suppose that (S, α) is a partial Galois extension. Then $S\tilde{x}$ is a Galois extension with Galois group $N_K(x)$.*

Proof Suppose that the $\alpha_{N_K(x)}$-partial orbit of x in R consists of $\alpha_{k_1}(x), \alpha_{k_2}(x), \ldots, \alpha_{k_l}(x)$, where $k_i \in N_K(x)$ such that $x \in R \cap S_{k_i^{-1}}$ for each $1 \le i \le l$. Let $\tilde{x} = \vee_{i=1}^{l} \alpha_{k_i}(x)$. By Theorem 4.4, it suffices to show that $\tilde{x} \in S^{\alpha_{N_K(x)}}$ and $N_K(x) \subseteq N(\tilde{x})$. Let $h \in N_K(x)$. Then for each $1 \le i \le l$, $k_i^{-1}h \in N_K(x)$ and so $\alpha_{k_i}(x) = \alpha_{k_i}(x1_{k_i^{-1}h}) = \alpha_{k_i}(x)1_h$. Hence $\tilde{x}1_h = \vee_{i=1}^{l}\alpha_{k_i}(x)1_h = \vee_{i=1}^{l}\alpha_{k_i}(x) = \tilde{x}$; that is, $h \in N(\tilde{x})$. We have shown that $N_K(x) \subseteq N(\tilde{x})$. Now similarly, we have $(hk_i)^{-1} \in N_K(x)$ and so $x \in R \cap S_{(hk_i)^{-1}}$ for each $1 \le i \le l$. Thus by Corollary 5.5, $\alpha_{hk_1}(x), \ldots, \alpha_{hk_l}(x)$ also constitute the $\alpha_{N_K(x)}$-partial orbit of x in R. Therefore, $\alpha_h(\tilde{x}1_{h^{-1}}) = \alpha_h(\tilde{x}) = \alpha_h(\vee_{i=1}^{l}\alpha_{k_i}(x)) = \vee_{i=1}^{l}\alpha_{hk_i}(x) = \tilde{x} = \tilde{x}1_h$, where we have applied the assumption that $N_K(x)$ is a subgroup and the previous result that $N_K(x) \subseteq N(\tilde{x})$. We conclude that $\tilde{x} \in S^{\alpha_{N_K(x)}}$. ◻

Corollary 6.8 *Suppose that (S, α) is a partial Galois extension. If e is a nonzero element of $(\mathscr{B}_K, \bullet)$, the Boolean semigroup generated by $\{1_k \mid k \in K\}$, such that $N_K(e)$ is a subgroup of K, then Se is a Galois extension with Galois group $N_K(e)$.*

Proof Note that $e = \prod_{k \in N_K(e)} 1_k$. Since $N_K(e)$ is a subgroup of K, it follows that for each $h \in N_K(e)$, $e1_{h^{-1}} = e$ and $\alpha_h(e) = \prod_{k \in N_K(e)} \alpha_h(1_k 1_{h^{-1}}) = \prod_{k \in N_K(e)} 1_{hk}1_h = \prod_{k \in N_K(e)} 1_k = e$. This means the $\alpha_{N_K(e)}$-partial orbit of e in $\mathscr{B}_K$ consists of e only. Therefore, by the preceding theorem, Se is a Galois extension with Galois group $N_K(e)$. ◻

Example 6.9 In Example 4.10, let $K = \{1, \sigma^2, \sigma^4\}$. For the central idempotent e_1 of S, $N_K(e_1) = K$ and the $\alpha_{N_K(e_1)}$-partial orbit of e_1 in $\mathscr{I}(S)$ consist of e_1, e_3, e_5. Hence by Theorem 6.7, $S\tilde{e_1} = S(e_1 \vee e_3 \vee e_5)$ is a Galois extension with Galois group K. This coincides with the conclusion of Example 4.10. On the other hand, e_1 is minimal in $\mathscr{I}(S)^\times$ and the α_K-partial orbit of e_1 in $\mathscr{M}\mathscr{I}(S)$ is $\{e_1, e_3, e_5\}$ with $N(\tilde{e_1}) = N(e_1 \vee e_3 \vee e_5) = K$. Hence the result also follows by Corollary 6.4.

Example 6.10 In Example 2.2, let $K = \{1, \sigma^3\}$. Then $f_3 \in (\mathscr{B}_K, \bullet)$ such that $N_K(f_3) = K$. Hence by Corollary 6.8, Sf_3 is a Galois extension with Galois group K. Notice that in this case $\mathscr{M}_K = \{f_3\}$ and $N(f_3) = K$, so Corollary 6.4 also assures this result.

We close this section with the following observation.

Proposition 6.11 *Let R be one of the four α_K-invariant subsets of S: $\mathscr{B}_K$, $\mathscr{M}_K$, $\mathscr{I}(S)$ or $\mathscr{M}\mathscr{I}(S)$. If $x, y \in R$ belong to the same α_K-partial orbit in R, then $Sx \cong Sy$.*

Proof Since x and y belong to the same α_K-partial orbit in R, there exists some $k \in K$ such that $x \in S_{k^{-1}}$ and $y = \alpha_k(x) \in S_k$. Let $\phi \colon Sx \to Sy$ be defined by $\phi(sx) = \alpha_k(s1_{k^{-1}})y$ for each $s \in S$. Then $\phi(sx) = \alpha_k(sx1_{k^{-1}})$ for all $s \in S$. It follows that ϕ is a ring isomorphism. We explain more why ϕ is surjective. For $s'y \in Sy$, by the surjectivity of α_k, there exists some $s \in S$ such that $s'1_k = \alpha_k(s1_{k^{-1}})$, so $s'y = s'1_k y1_k = \alpha_k(s1_{k^{-1}})\alpha_k(x1_{k^{-1}}) = \alpha_k(sx1_{k^{-1}}) = \phi(sx)$. □

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