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A remark on exceptional sets in Erdös-Rényi limit theorem

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Abstract Let $(x_i)_{i=1}^{+\infty}$ be the digits sequence in the unique terminating dyadic expansion of $x \in [0, 1)$. The run-length function $l_n(x)$ is defined by

$$l_n(x) := \max \left\{ j : x_{i+1} = x_{i+2} = \cdots = x_{i+j} = 1 \text{ for some } 0 \le i \le n - j \right\}.$$

Erdös and Rényi proved that

$$\lim_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = 1, \text{ a.e. } x \in [0, 1).$$

In this note, we show that for each pair of numbers $\alpha, \beta \in [0, +\infty]$ with $\alpha \le \beta$, the following exceptional set

$$E_{\alpha,\beta} = \left\{ x \in [0, 1) : \liminf_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = \alpha, \ \limsup_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = \beta \right\}$$

has Hausdorff dimension one.

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1 Introduction

For any $x \in [0, 1)$, it can be uniquely expanded into its terminating dyadic expansion:

$$x = \frac{x_1}{2} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \cdots,$$

where $x_n \in \{0, 1\}$ is called the digit of x. The run-length function $l_n(x)$ is the longest run of 1's in the first n digits of the dyadic expansion of x. A classic result due to Erdös and Rényi [5] asserts that

$$\lim_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = 1$$

for Lebesgue almost all $x \in [0, 1)$. Ma, Wen and Wen [13] proved that the set of all points in $[0, 1)$ for which the above Erdös-Rényi's theorem does not hold has Hausdorff dimension one. For an increasing integer sequence $(\delta_n)_{n\geq 1}$, Zou [16] considered the set of points whose run-length function behaves asymptotically as δ_n, that is

$$E(\{\delta_n\}) = \left\{x \in [0, 1) : \lim_{n\to+\infty} \frac{l_n(x)}{\delta_n} = 1\right\}.$$

He showed that the set $E(\{\delta_n\})$ has Hausdorff dimension one under the condition $\lim_{n\to+\infty} \frac{\delta_{n+\delta_n}}{\delta_n} = 1$. A similar result holds in an infinite symbolic system: continued fraction dynamical system, see [15].

Remark 1 Applying Zou's result to $\delta_n = [\alpha \log_2 n]$ with $\alpha \in (0, +\infty)$, $\delta_n = [\log_2 \log_2 n]$ and $\delta_n = [\sqrt{n}]$ respectively, we get for any $\alpha \in [0, +\infty]$,

$$\dim_H \left\{x \in [0, 1) : \lim_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = \alpha\right\} = 1,$$

in view of the inclusions

$$\left\{x \in [0, 1) : \lim_{n\to+\infty} \frac{l_n(x)}{[\log_2 \log_2 n]} = 1\right\} \subset \left\{x \in [0, 1) : \lim_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = 0\right\}$$

and

$$\left\{x \in [0, 1) : \lim_{n\to+\infty} \frac{l_n(x)}{[\sqrt{n}]} = 1\right\} \subset \left\{x \in [0, 1) : \lim_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = +\infty\right\},$$

where $[\cdot]$ denotes the integer part function and $\dim_H$ denotes the Hausdorff dimension.

Recently, Li and Wu [11,12] studied the extreme situation for general asymptotic behaviour of run-length function. More precisely, they proved that the set

$$\left\{x \in [0,1) \colon \liminf_{n\to+\infty} \frac{l_n(x)}{\varphi(n)} = 0,\ \limsup_{n\to+\infty} \frac{l_n(x)}{\varphi(n)} = +\infty\right\}$$

has Hausdorff dimension 0 or 1 according as $\limsup_{n\to+\infty} \frac{n}{\varphi(n)} < +\infty$ or $\limsup_{n\to+\infty} \frac{n}{\varphi(n)} = +\infty$, where $\varphi \colon \mathbb{N} \to \mathbb{R}^+$ is a monotonically positive increasing function with $\lim_{n\to+\infty} \varphi(n) = +\infty$.

Remark 2 If we take $\varphi(n) = \log_2 n$, it follows that

$$\dim_{\mathrm{H}} \left\{x \in [0,1) \colon \liminf_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = 0,\ \limsup_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = +\infty\right\} = 1$$

In this note, we would like to consider a subtle question: what is the Hausdorff dimension of the set

$$E_{\alpha,\beta} = \left\{x \in [0,1) \colon \liminf_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = \alpha,\ \limsup_{n\to+\infty} \frac{l_n(x)}{\log_2 n} = \beta\right\}$$

with $0 \le \alpha \le \beta \le +\infty$. We show

Theorem 1.1

$$\dim_{\mathrm{H}} E_{\alpha,\beta} = 1.$$

The first analogous investigation on the fractal sets of this type goes back to Besicovitch [3], where he considered the Hausdorff dimension of the level sets determined by the frequency of digits in dyadic system. Eggleston [4] generalised Besicovitch's result to base $m \ge 2$. Their results were recovered and generalized by Barreira, Saussol and Schmeling [2] using a multidimensional version of multifractal analysis. Similar questions had also been extensively studied for the recurrent sets in various dynamical system, see [1,7–10,14] and reference therein. For more details about Hausdorff dimension, we refer to the book of Falconer [6].

2 Proof of the main result

In this section, we will prove the main result of this note. The proof of Theorem 1.1 rests on the following proposition applied successfully in [11] and [13].

Proposition 2.1 [13] *Given a set of positive integers* $\mathcal{J} = \{j_1 < j_2 < j_3 < \cdots\}$ *and an infinite sequence* $\{a_i\}_{i\ge1}$ *of* 0*'s and* 1*'s, let*

$$E\left(\mathcal{J}, \{a_i\}\right) = \left\{x \in [0,1) \colon x = \sum_{i=1}^{+\infty} \frac{x_i}{2^i},\ x_i = a_i, \forall i \in \mathcal{J}\right\}.$$

If the density of $\mathcal{J}$ is zero, that is,

$$\lim_{n\to+\infty} \frac{\#\{i \le n : i \in \mathcal{J}\}}{n} = 0,$$

then $\dim_{\mathrm{H}} E(\mathcal{J}, \{a_i\}) = 1$, *where # denotes the number of elements in a set.*

Proof of Theorem 1.1. By Remarks 1 and 2, we need only to prove the theorem for the cases $0 < \alpha < \beta < +\infty$, $0 = \alpha < \beta < +\infty$ and $0 < \alpha < \beta = +\infty$. The whole proof is divided into two parts: a detailed proof for the case $0 < \alpha < \beta < +\infty$ and sketches of proof for the remaining cases. We now first restrict ourselves to the case $0 < \alpha < \beta < +\infty$. Our strategy is to construct a subset of real numbers for which the maximal lengths of blocks of digits 1 among the dyadic expansions reach at suitable scattered positions, which guarantee that the points are in $E_{\alpha,\beta}$ and also the subset with full Hausdorff dimension. Choose two subsequences $\{m_k\}_{k\ge1}$ and $\{n_k\}_{k\ge1}$ satisfying, for each $k \ge 1$,

$$n_1 = \left[2^{\frac{\beta}{\alpha}}\right],\ n_{k+1} = \left[n_k^{\frac{\beta}{\alpha}}\right],\ m_k = n_k + \left[\beta \log_2 n_k\right].$$

Clearly, $\{n_k\}_{k\ge1}$ increases super-exponentially, and there exists $K \ge 1$ such that for any $k \ge K$, we have $n_k < m_k < n_{k+1}$. For $k \ge K$, let t_k be the largest integer such that $m_k + t_k(m_k - n_k) < n_{k+1}$. Take

$$\begin{aligned}\mathcal{D} := \mathcal{D}(\{m_k\}, \{n_k\}) = \{&1, 2, \ldots, n_K - 1, \text{and } n_k, n_k + 1, \ldots, m_k - 1, m_k,\\ &m_k + (m_k - n_k), \ldots, m_k + t_k(m_k - n_k),\ \text{for } k \ge K\}.\end{aligned}$$

Define an infinite sequence $\{a_i\}_{i\ge1}$ as follows. For $1 \le i < n_K$, set

$$a_i = 0.$$

For $k \ge K$, set

$$a_{n_k} = 0,\ a_{n_k+1} = \cdots = a_{m_k-1} = 1,\ a_{m_k} = 0$$

and

$$a_{m_k+(m_k-n_k)} = a_{m_k+2(m_k-n_k)} = \cdots = a_{m_k+t_k(m_k-n_k)} = 0.$$

We consider the set E of real numbers $x \in [0, 1)$ whose dyadic expansion $x = \sum_{i=1}^{+\infty} \frac{x_i}{2^i}$ satisfies $x_i = a_i$ for all $i \in \mathcal{D}$, that is

$$E := E(\mathcal{D}, \{a_i\}) = \left\{x \in [0, 1) : x = \sum_{i=1}^{+\infty} \frac{x_i}{2^i},\ x_i = a_i, \forall i \in \mathcal{D}\right\}.$$

Now we prove $E \subset E_{\alpha,\beta}$. Fix $x \in E$, for any $n \geq n_{K+1}$, let k be the integer such that $n_k \leq n < n_{k+1}$. From the construction of the set E, we see that

$$l_n(x) = \begin{cases} m_{k-1} - n_{k-1} - 1 = \left[\beta \log_2 n_{k-1}\right] - 1, & \text{if } n_k \leq n \leq n_k + m_{k-1} - n_{k-1} - 1, \\ n - n_k, & \text{if } n_k + m_{k-1} - n_{k-1} \leq n \leq m_k - 1, \\ m_k - n_k - 1 = \left[\beta \log_2 n_k\right] - 1, & \text{if } m_k \leq n < n_{k+1}. \end{cases}$$

Thus

$$\begin{aligned} \liminf_{n\to+\infty} \frac{l_n(x)}{\log_2 n} &= \liminf_{k\to+\infty} \min\left\{ \frac{l_{n_k+m_{k-1}-n_{k-1}-1}(x)}{\log_2(n_k+m_{k-1}-n_{k-1}-1)}, \frac{l_{n_{k+1}-1}(x)}{\log_2(n_{k+1}-1)} \right\} \\ &= \liminf_{k\to+\infty} \min\left\{ \frac{[\beta \log_2 n_{k-1}]-1}{\log_2(n_k+[\beta\log_2 n_{k-1}]-1)}, \frac{[\beta\log_2 n_k]-1}{\log_2(n_{k+1}-1)} \right\} \\ &= \alpha, \end{aligned}$$

and

$$\begin{aligned} \limsup_{n\to+\infty} \frac{l_n(x)}{\log_2 n} &= \limsup_{k\to+\infty} \max\left\{ \frac{l_{n_k}(x)}{\log_2 n_k}, \frac{l_{m_k-1}(x)}{\log_2(m_k-1)} \right\} \\ &= \limsup_{k\to+\infty} \max\left\{ \frac{[\beta\log_2 n_{k-1}]-1}{\log_2 n_k}, \frac{[\beta\log_2 n_k]-1}{\log_2(n_k+[\beta\log_2 n_k]-1)} \right\} \\ &= \beta. \end{aligned}$$

Hence $x \in E_{\alpha,\beta}$.

In the following, we show that the density of $\mathcal{D} \subset \mathbb{N}$ is zero. Clearly, for any $n \geq n_{K+1}$, there exists $k \geq K+1$ such that $n_k \leq n < n_{k+1}$,

- if $n_k \leq n \leq m_k$, then

$$\#\{i \leq n,\ i \in \mathcal{D}\} = n_K + \sum_{j=K}^{k-1} \left[(m_j - n_j + 1) + t_j\right] + n - n_k;$$

- if $m_k + t(m_k - n_k) \leq n < m_k + (t+1)(m_k - n_k)$ for some $0 \leq t \leq t_k - 1$, then

$$\#\{i \leq n,\ i \in \mathcal{D}\} = n_K + \sum_{j=K}^{k-1} \left[(m_j - n_j + 1) + t_j\right] + m_k - n_k + t;$$

- if $m_k + t_k(m_k - n_k) \leq n < n_{k+1}$, then

$$\#\{i \leq n,\ i \in \mathcal{D}\} = n_K + \sum_{j=K}^{k-1} \left[(m_j - n_j + 1) + t_j\right] + m_k - n_k + t_k.$$

It follows that

$$\limsup_{n\to+\infty} \frac{1}{n} \#\{i \leq n,\ i \in \mathcal{D}\}$$

$$\le \limsup_{k\to+\infty} \max_{0\le t\le t_k} \left\{ \frac{n_K + \sum_{j=K}^{k-1}\left[(m_j - n_j + 1) + t_j\right] + m_k - n_k + t}{m_k + t(m_k - n_k)} \right\}$$

$$\le \limsup_{k\to+\infty} \left\{ \frac{n_K + \sum_{j=K}^{k-1}\left[(m_j - n_j + 1) + t_j\right] + m_k - n_k}{m_k} + \frac{1}{m_k - n_k} \right\}$$

$$= 0.$$

Therefore, by Proposition 2.1, we have $\dim_{\mathrm{H}} E = 1$.

Since the proof for the remaining cases is similar to the proof of the case $0 < \alpha < \beta < +\infty$. We will only give the constructions for the proper sequences $\{m_k\}_{k\ge1}$ and $\{n_k\}_{k\ge1}$. One can verify the corresponding $\mathcal{D}(\{m_k\}, \{n_k\})$ is of density zero and $E(\mathcal{D}, \{a_i\})$ with full Hausdorff dimension is a subset of $E_{\alpha,\beta}$ for different cases.

Case 1: $\alpha = 0$ and $\beta < +\infty$, take $n_k = 2^{2^{2^k}}$ and $m_k = n_k + \left[\beta \log_2 n_k\right]$ for $k \ge 1$.

Case 2: $\alpha > 0$ and $\beta = +\infty$, take $n_1 = 2$, $n_{k+1} = n_k^k$ and $m_k = n_k + \left[\alpha k \log_2 n_k\right]$ for $k \ge 1$.

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