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On a theorem of Reiter and spectral synthesis

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Abstract Motivated by and extending a theorem of Reiter on sets of synthesis in $\mathbb{R}^N$, we establish a general result for Fourier algebras of locally compact groups, even in the wider context of regular, semisimple and Tauberian commutative Banach algebras, which contains Reiter's theorem as a special case and explains why it holds. In addition, we give a number of examples and several further results on weak spectral sets.

Keywords Locally compact group · Fourier algebra · Set of synthesis · Ditkin set · Commutative Banach algebra · Structure space · Weak spectral set

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1 Introduction

By a classical result due to L. Schwartz the unit sphere S^{N-1} in $\mathbb{R}^N$ fails to be a set of synthesis for the Fourier algebra $A(\mathbb{R}^N)$ when $N \geq 3$. Subsequently, Reiter [20] proved an intriguing theorem saying that if F is any closed subset of $\mathbb{R}^N$, $N \geq 3$, then

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the set $F \cup S^{N-1}$ cannot be a set of synthesis unless F contains S^{N-1}. However, the sole fact that S^{N-1} disobeys synthesis does not explain why Reiter's theorem holds.

In this paper we prove a general result for the Fourier algebra $A(G)$ of an arbitrary locally compact group G, satisfying Ditkin's condition at infinity, which contains Reiter's theorem as a special case and at the same time explains why it holds (Sect. 3). Actually, our result is valid for general semisimple, regular, Tauberian commutative Banach algebras, which satisfy Ditkin's condition at infinity (Sect. 4). Moreover, we deal with weak spectral sets rather than just sets of synthesis. In Sect. 5 we present a number of examples illustrating the results of Sect. 4. These examples concern, in particular, conjugacy classes and double cosets of compact subgroups in G. Finally, in Sect. 6 we add a number of further results on weak spectral sets.

2 Preliminaries

Let A be a regular and semisimple commutative Banach algebra with structure space $\Delta(A)$ and Gelfand transform $a \to \widehat{a}$. For any subset M of A, the hull $h(M)$ of M is defined by $h(M) = \{\varphi \in \Delta(A) : \varphi(M) = \{0\}\}$. Associated to each closed subset E of $\Delta(A)$ are two distinguished ideals with hull equal to E, namely

$$k(E) = \{a \in A : \widehat{a}(\varphi) = 0 \quad \text{for all } \varphi \in E\}$$

and

$$j(E) = \{a \in A : \widehat{a} \text{ has compact support disjoint from } E\}.$$

Then $k(E)$ is the largest ideal with hull equal to E and $j(E)$ is the smallest such ideal, and consequently $J(E) = \overline{j(E)}$ is the smallest closed ideal with hull E. The set E is called a *set of synthesis* or *spectral set* if $k(E) = J(E)$. If $a \in \overline{aj(E)}$ for every $a \in k(E)$, then E is called a *Ditkin set*. Finally, A is said to satisfy *Ditkin's condition at infinity* if $\emptyset$ is a Ditkin set. As general references to spectral synthesis we mention [1,10,21,22].

For any Banach space X, the duality between X and its dual Banach space X^* is written as $\langle f, x\rangle$ or $\langle x, f\rangle$, $x \in X$, $f \in X^*$. For a subset M of X, $M^{\perp}$ will denote the annihilator of M in X^*.

Throughout Sects. 4 and 6 of the paper A will denote a commutative, regular, semisimple and Tauberian Banach algebra, which satisfies Ditkin's condition at infinity. We recall that A is said to be Tauberian if the set of all $a \in A$ such that $\widehat{a}$ has compact support is dense in A.

For $a \in A$ and $f \in A^*$, the functional $a \cdot f$ is defined by $\langle a \cdot f, b\rangle = \langle f, ab\rangle$, $b \in A$. It is clear that $\|a \cdot f\| \leq \|a\| \cdot \|f\|$ and, for $\gamma \in \Delta(A)$ and $a \in A$, $a \cdot \gamma = \widehat{a}(\gamma)\gamma = \langle \gamma, a\rangle\gamma$. As $\emptyset$ is a Ditkin set, $a \in \overline{aA}$ for each $a \in A$. This in turn implies that $\langle a, f\rangle = 0$ whenever $a \cdot f = 0$.

For $f \in A^*$, the *spectrum* $\sigma(f)$ of f is defined to be

$$\sigma(f) = \{\gamma \in \Delta(A) : \text{for each } a \in A, a \cdot f = 0 \text{ implies } \widehat{a}(\gamma) = 0\}.$$

The following properties of the spectrum $\sigma(f)$ are used throughout the paper:

(1) $\sigma(f) = \emptyset$ if and only if $f = 0$.
(2) For any $f \in A^*$ and $a \in A$, $\sigma(a \cdot f) \subseteq \sigma(f) \cap \operatorname{supp} \widehat{a}$.
(3) For any closed subset E of $\Delta(A)$, $\sigma(f) \subseteq E$ if and only if $f \in J(E)^\perp$.
(4) If $(f_\alpha)_\alpha$ is a net in A^* converging to f in the w^*-topology and $\sigma(f_\alpha) \subseteq E$ for some closed subset E of $\Delta(A)$, then $\sigma(f) \subseteq E$.

For closed ideals I and J of A, IJ denotes the closed ideal

$$\overline{\left\{\sum_{i=1}^{n} a_i b_i : a_i \in I,\ b_i \in J,\ 1 \le i \le n,\ n \in \mathbb{N}\right\}}.$$

Accordingly, for $k \in \mathbb{N}$, $k \geq 2$, I^k is inductively defined by $I^k = I^{k-1}I$. Using identities such as $4ab = (a+b)^2 - (a-b)^2$, a straightforward inductive argument shows that I^k is also the closed linear span of all the elements of the form a^k, $a \in I$.

3 Reiter's theorem for the Fourier algebra

As mentioned in the introduction, according to a classical result due to L. Schwartz, the unit sphere S^{N-1} in $\mathbb{R}^N = \Delta(L^1(\mathbb{R}^N))$ fails to be a set of synthesis for $N \geq 3$ (see [22, 7.3.1 and 7.3.2]). In an attempt to construct examples of functions in $k(S^{N-1}) \setminus J(S^{N-1})$, Reiter proved the intriguing result that if F is any closed subset of $\mathbb{R}^N$ which does not contain S^{N-1}, then $S^{N-1} \cup F$ fails to be a set of synthesis [20, Theorem 2]. In this section, we present a generalization of Reiter's theorem to Fourier algebras of arbitrary locally compact groups and explain why Reiter's theorem holds.

Let G be a locally compact group and $A(G)$ the Fourier algebra of G as introduced and extensively studied by Eymard [4]. This is a commutative, semisimple, regular and Tauberian Banach algebra of functions on G, which has since become one of the main objects of study in abstract harmonic analysis. The Gelfand spectrum of $A(G)$ can be identified with G, via the evaluation functionals. If G is abelian, then $A(G)$ is isometrically isomorphic to the L^1-algebra of the dual group $\widehat{G}$ of G. Recall that $A(G)^*$ is isomorphic to the $VN(G)$, the von Neumann algebra generated by the left regular representation of G on the Hilbert space $L^2(G)$. We will throughout assume that $u \in \overline{uA(G)}$ for every $u \in A(G)$. No locally compact group seems to be known for which this condition is not satisfied.

To every closed subset E of $G = \Delta(A(G))$ we associate the closed subset

$$\sigma_1(E) = \overline{\bigcup \{\sigma(u \cdot f) : u \in k(E),\ f \in J(E)^\perp\}}$$

of E and the closed ideal

$$I_1(E) = \{u \in A(G) : uk(E) \subseteq J(E)\}$$

of $A(G)$. The set $\sigma_1(E)$, which has been introduced in [24] and used there to study synthesis problems, is the hull of $I_1(E)$ and always contained in the boundary $\partial(E)$

of E. Moreover, by its very definition, $\sigma_1(E) = \emptyset$ if and only if the set E is a set of synthesis.

Let Φ be a group of homeomorphisms of G that acts continuously on $A(G)$ in the following sense.

(1) For each $u \in A(G)$ and $\phi \in \Phi$, the function $\phi(u)$ defined by $\phi(u)(x) = u(\phi^{-1}(x))$, $x \in G$, belongs to $A(G)$.
(2) If a sequence $(u_n)_n$ in $A(G)$ converges to u for some $u \in A(G)$, then $\phi(u_n) \to \phi(u)$ for every $\phi \in \Phi$.

Recall that Φ acts transitively on a subset E of G if for some (and hence every) $x \in E$, $E = \{\phi(x) : \phi \in \Phi\}$. In this case, $\phi(E) = E$ for each $\phi \in \Phi$.

The following theorem is a dichotomy result for such a set E. Concerning synthesibility, it behaves either very good or as bad as possible.

Theorem 3.1 *Let G be a locally compact group and suppose that Φ is a group of homeomorphisms of G which acts continuously on $A(G)$. Let E be a closed subset of G such that Φ acts transitively on E. Then either E is a set of synthesis or $\sigma_1(E) = E$. Moreover, E is a set of synthesis if and only if E contains a set of synthesis F whose relative interior is nonempty (i.e. $\overline{E \setminus F} \neq E$).*

Let $G = \mathbb{R}^N$, $N \geq 3$, $\Phi = SO(N)$, acting on $\mathbb{R}^N$ by rotation, and $E = S^{N-1}$. Then Φ acts transitively on E. Since S^{N-1} is not of synthesis for $A(\mathbb{R}^N)$, it follows from the preceding theorem that $\sigma_1(S^{N-1}) = S^{N-1}$. Note that in [13, Example 6.6] this was obtained as a consequence of Varopoulos's work [25]. Since, as shown by Herz [8], the unit circle is a set of synthesis for $A(\mathbb{R}^2)$, we see that both alternatives in Theorem 3.1 occur.

We now present a class of sets E for which $\sigma_1(E) = E$. Contrary to the sphere, these are thin sets. For any $f \in VN(G)$, set

$$X_f = \{u \cdot f : u \in A(G)\}.$$

Then X_f is a (not necessarily closed) $A(G)$-invariant subspace of $VN(G)$.

Proposition 3.2 *Let F be a closed subset of G and let $f \in VN(G)$ be such that $J(F)^\perp \cap X_f \neq \{0\}$, but $k(F)^\perp \cap X_f = \{0\}$. Then F contains a nonempty set E such that $\sigma_1(E) = E$.*

Proof Let $g \in J(F)^\perp \cap X_f$, $g \neq 0$, and set $E = \sigma(g)$. As

$$J(\sigma_1(E))k(E) \subseteq I_1(E)k(E) \subseteq J(E) \subseteq J(\sigma_1(E)),$$

we have $J(\sigma_1(E))k(E) = J(E)$. This equality implies that, for any $u \in J(\sigma_1(E))$, $u \cdot g \in k(E)^\perp \cap X_f$. However, since $E \subseteq F$, $k(E)^\perp \cap X_f = \{0\}$ and therefore $u \cdot g = 0$. As $u \in \overline{uA(G)}$, we get that $\langle u, g \rangle = 0$. This proves that $g \in J(\sigma_1(E))^\perp$, so that $E = \sigma(g) \subseteq \sigma_1(E)$ and hence $\sigma_1(E) = E$. □

In the context of metrizable locally compact abelian groups, for the existence of sets F satisfying the hypothesis of the preceding proposition, we refer the reader to [23].

Theorem 3.3 *Let E be a closed subset of G and suppose that $\sigma_1(E) = E$. Then, for any closed subset F of G, $E \cup F$ is a set of synthesis (if and) only if $E \subseteq F$ and F is a set of synthesis. Moreover, in this case, $\overline{F \backslash E} = F$.*

Reiter's theorem is now an immediate consequence of Theorem 3.3 since $\sigma_1(S^{N-1}) = S^{N-1}$, and we note that this equality is the reason for Reiter's theorem to hold. The next corollary follows from Theorem 3.1.

Corollary 3.4 *Let G, ϕ and E be as in Theorem 3.1. If E fails to be a set of synthesis, then E cannot be written as a union of countably many sets of synthesis.*

As do the previous results, the following corollary, which is an immediate consequence of Theorems 3.1 and 3.3, also holds in a more general setting (see Corollary 4.4)

Corollary 3.5 *Let E and F be closed subsets of $\mathbb{R}^N$, $N \geq 3$, such that $E \subseteq S^{N-1} \subseteq F$.*

(i) *If E is a set of synthesis for $A(\mathbb{R}^N)$, then $\overline{S^{N-1} \backslash E} = S^{N-1}$.*
(ii) *If F is a set of synthesis for $A(\mathbb{R}^N)$, then $\overline{F \backslash S^{N-1}} = F$.*

We refrain from presenting proofs of Theorems 3.1 and 3.3 and Corollaries 3.4 and 3.5 here because in the next section we shall prove abstract versions of these results in the general setting of semisimple, regular and Tauberian commutative Banach algebras. The purpose of stating the above results separately is to emphasize the motivation for what will be accomplished in Sect. 4. We continue with an example taken from [14].

Example 3.6 Let $T : \mathbb{R}^N \to \mathbb{R}^N$ be a linear map and let $\Phi = \{e^{sT} : s \in \mathbb{R}\}$ be the associated one-parameter group. Suppose that Φ is closed in $GL(N, \mathbb{R})$. Then, as proved in [14], for any compact connected subset H of Φ and any $x \in \mathbb{R}^N$, the set $H(x)$ is a set of synthesis for $A(\mathbb{R}^N)$ (more generally, for the so-called Figà-Talamanca-Herz algebras $A_p(\mathbb{R}^N)$, $1 < p < \infty$). Now, let $H_n = \{e^{sT} : -n \leq s \leq n\}$, $n \in \mathbb{N}$. Then, since Φ is closed in $GL(N, \mathbb{R})$, each orbit $\Phi(x)$ is closed in $\mathbb{R}^N$ (for instance, see [7, p.150, Exercise 5]). It now follows from Corollary 3.4 that $\Phi(x) = \bigcup_{n=1}^{\infty} H_n(x)$ is a set of synthesis.

4 An abstract version of Reiter's theorem and some consequences

From now on we allow the more general setting of a regular, semisimple and Tauberian commutative Banach algebra A with structure space $\Delta(A)$ and Gelfand transform $a \to \widehat{a}$. We assume in addition that $\emptyset$ is a Ditkin set. We also consider weak spectral sets rather than just sets of synthesis. These sets were introduced and first studied by Warner [26] in connection with the union problem for sets of synthesis. A closed subset E of $\Delta(A)$ is called a *weak spectral set* or *set of weak synthesis* if there exists $n \in \mathbb{N}$ such that $a^n \in J(E)$ for every $a \in k(E)$. The smallest such number n are denoted $\xi(E)$ and called the characteristic of E. Thus $\xi(E) = 1$ if and only if E is a set of synthesis. Using this terminology, the main result of [25] says that S^{N-1} is a weak spectral set for $A(\mathbb{R}^N)$ with $\xi(S^{N-1}) = \lfloor \frac{N+1}{2} \rfloor$.

We say that *weak spectral synthesis* holds for A if $\xi(E) < \infty$ for every closed subset E of $\Delta(A)$. Warner proved that the union of two weak spectral sets E and F is again a weak spectral set and that $\xi(E \cup F) \leq \xi(E) + \xi(F)$. Subsequently, weak spectral sets and the weak synthesis problem gained considerable attention and have been studied by several authors [9,11–13,16,17,19,26,27]. There are several important Banach algebras for which weak synthesis holds, whereas spectral synthesis fails. In contrast, as independently shown in [9,17], for a locally compact group G, weak spectral synthesis holds for the Fourier algebra $A(G)$ if and only if G is discrete.

We define, for each $n \in \mathbb{N}$, a closed ideal $I_n(E)$ of A by

$$I_n(E) = \{a \in A : ak(E)^n \subseteq J(E)\}.$$

Then $I_n(E)$ is the largest closed ideal I of A such that $Ik(E)^n \subseteq J(E)$. Moreover, for each n we define a subset $\sigma_n(E)$ of $\Delta(A)$ by

$$\sigma_n(E) = \overline{\bigcup \{\sigma(a \cdot f) : a \in k(E)^n, f \in J(E)^{\perp}\}}^{w^*},$$

where the w^*-closure is taken in $\Delta(A)$. The set $\sigma_1(E)$ was introduced in [24] to study sets of synthesis, and subsequently the decreasing sequence of sets $\sigma_n(E)$ and the increasing sequence of ideals $I_n(E)$, $n \in \mathbb{N}$, have been defined and employed in [13]. For some examples of Banach algebras they have been determined explicitly [13, Section 6]. Moreover, for every $n \in \mathbb{N}$, $\sigma_n(E) \subseteq \overline{\Delta_n(E)}$, where $\Delta_n(E)$ denotes the n-difference spectrum, which was introduced in [16] as a tool to study weak spectral synthesis. The following facts will be used several times in the sequel.

(i) For any closed subset E of $\Delta(A)$,

$$J(\sigma_n(E)) \subseteq I_n(E) \subseteq k(\sigma_n(E))$$

and hence $h(I_n(E)) = \sigma_n(E)$ [13, Lemma 2.2].

(ii) The set E is a weak spectral set if and only if $\sigma_n(E) = \emptyset$ for some $n \in \mathbb{N}$. Moreover, in this case $\xi(E)$ is the smallest such number n. In particular, E is a set of synthesis if and only if $\sigma_1(E) = \emptyset$ [13, Proposition 2.3]

Let now Φ be a group of homeomorphisms of $\Delta(A)$ such that

(1) for each $a \in A$ and $\phi \in \Phi$, there exists an element $\phi(a)$ of A such that $\widehat{\phi(a)}(\gamma) = \widehat{a}(\phi^{-1}(\gamma))$ for all $\gamma \in \Delta(A)$;
(2) if $\|a_n - a\| \to 0$, then $\|\phi(a_n) - \phi(a)\| \to 0$ for every $\phi \in \Phi$.

We then say that Φ acts continuously on A. Note that, since A is semisimple, $\phi(a)$ is uniquely determined and the map $a \to \phi(a)$ is an automorphism of A.

The following theorem is the abstract version of Theorem 3.1.

Theorem 4.1 *Let Φ be as above and let E be a closed subset of $\Delta(A)$ such that Φ acts transitively on E. Then either E is a weak spectral set or $\sigma_n(E) = E$ for every $n \in \mathbb{N}$. Furthermore, the set E is a weak spectral set with $\xi(E) \leq m$ for some $m \in \mathbb{N}$ if and only if E contains a weak spectral set F with $\xi(F) \leq m$ whose relative interior is nonempty.*

Proof For $a \in k(E)$, $\phi \in \Phi$ and $\gamma \in E$, we have $\widehat{\phi(a)}(\gamma) = \widehat{a}(\phi^{-1}(\gamma)) = 0$ since E if Φ-invariant. Thus the ideal $k(E)$ is Φ-invariant and hence so is $k(E)^m$ for every $m \in \mathbb{N}$ because each $\phi \in \Phi$ is a continuous automorphism of A. Moreover, if C is a compact subset of $\Delta(A)$ such that $C \cap E = \emptyset$, then $\phi(C) \cap E = \phi(E \cap C) = \emptyset$. It follows that $j(E)$, and consequently $J(E)$, is Φ-invariant. Now, for every $\phi \in \Phi$,

$$k(E)^m \phi(I_m(E)) = \phi(k(E)^m I_m(E)) \subseteq \phi(J(E)) = J(E).$$

Since $I_m(E)$ is the largest closed ideal I of A such that $k(E)^m I \subseteq J(E)$, we conclude that $\phi(I_m(E)) = I_m(E)$ for every $\phi \in \Phi$.

Now assume that $\sigma_m(E) \neq \emptyset$ and choose $\gamma \in \sigma_m(E)$. Then, since Φ acts transitively on E, $E = \{\phi(\gamma) : \phi \in \Phi\}$. Finally, since $h(I_m(E)) = \sigma_m(E)$ and the ideal $I_m(E)$ is Φ-invariant, it follows that $\widehat{a}$ vanishes on E for every $a \in I_m(E)$ and therefore $\sigma_m(E) = E$.

For the last assertion, assume that E fails to be a weak spectral set with $\xi(E) \leq m$. If $F \subseteq E$ is any weak spectral set with $\xi(F) \leq m$, then $E = \sigma_m(E) \subseteq \overline{E \backslash F}$ by [13, Lemma 3.1], whence F has empty interior in E. □

The next theorem is an abstract version of Theorem 3.3 and therefore of Reiter's theorem.

Theorem 4.2 *Let E be a closed subset of $\Delta(A)$ and suppose that, for some $m \in \mathbb{N}$, $\sigma_m(E) = E$. Then, for any closed subset F of $\Delta(A)$, $E \cup F$ is a weak spectral set with $\xi(E \cup F) \leq m$ (if and) only if $E \subseteq \overline{F \backslash E}$ and F is a weak spectral set with $\xi(F) \leq m$. Moreover, in this case $\overline{F \backslash E} = F$.*

Proof The first inclusion below being obvious, by hypothesis we have

$$k(E)^m k(F \backslash E)^m \subseteq k(E \cup F)^m \subseteq J(E \cup F) \subseteq J(E).$$

Hence, since $I_m(E)$ is the largest ideal I of A with $Ik(E)^m \subseteq J(E)$,

$$k(F \backslash E)^m \subseteq I_m(E) \subseteq k(\sigma_m(E)) = k(E).$$

Thus $E \subseteq \overline{F \backslash E}$, and since $E \cup F = E \cup (F \backslash E) \subseteq \overline{F \backslash E}$, we conclude that $F = \overline{F \backslash E}$. It follows that

$$k(F)^m = k(\overline{F \backslash E})^m = k(E \cup F)^m \subseteq J(E \cup F) \subseteq J(F).$$

This proves that $\xi(F) \leq m$ and also that $\overline{F \backslash E} = F$. □

Corollary 4.3 *Let E and Φ be as in Theorem 4.1, and let F and S be closed subsets of $\Delta(A)$ such that $E \subseteq S \subseteq E \cup F$. Suppose that S is a weak spectral set and that $\xi(E) = \infty$ or $\xi(E) > \xi(S)$. Then $E \subseteq \overline{F \backslash E}$ and hence $F = \overline{F \backslash E}$.*

Proof Let $m = \xi(S)$. Since $\xi(E) > m$, $\sigma_m(E) \neq \emptyset$ and hence $\sigma_m(E) = E$ by Theorem 4.1 . The statement now follows from Theorem 4.2. □

Corollary 4.4 *Let Φ be as above and let E be a closed subset of $\Delta(A)$ such that Φ acts transitively on E. Then the following two conditions are equivalent.*

(i) *E is a set of weak synthesis.*

(ii) *E can be written as a countable union of weak spectral sets.*

Moreover, if $E = \bigcup_{n=1}^{\infty} E_n$, where each E_n is a weak spectral set, and if $m \in \mathbb{N}$ is such that E_m has nonempty relative interior E_m° in E, then $\xi(E) \leq \xi(E_m)$.

Proof (i) $\Rightarrow$ (ii) is trivial. For (ii) $\Rightarrow$ (i), assume that $E = \bigcup_{n=1}^{\infty} E_n$, where each E_n is weak spectral set, and nevertheless E is not a set of weak synthesis. Then $\sigma_m(E) = E$ for all $m \in \mathbb{N}$ by Theorem 4.1. Since each E_n is of weak synthesis, by [13, Lemma 3.1] we have $\sigma_{\xi(E_n)}(E) \subseteq \overline{E\backslash E_n}$ for every n. Since $E = \bigcup_{n=1}^{\infty} E_n$ and E is a Baire space, at least one of the sets E_n has nonempty interior in E. However, this contradicts $E = \sigma_{\xi(E_n)}(E) \subseteq \overline{E\backslash E_n}$.

For the additional statement, note that we have $\xi(\phi(E_m)) = \xi(E_m)$ for each $\phi \in \Phi$, and since Φ acts transitively on E, $E = \bigcup_{\phi \in \Phi} \phi(E_m^\circ)$ whenever $E_m^\circ \neq \emptyset$. Thus every point of E has a closed relative neighbourhood in E with characteristic equal to $\xi(E_m)$. The statement now follows from [9, Proposition 1.6 and the remark following it]. □

Corollary 4.5 *Let E, S and F be closed subsets of $\Delta(A)$ such that $E \subseteq S \subseteq F$, and suppose that $\sigma_m(S) = S$ for some $m \in \mathbb{N}$.*

(i) *If E is a weak spectral set with $\xi(E) \leq m$, then $\overline{S\backslash E} = S$.*

(ii) *If F is a weak spectral set with $\xi(F) \leq m$, then $\overline{F\backslash S} = F$.*

Proof (i) Since $\sigma_m(S) = S$, S is not a weak spectral set with $\xi(S) \leq m$ and therefore, by Theorem 4.1, the relative interior of the subset E of S must be empty.

(ii) follows from Theorem 4.2 taking $E = S$ since $F \supseteq S$.

5 Examples

In this section we exhibit a number of examples in which Theorem 4.1 can be used to conclude that $\sigma_n(E) = E$. These examples concern closures of conjugacy classes and double cosets KaK, $a \in G$, where K is a compact subgroup of G, in $G = \Delta(A(G))$ and also the Fourier algebra of coset spaces G/K.

Let K be a compact subgroup of G. Then $K \times K$ acts on G by $(x, y) \cdot z = x^{-1}zy$, $x, y \in K$, $z \in G$, and it acts transitively on each double coset KzK. It is clear that $K \times K$ acts continuously on $A(G)$ by $(x, y)(u)(z) = u(x^{-1}zy)$.

Example 5.1 Consider $SO(N)$, $N \geq 3$, and identify $SO(N-1)$ with the subgroup of all elements of $SO(N)$ which fix the vector $(1, 0, \ldots, 0)^t \in \mathbb{R}^N$. Note that there is a bijection

$$[-1, 1] \to SO(N)//SO(N-1), \quad t \to SO(N-1)A(t)SO(N-1),$$

where

$$A(t) = \begin{pmatrix} \begin{pmatrix} t & -\sqrt{1-t^2} \\ \sqrt{1-t^2} & t \end{pmatrix} & 0_{2,d-2} \\ 0_{d-2,2} & 1_{d-2,d-2} \end{pmatrix}$$

and $O_{n,m}$ and $1_{n,m}$ denotes the zero and identity matrix of size $n \times m$, respectively. Let $E_t = SO(N-1)A(t)SO(N-1)$. Then, by [3, Corollary 3.4], $\xi(E_t) \geq \lfloor \frac{N+1}{2} \rfloor$ for every $t \in]-1, 1[$. Since $SO(N-1) \times SO(N-1)$ acts transitively on E_t, we conclude that $\sigma_n(E_t) = E_t$ for all $n < \lfloor \frac{N+1}{2} \rfloor$ and $t \in]-1, 1[$. However, we do not know whether these sets E_t are weak spectral sets. In contrast, the sets E_1 and E_{-1} are sets of synthesis, because $E_1 = SO(N-1)$ and $E_{-1} = A(-1)SO(N-1)$ since $A(-1)$ commutes with all matrices in $SO(N-1)$.

In the next example, for any $a \in G$, let $C(a) = \overline{\{xax^{-1} : x \in G\}}$, the closure of the conjugacy class of a.

Example 5.2 (1) Let G be a 2-step nilpotent locally compact group and let Z denote its centre. For $x, y \in G$, let $[y, x] = yxy^{-1}x^{-1}$ denote the commutator of y and x. Then, for $a \in G$, $C(a) = a\overline{\{[a^{-1}, x] : x \in G\}}$ and the map $x \to [a^{-1}, x]$ is a homomorphism from G into Z. Thus $\overline{C(a)}$ is a coset of some closed subgroup of G and hence a set of synthesis.

(2) In [15] Meaney has investigated the problem of whether conjugacy classes in compact connected Lie groups are sets of synthesis. We remind the reader that an element of a compact connected Lie group G is called regular if it is not contained in two distinct maximal tori and that the set of regular elements of G has full Haar measure. One of the main results of [15] says that if G is semisimple and a is a regular element of G, then $C(a)$ fails to be a set of synthesis [15, Theorem 3.3]. From this it can be deduced that if G is any nonabelian compact connected Lie group, then there exist elements of G whose conjucacy classes are not of synthesis [15, Corollary 3.6].

(3) Let G be a semisimple compact connected Lie group and fix a maximal torus T of G. Then, for any regular element a of T, $\xi(C(a)) > \frac{1}{2} \dim C(a)$ [15, Corollary 3.5]. Since G acts transitively on $C(a)$, we get $\sigma_n(C(a)) = C(a)$ for all $n \leq \frac{1}{2} \dim C(a)$ (Theorem 4.1). We note that the regular elements a of T are characterized by the equation $\dim C(a) = \dim G - \dim T$ [2, Theorem 2.11].

Now consider the special case $G = SO(N)$, $N \geq 3$, and let $m = \lfloor \frac{N}{2} \rfloor$. Then a maximal torus $T(N)$ of G is given by $T(N) = SO(2) \times \cdots \times SO(2)$ (m-fold direct product), where $T(N)$ acts on $\mathbb{R}^{2m} = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ and $\mathbb{R}^{2m+1} = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R}$ in the obvious manner [2, Theorem 3.4]. Since $\dim SO(N) = \frac{1}{2}N(N-1)$, we obtain

$$\frac{1}{2}(\dim SO(N) - \dim T(N)) = \begin{cases} m^2 & : \text{ if } N = 2m+1 \\ m(m-1) & : \text{ if } N = 2m \end{cases}.$$

Therefore, for any regular element $a \in T(N)$, $\sigma_n(C(a)) = C(a)$ at least for all $n \leq \lfloor \frac{N}{2} \rfloor \left(\lfloor \frac{N}{2} \rfloor - 1 \right)$.

Remark 5.3 Let G be a locally compact group and K a compact subgroup of G. The Fourier algebra $A(G/K)$ of the space of left cosets of K in G was introduced by Forrest [6]. It can be identified with the subalgebra of $A(G)$ consisting of all functions in $A(G)$ which are constant on left cosets of K, and then point evaluations provide a homeomorphism between G/K and $\Delta(A(G/K))$. For further results on $A(G/K)$ see [18]. Note that in general $A(G/K)$ is much smaller than $A(G)$. In fact,

$u \in A(G)$ belongs to $A(G/K)$ if and only if u can be represented as $u = f * \check{g}$, where $f, g \in L^2(G)$ and $L_k g = g$ for all $k \in K$ [12, Lemma 5.1].

Let H be a closed subgroup of G. Then H acts continuously on G/K and on $A(G/K)$ by $h(u)(xK) = u(h^{-1}xK)$. Let $x \in G$ and $E = H(xK) = \{hxK : h \in H\}$. Then H acts transitively on E and hence, for each $m \in \mathbb{N}$, either $\sigma_m(E) = E$ or $\sigma_m(E) = \emptyset$ by Theorem 4.1.

Now suppose that G is a semidirect product $G = N \rtimes K$, and identify N with the normal subgroup $N \times \{e_K\}$ and K with the subgroup $\{e_N\} \times K$ of G. Then $A(G/K)$ can equally well be viewed as a subalgebra of $A(N)$, and we have $\Delta(A(G/K)) = N$. Indeed, the restriction map $u \to u|_N$ is an isometric isomorphism of $A(G/K)$ onto its range B, say. For $\|u|_N\| = \|u\|$, it suffices to show that $\|u\| \leq \|u|_N\|$. There exists $v \in A(G)$ such that $v|_N = u$ and $\|v\| = \|u|_N\|$. Then define $w \in A(G/K)$ by $w(x) = \int_K v(xk)dk$, where dk is normalized Haar measure of K, and notice that $\|w\| \leq \|v\|$ and $w|_N = v|_N = u|_N$, whence $w = u$.

In our final example, we take for G the motion group of $\mathbb{R}^N$ and for both, K and H, the subgroup $SO(N)$.

Example 5.4 Let $G_N = \mathbb{R}^N \rtimes SO(N)$, $N \geq 2$, and let B denote the algebra $A(G_N/SO(N))$, viewed as a subalgebra of $A(\mathbb{R}^N)$ (Remark 5.3). We want to determine the sets $\sigma_n(S^{N-1})$ for $S^{N-1} \subseteq \Delta(B)$. This will be done by exploiting the main result of [25].

Let R denote the subalgebra of all radial functions in $A(\mathbb{R}^N)$, and for any $u \in R$ define $\widetilde{u}$ on G_N by $\widetilde{u}(x, T) = u(x)$. If u is positive definite, then so is $\widetilde{u}$. In fact, for any $x_1, \ldots, x_n \in \mathbb{R}^N$, $T_1, \ldots, T_n \in SO(N)$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, we have

$$\begin{aligned}\sum_{i,j=1}^n \lambda_i \overline{\lambda_j}\, \widetilde{u}((x_j, T_j)^{-1}(x_i, T_i)) &= \sum_{i,j=1}^n \lambda_i \overline{\lambda_j}\, \widetilde{u}(T_j^{-1}(x_i - x_j), T_j^{-1}T_i) \\ &= \sum_{i,j=1}^n \lambda_i \overline{\lambda_j}\, u(T_J^{-1}(x_i - x_j)) \\ &= \sum_{i,j=1}^n \lambda_i \overline{\lambda_j}\, u(x_i - x_j) \geq 0.\end{aligned}$$

Since every function in R is a finite linear combination of positive definite functions in R, it follows that the map $u \to \widetilde{u}$ is an embedding of R into $A(G_N/SO(N))$. In this manner, we view R as a subalgebra of B. Then, for each $n \in \mathbb{N}$,

$$(k(S^{N-1}) \cap R)^n \subseteq (k(S^{N-1}) \cap B)^n \subseteq k(S^{N-1}).$$

Now, by [25, Theorem 3 and Lemma 2(iv)], each inclusion in the following decreasing chain of ideals

$$k(S^{N-1}) \cap R \supseteq (k(S^{N-1}) \cap R)^2 \supseteq \cdots \supseteq (k(S^{N-1}) \cap R)^{\lfloor \frac{N+1}{2} \rfloor} = J(S^{N-1}) \cap R$$

is proper. It follows from this that the chain of ideals $(k(S^{N-1}) \cap B)^n$ has the same length. Indeed, if $(k(S^{N-1}) \cap B)^m = (k(S^{N-1}) \cap B)^{m+1}$ for some $1 \leq m < \lfloor \frac{N+1}{2} \rfloor$, then

$$(k(S^{N-1}) \cap B)^m = (k(S^{N-1}) \cap B)^{\lfloor \frac{N+1}{2} \rfloor} \subseteq k(S^{N-1})^{\lfloor \frac{N+1}{2} \rfloor} = J(S^{N-1}),$$

because the characteristic of $S^{N-1} \subseteq \Delta(A(\mathbb{R}^N)$ equals $\lfloor \frac{N+1}{2} \rfloor$. Hence $(k(S^{N-1}) \cap R)^m \subseteq J(S^{N-1})$, which is a contradiction. Since $SO(N)$ acts transitively on $S^{N-1} \subseteq \Delta(B)$, we conclude that

$$\sigma_n(S^{N-1}) = S^{N-1} \quad \text{for} 1 \leq n < \left\lfloor \frac{N+1}{2} \right\rfloor \quad \text{and} \quad \sigma_{\lfloor \frac{N+1}{2} \rfloor}(S^{N-1}) = \emptyset.$$

6 Further results on weak spectral sets in the spectrum of a commutative Banach algebra

In this section, as an application of our tools, $\sigma_n(E)$ and $I_n(E)$, we present a series of results on weak spectral sets which do not follow from those in the previous sections. The algebra A is assumed to have the properties specified at the beginning of Sect. 4.

Lemma 6.1 *Let E and D be closed subsets of $\Delta(A)$ such that $D \subseteq E$ and D is a Ditkin set. Then, for every $n \in \mathbb{N}$,*

$$\sigma_n(E) \subseteq \sigma_n(\overline{E \setminus D}) \quad \text{and} \quad \overline{\sigma_n(E) \setminus D} = \sigma_n(E).$$

In particular, if there is a family of Ditkin sets $D_\lambda \subseteq E$, $\lambda \in \Lambda$, such that $\bigcap_{\lambda \in \Lambda} \sigma_n(\overline{E \setminus D_\lambda}) = \emptyset$, then E is of weak synthesis with $\xi(E) \leq n$.

Proof Since D is a Ditkin set, $k(E) = k(\overline{E \setminus D})k(D)$ (see Corollary 6.4 below). So $k(E)^n = k(\overline{E \setminus D})^n k(D)^n = k(\overline{E \setminus D})^n k(D)$. For the first asserted inclusion it suffices to show that $\sigma((ab) \cdot f) \subseteq \sigma_n(\overline{E \setminus D})$ for every $a \in k(\overline{E \setminus D})^n$, $b \in k(D)$ and $f \in J(E)^\perp$. Fix such a, b and f, and observe first that $\sigma(b \cdot f) \subseteq \overline{E \setminus D}$. In fact, since D is a set of synthesis, there exists a sequence $(b_j)_j$ in $j(D)$ converging to b. Then

$$\sigma(b_j \cdot f) \subseteq \sigma(f) \cap \text{supp}\widehat{b_j} \subseteq E \cap (\Delta(A) \setminus D) = E \setminus D$$

and therefore $\sigma(b \cdot f) \subseteq \overline{E \setminus D}$. As $a \in k(\overline{E \setminus D})^n$, it follows that $\sigma((ab) \cdot f) \subseteq \sigma_n(\overline{E \setminus D})$.

For $\sigma_n(E) \subseteq \overline{\sigma_n(E) \setminus D}$, it is enough to verify that if $a \in k(E)^n$ and $f \in J(E)^\perp$, then $\sigma(a \cdot f) \subseteq \overline{\sigma_n(E) \setminus D}$. Since $a \in k(D)$ and D is a Ditkin set, $a = \lim_{i \to \infty}(ab_i)$, where $b_i \in j(D)$. Then, because $\sigma(a \cdot f) \subseteq \sigma_n(E)$, for each i,

$$\sigma((ab_i) \cdot f) \subseteq \sigma(a \cdot f) \cap \text{supp}\widehat{b_i} \subseteq \sigma_n(E) \cap (\Delta(A) \setminus D) = \sigma_n(E) \setminus D.$$

Consequently, $\sigma(a \cdot f) \subseteq \overline{\sigma_n(E) \setminus D}$. This proves $\sigma_n(E) \subseteq \overline{\sigma_n(E) \setminus D}$ and hence equality. □

Next we give a sufficient condition for $k(E \cup F) = k(E)k(F)$ to hold.

Lemma 6.2 *Let E and F be closed subsets of $\Delta(A)$ and suppose that there exists a Ditkin set D such that $E \cap F \subseteq D \subseteq E \cup F$. Then*

$$k(E \cup F) = k(E)k(F).$$

Proof It suffices to show that $k(E \cup F) \subseteq k(E)k(F)$. To that end, note first that, since D is a set of synthesis,

$$k(D) = J(D) \subseteq J(E \cap F) = \overline{J(E) + J(F)} \subseteq \overline{k(E) + k(F)}.$$

Thus using that D is a Ditkin set,

$$a \in \overline{a\overline{(k(E) + k(F))}} = \overline{a(k(E) + k(F))}$$

for every $a \in k(D)$. Therefore, given $a \in k(E \cup F) \subseteq k(D)$, there are sequences $(a_i)_i \subseteq k(E)$ and $(b_i)_i \subseteq k(F)$ such that $\|a - a(a_i + b_i)\| \to 0$. As $aa_i \in k(E)k(F)$ and $ab_i \in k(E)k(F)$, we get that $a \in k(E)k(F)$. The reverse inclusion being trivial, we get $k(E \cup F) = k(E)k(F)$, . □

Remark 6.3 The conclusion of Lemma 6.2 also holds for closed subsets E and F of $\Delta(A)$ satisfying $k(E \cup F)^2 = k(E \cup F)$. To see this, let $a \in k(E \cup F)$ be given. Then, given $\epsilon > 0$, there exist $a_i, b_i \in k(E \cup F)$, $1 \leq i \leq n$, such that $\|a - \sum_{i=1}^n (a_i b_i)\| \leq \epsilon$. Since $k(E \cup F) = k(E) \cap k(F)$, each $a_i b_i$ belongs to $k(E)k(F)$. Since $\epsilon > 0$ is arbitrary, it follows that $a \in k(E)k(F)$. So $k(E \cup F) \subseteq k(E)k(F)$, as was to be shown.

We also note that if one of the ideals $k(E)$, $k(F)$ or $k(E \cup F)$ has an approximate identity, then $k(E \cup F)^2 = k(E \cup F)$.

As an immediate consequence of Lemma 6.2 we obtain

Corollary 6.4 *For any closed subset E of $\Delta(A)$ and Ditkin set $D \subseteq \Delta(A)$, $k(E \cup D) = k(E)k(D)$.*

Theorem 6.5 *Let E and F be closed subsets of $\Delta(A)$. If F is a weak spectral set and $m = \xi(F)$, then $E \cup F$ is a weak spectral set with $\xi(E \cup F) \leq m$ if and only if $\sigma_m(E) \subseteq F$ and $k(E \cup F)^m = k(E)^m k(F)^m$.*

Proof Suppose first that $E \cup F$ is a weak spectral set with $n = \xi(E \cup F) \leq m$. Then

$$k(E \cup F)^m \subseteq k(E \cup F)^n = J(E \cup F) = J(E)J(F) \subseteq k(E)^m k(F)^m$$

and hence $k(E \cup F)^m = k(E)^m k(F)^m$. Moreover,

$$J(F)k(E)^m \subseteq k(F)^m k(E)^m = k(E \cup F)^m = J(E \cup F) \subseteq J(E).$$

By the very definition of the ideal $I_m(E)$, this means that $J(F) \subseteq I_m(E)$. This implies $\sigma_m(E) = h(I_m(E)) \subseteq h(J(F)) = F$.

Conversely, assume that $\sigma_m(E) \subseteq F$ and $k(E \cup F)^m = k(E)^m k(F)^m$. Then $J(F)k(E)^m \subseteq J(E)$ and hence

$$\begin{aligned} k(E \cup F)^m &= k(E)^m k(F)^m = k(E)^m J(F) = k(E)^m J(F)^2 \\ &\subseteq J(E)J(F) = J(E \cup F). \end{aligned}$$

Thus $E \cup F$ is of weak synthesis with $\xi(E \cup F) \leq m$. □

It is worth pointing out that the condition $k(E)^n = k(E)$ for all $n \in \mathbb{N}$ does not imply that E is of weak synthesis. In fact, in [5, Theorem 5.3] an example is given of a regular uniform algebra on a compact Hausdorff space X such that $\Delta(A) = X$ and there exists a point $x \in X$ such that the singleton $\{x\}$ fails to be a set of synthesis, but $k(\{x\})^n = k(\{x\})$ holds for all $n \in \mathbb{N}$.

Corollary 6.6 *Let E and D be closed subsets of $\Delta(A)$ such that D is a Ditkin set. Then $E \cup D$ is a set of synthesis if and only if $\sigma_1(E) \subseteq D$.*

Proof Taking $F = D$ in Theorem 6.5, we see that $E \cup D$ being a set of synthesis forces $\sigma_1(E) \subseteq D$. Conversely, suppose that $\sigma_1(E) \subseteq D$. Since D is a Ditkin set, we have $k(E \cup D) = k(E)k(D)$ by Corollary 6.4. Another application of Theorem 6.5 shows that $E \cup D$ is a set of synthesis. □

Another immediate consequence of Theorem 6.5 is the following

Corollary 6.7 *Let E and F be closed subsets of $\Delta(A)$ such that F is of synthesis. If $k(E)$ has an approximate identity and $\sigma_1(E) \subseteq F$, then $E \cup F$ is a set of synthesis.*

This result shows that synthesibility of $E \cup F$ depends substantially on the position of both sets to each other.

It is well-known that if E is a closed subset of $\Delta(A)$ and if there exists a Ditkin set D such that $\partial(E) \subseteq D \subseteq E$, then E is a set of synthesis. In fact, this is a simple application of the local membership principle. We continue with a generalization in the context of weak spectral synthesis.

Theorem 6.8 *Let E be a closed subset of $\Delta(A)$ and $m \in \mathbb{N}$. Suppose that there are countably many Ditkin sets D_n, $n \in \mathbb{N}$, such that $\sigma_m(E) \subseteq \bigcup_{n=1}^{\infty} D_n \subseteq E$. Then E is a set of weak synthesis with $\xi(E) \leq m$.*

Proof By Lemma 6.1, each of the sets $V_n = \sigma_m(E) \backslash (\sigma_m(E) \cap D_n)$ is open and dense in $\sigma_m(E)$. Since $\sigma_m(E)$ is a Baire space, $\bigcap_{n=1}^{\infty} V_n$ is dense in $\sigma_m(E)$. However, since $\bigcup_{n=1}^{\infty} D_n \supseteq \sigma_m(E)$, $\bigcap_{n=1}^{\infty} V_n = \emptyset$. Thus $\sigma_m(E) = \emptyset$, whence E is a weak spectral set with $\xi(E) \leq m$.

If $D = \bigcup_{n=1}^{\infty} D_n$ is closed, then it is also a Ditkin set. Therefore, the main issue of the preceding theorem is that D is not assumed to be closed in $\Delta(A)$. The following corollary is a considerable extension of a result due to Warner [26].

Corollary 6.9 *Let E and F be closed subsets of $\Delta(A)$ and suppose that there exists a sequence $(D_n)_n$ of Ditkin sets such that $\partial(E) \cap F \subseteq \bigcup_{n=1}^{\infty} D_n \subseteq E$.*

(i) *If, for some $m \in \mathbb{N}$, $k(E)^m J(F) \subseteq J(E)$, then E is a weak spectral set with $\xi(E) \leq m$.*
(ii) *If $E \cup F$ is a weak spectral set, then so is E and $\xi(E) \leq \xi(E \cup F)$.*

Proof (i) Since $k(E)^m J(F) \subseteq J(E)$, we have $\sigma_m(E) \subseteq F$, and hence by hypothesis

$$\sigma_m(E) \subseteq \partial(E) \cap F \subseteq \bigcup_{n=1}^{\infty} D_n \subseteq E.$$

It follows now from Theorem 6.8 that E is of weak synthesis with $\xi(E) \leq m$.
(ii) Let $m = \xi(E \cup F)$. Then

$$k(E)^m J(F) \subseteq k(E)^m k(F)^m \subseteq k(E \cup F)^m = J(E \cup F) \subseteq J(E),$$

and hence the statement follows from (i) □

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