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Universal inequalities for eigenvalues of a system of elliptic equations of the drifting Laplacian

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Abstract Let Ω be a bounded domain in a n-dimensional Euclidean space $\mathbb{R}^n$. We study eigenvalues of an eigenvalue problem of a system of elliptic equations of the drifting Laplacian

$$\begin{cases} \mathbb{L}_\phi \mathbf{u} + \alpha(\nabla(\mathrm{div}\mathbf{u}) - \nabla\phi\mathrm{div}\mathbf{u}) = -\bar{\sigma}\mathbf{u}, \text{ in } \Omega; \\ \mathbf{u}|_{\partial\Omega} = 0. \end{cases}$$

Estimates for eigenvalues of the above eigenvalue problem are obtained. Furthermore, a universal inequality for lower order eigenvalues of the problem is also derived. Finally, we prove an universal inequality type Ashbaugh and Benguria for the drifting Laplacian on Riemannian manifold immersed in an unit sphere or a projective space.

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1 Introduction

Let $(M, <, >)$ be an n-dimensional compact Riemannian manifold with boundary (possibly empty), $\phi \in C^2(M)$ and $d\mu = e^{-\phi}d\nu$, where $d\nu$ is the Riemannian volume measure on M. The drifting Laplacian with respect to the weighted volume measure μ is given by

$$\mathbb{L}_\phi = \Delta - \nabla\phi\nabla.$$

Interesting results for the eigenvalues of the drifting Laplacian have been obtained in recent years, for example in the works of Ma-Liu [13,14] and Ma-Du [12]. In 2014, Xia-Xu [19] have investigated the eigenvalues of the Dirichlet problem of the drifting Laplacian on compact manifolds and got some universal inequalities for them. Besides, at the same year, Xia et al. [6] have drawn universal inequalities of Yang type for eigenvalues of the bi-drifting Laplacian problem on a compact Riemannian manifold with boundary (possibly empty) immersed in: an Euclidean space, a unit sphere or a projective space. Recently, Pereira et al. [15] have given some universal inequalities for the poly-drifting laplacian on bounded domains in a Euclidean space or a unit sphere.

Let Ω a bounded domain with smooth boundary in an n-dimensional Euclidean space $\mathbb{R}^n$. Consider an eigenvalue problem of a system of n elliptic equations

$$\begin{cases} \mathbb{L}_\phi\mathbf{u} + \alpha(\nabla(\text{div}\mathbf{u}) - \nabla\phi\ \text{div}\mathbf{u}) = -\bar{\sigma}\mathbf{u}, \ \text{in}\ \Omega; \\ \mathbf{u}|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where $\mathbb{L}_\phi$ is the drifting laplacian in $\mathbb{R}^n$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is a vector-valued function from Ω to $\mathbb{R}^n$, α is a non-negative constant, div$\mathbf{u}$ denotes the divergence of $\mathbf{u}$ and ∇f is the gradient of a function f.

Let

$$0 < \bar{\sigma}_1 \leq \bar{\sigma}_2 \leq \cdots \leq \bar{\sigma}_k \leq \cdots \to +\infty$$

be the eigenvalues of the problem (1). Here each eigenvalue is repeated according to its multiplicity. When $\phi \equiv 0$ and $n = 3$, the problem (1) describes the behavior of the elastic vibration [17]. The literature about this eigenvalue problem is extensive, more informations can be found in [4,5,7,10].

This paper is organized as follows. In Sect. 2, we shall establish some general estimates for eigenvalues of the problem (1), in particular, we shall give an universal inequality wich for $\phi \equiv 0$ is the same got in [4]. In the last section, we shall use an algebraic argument for to prove a universal inequality, without Rayleigh-Ritz, for the problem (1) and for the drifting-Dirichlet problem. Besides, we shall prove this inequality for Riemannian manifold immersed in either an euclidean space, a unit sphere or a projective space.

2 The first results

In this section we prove some general estimates for the problem (1). We shall prove the inequality wich for $\phi \equiv 0$ coincides with the inequality obtained in [4].

Lemma 2.1 *Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Denote by $\mathbb{L}_\phi$ the drifting operator of $\mathbb{R}^n$. Let $\bar{\sigma}_i$ denote the i-th eigenvalue of the eigenvalue problem (1) and $\mathbf{u_i}$ be the orthonormal vector-valued eigenfunction corresponding to $\bar{\sigma}_i$. For any function $f \in C^2(\Omega) \cap C^1(\partial\Omega)$, we have*

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 \left\{ \int_\Omega |\nabla f|^2 |\mathbf{u}_i|^2 d\mu + \alpha \int_\Omega |\nabla f \cdot \mathbf{u}_i|^2 d\mu \right\} \leq \sum_{i=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)||P_i(f)||^2, \tag{2}$$

and, for any positive constant δ,

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 \left\{ (1-\delta) \int_\Omega |\nabla f|^2 |\mathbf{u}_i|^2 d\mu - \delta\alpha \int_\Omega |\nabla f \cdot \mathbf{u}_i|^2 d\mu \right\} \leq \frac{1}{\delta}\sum_{i=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)||\nabla f \cdot \nabla(\mathbf{u}_i) + \frac{1}{2}\mathbf{u}_i \mathbb{L}_\phi f||^2, \tag{3}$$

where $P_i(f) = 2\nabla f \cdot \nabla(\mathbf{u}_i) + \mathbf{u}_i \mathbb{L}_\phi f + \alpha\{\nabla(\nabla f \cdot \mathbf{u}_i) + \nabla f \operatorname{div}(\mathbf{u}_i) - \nabla\phi(\nabla f \cdot \mathbf{u}_i)\}$ and $\nabla f \cdot \nabla(\mathbf{u}_i)$ is defined by

$$\nabla f \cdot \nabla(\mathbf{u}_i) = (\nabla f \cdot \nabla {u_i}^1, \nabla f \cdot \nabla {u_i}^2, \ldots, \nabla f \cdot \nabla {u_i}^n).$$

Proof We define the vector-valued functions $\mathbf{v}_i$ by $\mathbf{v}_i = f\mathbf{u}_i - \sum_{j=1}^{k} a_{ij}\mathbf{u}_j$, where $a_{ij} = \int_\Omega f\mathbf{u}_i\mathbf{u}_j d\mu$.

Since $\mathbf{v}_i|_{\partial\Omega} = 0$ and $\int_\Omega \mathbf{v}_i\mathbf{u}_j d\mu = 0$, then it follows from the Rayleigh-Ritz inequality [9] that

$$\bar{\sigma}_{k+1} \leq \frac{\int_\Omega \{-\mathbf{v}_i \mathbb{L}_\phi \mathbf{v}_i + \alpha(\operatorname{div}\mathbf{v}_i)^2\} d\mu}{\int_\Omega |\mathbf{v}_i|^2 d\mu}. \tag{4}$$

From the definition of $\mathbf{v}_i$, we derive

$$
\begin{aligned}
-\int_\Omega \mathbf{v}_i \mathbb{L}_\phi \mathbf{v}_i d\mu &= -\int_\Omega \mathbf{v}_i \mathbb{L}_\phi (f\mathbf{u}_i) d\mu + \sum_{j=1}^k a_{ij} \int_\Omega \mathbf{v}_i \mathbb{L}_\phi \mathbf{u}_j d\mu \\
&= -\int_\Omega \mathbf{v}_i \mathbb{L}_\phi (f\mathbf{u}_i) d\mu + \sum_{j=1}^k a_{ij} \int_\Omega \mathbf{v}_i \{-\bar{\sigma}_j \mathbf{u}_j - \alpha(\nabla(\operatorname{div}\mathbf{u}_j) - \nabla\phi \operatorname{div}\mathbf{u}_j)\} d\mu \\
&= -\int_\Omega \mathbf{v}_i (f\mathbb{L}_\phi \mathbf{u}_i + \mathbf{u}_i \mathbb{L}_\phi f + 2\nabla f \cdot \nabla \mathbf{u}_i) d\mu - \alpha \sum_{j=1}^k a_{ij} \int_\Omega \mathbf{v}_i (\nabla(\operatorname{div}\mathbf{u}_j) \\
&\quad -\nabla\phi \operatorname{div}\mathbf{u}_j) d\mu = -\int_\Omega \mathbf{v}_i f \{-\bar{\sigma}_i \mathbf{u}_i - \alpha(\nabla(\operatorname{div}\mathbf{u}_i) - \nabla\phi \operatorname{div}\mathbf{u}_i)\} d\mu \\
&\quad -\int_\Omega \mathbf{v}_i (\mathbf{u}_i \mathbb{L}_\phi f + 2\nabla f \cdot \nabla \mathbf{u}_i) d\mu - \alpha \sum_{j=1}^k a_{ij} \int_\Omega \mathbf{v}_i (\nabla(\operatorname{div}\mathbf{u}_j) - \nabla\phi \operatorname{div}\mathbf{u}_j) d\mu \\
&= \bar{\sigma}_i ||\mathbf{v}_i||^2 + \alpha \left\{ \int_\Omega \mathbf{v}_i f (\nabla(\operatorname{div}\mathbf{u}_i) - \nabla\phi \operatorname{div}\mathbf{u}_i) d\mu - \sum_{j=1}^k a_{ij} \right. \\
&\quad \left. \times \int_\Omega \mathbf{v}_i (\nabla(\operatorname{div}\mathbf{u}_j) - \nabla\phi \operatorname{div}\mathbf{u}_j) d\mu \right\} - \int_\Omega \mathbf{v}_i (\mathbf{u}_i \mathbb{L}_\phi f + 2\nabla f \cdot \nabla \mathbf{u}_i) d\mu. \qquad (5)
\end{aligned}
$$

We have that

$$
\begin{aligned}
&\int_\Omega \mathbf{v}_i f (\nabla(\operatorname{div}\mathbf{u}_i) - \nabla\phi \operatorname{div}\mathbf{u}_i) d\mu - \sum_{j=1}^k a_{ij} \int_\Omega \mathbf{v}_i (\nabla(\operatorname{div}\mathbf{u}_j) - \nabla\phi \operatorname{div}\mathbf{u}_j) d\mu \\
&\quad = -\int_\Omega \operatorname{div}\mathbf{u}_i (\operatorname{div}(f\mathbf{v}_i) - \nabla\phi \cdot (f\mathbf{v}_i)) d\mu - \int_\Omega f\mathbf{v}_i \nabla\phi \operatorname{div}\mathbf{u}_i d\mu \\
&\qquad + \sum_{j=1}^k a_{ij} \int_\Omega \operatorname{div}\mathbf{u}_j (\operatorname{div}\mathbf{v}_i - \nabla\phi \cdot \mathbf{v}_i) d\mu + \sum_{j=1}^k a_{ij} \int_\Omega \mathbf{v}_i \nabla\phi \operatorname{div}\mathbf{u}_j d\mu \\
&\quad = -\int_\Omega \operatorname{div}\mathbf{u}_i (f \operatorname{div}\mathbf{v}_i + \nabla f \cdot \mathbf{v}_i) d\mu + \sum_{j=1}^k a_{ij} \int_\Omega \operatorname{div}\mathbf{u}_j \operatorname{div}\mathbf{v}_i d\mu \\
&\quad = -\int_\Omega \operatorname{div}\mathbf{v}_i (f \operatorname{div}\mathbf{u}_i + \nabla f \cdot \mathbf{u}_i - \nabla f \cdot \mathbf{u}_i) d\mu - \int_\Omega \operatorname{div}\mathbf{u}_i (\nabla f \cdot \mathbf{v}_i) d\mu \\
&\qquad + \sum_{j=1}^k a_{ij} \operatorname{div}\mathbf{u}_j \operatorname{div}\mathbf{v}_i d\mu = -\int_\Omega \operatorname{div}\mathbf{v}_i \operatorname{div}(f\mathbf{u}_i) d\mu + \int_\Omega \operatorname{div}\mathbf{v}_i (\nabla f \cdot \mathbf{u}_i) d\mu \\
&\qquad - \int_\Omega \operatorname{div}\mathbf{u}_i (\nabla f \cdot \mathbf{v}_i) d\mu + \sum_{j=1}^k a_{ij} \int_\Omega \operatorname{div}\mathbf{u}_j \operatorname{div}\mathbf{v}_i d\mu
\end{aligned}
$$

$$= -\int_\Omega \mathrm{div}\mathbf{v}_i \mathrm{div}(f\mathbf{u}_i - \sum_{j=1}^k a_{ij}\mathbf{u}_j)d\mu + \int_\Omega (\nabla f\cdot\mathbf{u}_i)(\nabla\phi\cdot\mathbf{v}_i)d\mu$$
$$-\int_\Omega \mathbf{v}_i\nabla(\nabla f\cdot\mathbf{u}_i)d\mu - \int_\Omega \mathrm{div}\mathbf{u}_i(\nabla f\cdot\mathbf{v}_i)d\mu = -\int_\Omega (\mathrm{div}\mathbf{v}_i)^2 d\mu$$
$$-\int_\Omega \mathbf{v}_i(\nabla(\nabla f\cdot\mathbf{u}_i) + \nabla f\mathrm{div}\mathbf{u}_i - \nabla\phi(\nabla f\cdot\mathbf{u}_i)d\mu. \tag{6}$$

From (5) and (6), we derive

$$-\int_\Omega \mathbf{v}_i\mathbb{L}_\phi\mathbf{v}_i d\mu = \bar{\sigma}_i||\mathbf{v}_i||^2 - \alpha||\mathrm{div}\mathbf{v}_i||^2 - \alpha\int_\Omega \mathbf{v}_i\{\nabla(\nabla f\cdot\mathbf{u}_i) + \nabla f\mathrm{div}\mathbf{u}_i$$
$$-\nabla\phi(\nabla f\cdot\mathbf{u}_i)\}d\mu - \int_\Omega \mathbf{v}_i(\mathbf{u}_i\mathbb{L}_\phi f + 2(\nabla f\cdot\nabla\mathbf{u}_i))d\mu.$$

Therefore, from (4), we have

$$(\bar{\sigma}_{k+1} - \bar{\sigma}_i)||\mathbf{v}_i||^2 \le -\int_\Omega \mathbf{v}_i\{2(\nabla f\cdot\nabla\mathbf{u}_i) + \mathbf{u}_i\mathbb{L}_\phi f + \alpha[\nabla(\nabla f\cdot\mathbf{u}_i) + \nabla f\mathrm{div}\mathbf{u}_i$$
$$-\nabla\phi(\nabla f\cdot\mathbf{u}_i)]\}d\mu = -\int_\Omega \mathbf{v}_i P_i(f)d\mu.$$

Define $b_{ij} = \int_\Omega \left(\nabla f\cdot\nabla\mathbf{u}_i + \frac{1}{2}\mathbf{u}_i\mathbb{L}_\phi f\right)\mathbf{u}_j d\mu$. From Stokes' theorem, we infer

$$-\int_\Omega \mathbf{v}_i(2(\nabla f\cdot\nabla\mathbf{u}_i) + \mathbf{u}_i\mathbb{L}_\phi f)d\mu = -\int_\Omega f\mathbf{u}_i(2(\nabla f\cdot\nabla\mathbf{u}_i) + \mathbf{u}_i\mathbb{L}_\phi f)d\mu$$
$$+2\sum_{j=1}^k a_{ij}b_{ij} = -2\int_\Omega f\mathbf{u}_i(\nabla f\cdot\nabla\mathbf{u}_i)d\mu - \int_\Omega f\mathbf{u}_i^2\mathbb{L}_\phi f d\mu + 2\sum_{j=1}^k a_{ij}b_{ij}$$
$$= \int_\Omega |\mathbf{u}_i|^2|\nabla f|^2 d\mu + 2\sum_{j=1}^k a_{ij}b_{ij} \tag{7}$$

and

$$-\alpha\int_\Omega \mathbf{v}_i\{\nabla(\nabla f\cdot\mathbf{u}_i) + \nabla f\mathrm{div}\mathbf{u}_i - \nabla\phi(\nabla f\cdot\mathbf{u}_i)\}d\mu$$
$$= -\alpha\int_\Omega f\mathbf{u}_i\{\nabla(\nabla f\cdot\mathbf{u}_i) + \nabla f\mathrm{div}\mathbf{u}_i - \nabla\phi(\nabla f\cdot\mathbf{u}_i)\}d\mu$$
$$+\alpha\sum_{j=1}^k a_{ij}\int_\Omega \mathbf{u}_j\{\nabla(\nabla f\cdot\mathbf{u}_i) + \nabla f\mathrm{div}\mathbf{u}_i - \nabla\phi(\nabla f\cdot\mathbf{u}_i)\}d\mu$$
$$= \alpha\int_\Omega (\nabla f\cdot\mathbf{u}_i)[\mathrm{div}(f\mathbf{u}_i) - \nabla\phi\cdot f\mathbf{u}_i]d\mu - \alpha\int_\Omega f\mathbf{u}_i\nabla f\mathrm{div}\mathbf{u}_i d\mu$$

$$
\begin{aligned}
&+\alpha\int_\Omega f\mathbf{u}_i(\nabla\phi(\nabla f\cdot\mathbf{u}_i))d\mu-\alpha\sum_{j=1}^k a_{ij}\int_\Omega(\nabla f\cdot\mathbf{u}_i)[\mathrm{div}\mathbf{u}_j-\nabla\phi\cdot\mathbf{u}_j]d\mu\\
&+\alpha\sum_{j=1}^k a_{ij}\int_\Omega\mathbf{u}_j\nabla f\mathrm{div}\mathbf{u}_i d\mu-\alpha\sum_{j=1}^k a_{ij}\int_\Omega\mathbf{u}_j\nabla\phi(\nabla f\cdot\mathbf{u}_i)d\mu\\
=&\,\alpha\int_\Omega(\nabla f\cdot\mathbf{u}_i)[f\mathrm{div}\mathbf{u}_i+\nabla f\cdot\mathbf{u}_i]d\mu-\alpha\int_\Omega f\mathbf{u}_i\nabla f\mathrm{div}\mathbf{u}_i d\mu\\
&-\alpha\sum_{j=1}^k a_{ij}\int_\Omega((\nabla f\cdot\mathbf{u}_i)\mathrm{div}\mathbf{u}_j-(\nabla f\cdot\mathbf{u}_j)\mathrm{div}\mathbf{u}_i)d\mu=\alpha\int_\Omega(\nabla f\cdot\mathbf{u}_i)^2d\mu\\
&-\alpha\sum_{j=1}^k a_{ij}\int_\Omega((\nabla f\cdot\mathbf{u}_i)\mathrm{div}\mathbf{u}_j-(\nabla f\cdot\mathbf{u}_j)\mathrm{div}\mathbf{u}_i)d\mu. \qquad (8)
\end{aligned}
$$

With the inequalities obtained in (7) and (8), we have

$$
\begin{aligned}
(\bar\sigma_{k+1}-\bar\sigma_i)^2||\mathbf{v}_i||^2\le&\int_\Omega|\mathbf{u}_i|^2|\nabla f|^2d\mu+2\sum_{j=1}^k a_{ij}b_{ij}+\alpha\int_\Omega(\nabla f\cdot\mathbf{u}_i)^2d\mu\\
&-\alpha\sum_{j=1}^k a_{ij}\int_\Omega((\nabla f\cdot\mathbf{u}_i)\mathrm{div}\mathbf{u}_j-(\nabla f\cdot\mathbf{u}_j)\mathrm{div}\mathbf{u}_i)d\mu.
\end{aligned}
$$

Moreover, we derive

$$
\begin{aligned}
b_{ij}=&\int_\Omega\mathbf{u}_j\left((\nabla f\cdot\nabla\mathbf{u}_i)+\frac{1}{2}\mathbf{u}_i\mathbb{L}_\phi f\right)d\mu=\frac{1}{2}\int_\Omega\mathbf{u}_j[\mathbb{L}_\phi(f\mathbf{u}_i)-f\mathbb{L}_\phi\mathbf{u}_i]d\mu\\
=&\frac{1}{2}\int_\Omega f\mathbf{u}_i\mathbb{L}_\phi\mathbf{u}_jd\mu-\frac{1}{2}\int_\Omega f\mathbf{u}_j\mathbb{L}_\phi\mathbf{u}_id\mu=\frac{1}{2}\int_\Omega f\mathbf{u}_i[-\bar\sigma_j\mathbf{u}_j-\alpha(\nabla(\mathrm{div}\mathbf{u}_j)\\
&-\nabla\phi\mathrm{div}\mathbf{u}_j)]d\mu-\frac{1}{2}\int_\Omega f\mathbf{u}_j[-\bar\sigma_i\mathbf{u}_i-\alpha(\nabla(\mathrm{div}\mathbf{u}_i)-\nabla\phi\mathrm{div}\mathbf{u}_i)]d\mu\\
=&\frac{1}{2}\int_\Omega(\bar\sigma_i-\bar\sigma_j)f\mathbf{u}_i\mathbf{u}_jd\mu+\frac{\alpha}{2}\int_\Omega\mathrm{div}\mathbf{u}_j[\mathrm{div}(f\mathbf{u}_i)-\nabla\phi\cdot(f\mathbf{u}_i)]d\mu\\
&+\frac{\alpha}{2}\int_\Omega f\mathbf{u}_i\nabla\phi\mathrm{div}\mathbf{u}_jd\mu-\frac{\alpha}{2}\int_\Omega\mathrm{div}\mathbf{u}_i[\mathrm{div}(f\mathbf{u}_j)-\nabla\phi\cdot(f\mathbf{u}_j)]d\mu\\
&-\frac{\alpha}{2}\int_\Omega f\mathbf{u}_j\nabla\phi\mathrm{div}\mathbf{u}_id\mu=\frac{1}{2}(\bar\sigma_i-\bar\sigma_j)a_{ij}+\frac{\alpha}{2}\int_\Omega\mathrm{div}\mathbf{u}_j[f\mathrm{div}\mathbf{u}_i\\
&+\nabla f\cdot\mathbf{u}_i]d\mu-\frac{\alpha}{2}\int_\Omega\mathrm{div}\mathbf{u}_i[f\mathrm{div}\mathbf{u}_j+\nabla f\cdot\mathbf{u}_j]d\mu\\
=&\frac{1}{2}(\bar\sigma_i-\bar\sigma_j)a_{ij}-\frac{\alpha}{2}\int_\Omega(\mathrm{div}\mathbf{u}_i(\nabla f\cdot\mathbf{u}_j)-\mathrm{div}\mathbf{u}_j(\nabla f\cdot\mathbf{u}_i))d\mu.
\end{aligned}
$$

Thus,

$$2b_{ij} = (\bar{\sigma}_i - \bar{\sigma}_j)a_{ij} - \alpha \int_\Omega (\text{div}\mathbf{u}_i(\nabla f \cdot \mathbf{u}_j) - \text{div}\mathbf{u}_j(\nabla f \cdot \mathbf{u}_i))d\mu. \tag{9}$$

Hence,

$$(\bar{\sigma}_{k+1} - \bar{\sigma}_i)||\mathbf{v}_i||^2 \leq \int_\Omega |\mathbf{u}_i|^2|\nabla f|^2 d\mu + \sum_{j=1}^{k}(\bar{\sigma}_i - \bar{\sigma}_j)a_{ij}^2 + \alpha \int_\Omega (\nabla f \cdot \mathbf{u}_i)^2 d\mu.$$

From (7) and (9), we have

$$\begin{aligned}
&(\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 \left(\int_\Omega |\mathbf{u}_i|^2|\nabla f|^2 d\mu + \sum_{j=1}^{k}(\bar{\sigma}_i - \bar{\sigma}_j)a_{ij}^2 + \alpha \int_\Omega (\nabla f \cdot \mathbf{u}_i)^2 d\mu\right)^2 \\
&= (\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 \left(-\int_\Omega \mathbf{v}_i P_i(f) d\mu\right)^2 \\
&= (\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 \left(-\int_\Omega \mathbf{v}_i [P_i(f) - \sum_{j=1}^{k}(\bar{\sigma}_i - \bar{\sigma}_j)a_{ij}\mathbf{u}_j] d\mu\right)^2 \\
&\leq (\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2||\mathbf{v}_i||^2 \left(\int_\Omega (p_i(f) - \sum_{j=1}^{k}(\bar{\sigma}_i - \bar{\sigma}_j)a_{ij}\mathbf{u}_j)^2 d\mu\right) \\
&= (\bar{\sigma}_i - \bar{\sigma}_j)^2||\mathbf{v}_i||^2 \left(||P_i(f)||^2 - \sum_{j=1}^{k}(\bar{\sigma}_i - \bar{\sigma}_j)^2 a_{ij}^2\right) \\
&\leq (\bar{\sigma}_i - \bar{\sigma}_j) \left(\int_\Omega |\mathbf{u}_i|^2|\nabla f|^2 d\mu + \sum_{j=1}^{k}(\bar{\sigma}_i - \bar{\sigma}_j)a_{ij}^2 + \alpha \int_\Omega (\nabla f \cdot \mathbf{u}_i)^2 d\mu\right) \\
&\quad \times \left(||P_i(f)||^2 - \sum_{j=1}^{k}(\bar{\sigma}_i - \bar{\sigma}_j)^2 a_{ij}^2\right).
\end{aligned}$$

Summing over $i = 1, \ldots, k$, we infer

$$\begin{aligned}
&\sum_{i=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 \left(\int_\Omega |\mathbf{u}_i|^2|\nabla f|^2 d\mu + \alpha \int_\Omega (\nabla f \cdot \mathbf{u}_i)^2 d\mu\right) + \sum_{i,j=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 \\
&\quad \times (\bar{\sigma}_i - \bar{\sigma}_j)a_{ij}^2 \leq \sum_{i=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)||P_i(f)||^2 - \sum_{i,j=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)(\bar{\sigma}_i - \bar{\sigma}_j)^2 a_{ij}^2.
\end{aligned} \tag{10}$$

Since a_{ij} is anti-symmetric, it follows

$$\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2(\bar{\sigma}_i-\bar{\sigma}_j)a_{ij}^2 = \sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_{k+1}-\bar{\sigma}_j)(\bar{\sigma}_i-\bar{\sigma}_j)a_{ij}^2 - \sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_i-\bar{\sigma}_j)^2a_{ij}^2 = -\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_i-\bar{\sigma}_j)^2a_{ij}^2.$$

This finishes the proof of inequality (2).
In order to obtain the inequality (3), we derive

$$\begin{aligned}
&(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2\left(\int_\Omega |\mathbf{u}_i|^2|\nabla f|^2 d\mu + 2\sum_{j=1}^{k} a_{ij}b_{ij}\right)\\
&= (\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2\left(-\int_\Omega \mathbf{v}_i\{2\,(\nabla f\cdot\nabla\mathbf{u}_i)+\mathbf{u}_i\mathbb{L}_\phi f\}d\mu\right)\\
&= (\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2\left(-2\int_\Omega \mathbf{v}_i\left\{(\nabla f\cdot\nabla\mathbf{u}_i)+\frac{1}{2}\mathbf{u}_i\mathbb{L}_\phi f-\sum_{j=1}^{k}b_{ij}\mathbf{u}_j\right\}d\mu\right)\\
&\le \delta(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^3||\mathbf{v}_i||^2+\frac{1}{\delta}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)\left(||(\nabla f\cdot\nabla\mathbf{u}_i)+\frac{1}{2}\mathbf{u}_i\mathbb{L}_\phi f||^2-\sum_{j=1}^{k}b_{ij}^2\right)\\
&\le \delta(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2\left(\int_\Omega |\mathbf{u}_i|^2|\nabla f|^2 d\mu+\sum_{j=1}^{k}(\bar{\sigma}_i-\bar{\sigma}_j)a_{ij}^2+\alpha\int_\Omega(\nabla f\cdot\mathbf{u}_i)^2 d\mu\right)\\
&\quad+\frac{1}{\delta}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)\left(||(\nabla f\cdot\nabla\mathbf{u}_i)+\frac{1}{2}\mathbf{u}_i\mathbb{L}_\phi f||^2-\sum_{j=1}^{k}b_{ij}^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2\left(\int_\Omega |\mathbf{u}_i|^2|\nabla f|^2 d\mu\right)+2\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2a_{ij}b_{ij}\\
&\quad\le \delta(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2\left(\int_\Omega |\mathbf{u}_i|^2|\nabla f|^2 d\mu+\alpha\int_\Omega(\nabla f\cdot\mathbf{u}_i)^2 d\mu\right)
\end{aligned}$$

$$+\delta\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2(\bar{\sigma}_i-\bar{\sigma}_j)a_{ij}^2+\frac{1}{\delta}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)$$
$$\times\left(||(\nabla f\cdot\nabla\mathbf{u}_i)+\frac{1}{2}\mathbf{u}_i\mathbb{L}_\phi f||^2\right)-\frac{1}{\delta}\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)b_{ij}^2.$$

Since a_{ij} is symmetric and b_{ij} is anti-symmetric, we have

$$2\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2a_{ij}b_{ij}=2\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_{k+1}-\bar{\sigma}_j)a_{ij}b_{ij}$$
$$-2\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_i-\bar{\sigma}_j)a_{ij}b_{ij}=-2\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_i-\bar{\sigma}_j)a_{ij}b_{ij}, \tag{11}$$

$$\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2(\bar{\sigma}_i-\bar{\sigma}_j)a_{ij}^2=\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_{k+1}-\bar{\sigma}_j)(\bar{\sigma}_i-\bar{\sigma}_j)a_{ij}^2$$
$$-\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_i-\bar{\sigma}_j)^2a_{ij}^2=-\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_i-\bar{\sigma}_j)^2a_{ij}^2. \tag{12}$$

With the results obtained in (11) and (12), moreover that

$$-\delta\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_i-\bar{\sigma}_j)a_{ij}^2-\frac{1}{\delta}\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)b_{ij}^2$$
$$\leq -2\sum_{i,j=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)(\bar{\sigma}_i-\bar{\sigma}_j)a_{ij}b_{ij}$$

we infer that

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2\left(\int_\Omega|\mathbf{u}_i|^2|\nabla f|^2d\mu\right)\leq\delta\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2$$
$$\times\left(\int_\Omega|\mathbf{u}_i|^2|\nabla f|^2d\mu+\alpha\int_\Omega(\nabla f\cdot\mathbf{u}_i)^2d\mu\right)$$
$$+\frac{1}{\delta}\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)||(\nabla f\cdot\nabla\mathbf{u}_i)+\frac{1}{2}\mathbf{u}_i\mathbb{L}_\phi f||^2.$$

By a simple computation, we get (3). □

The last lemma gives us the tools for the proof of next theorem.

Theorem 2.2 *Let Ω be a bounded domain in an n- dimensional Euclidean space $\mathbb{R}^n$. We consider ϕ a smooth function in Ω with $C_0 = \max_{\bar{\Omega}} |\nabla\phi|$. Let $\mathbb{L}_\phi$ be the drifting laplacian in $\mathbb{R}^n$. Denote by $\bar{\sigma}_i$ the ith eigenvalue of the problem (1) and $\mathbf{u}_i$ the orthonormal vector-valued eigenfunction corresponding to $\bar{\sigma}_i$. Then*

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2 \leq \sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)\left(\min\left\{\frac{4(n+\alpha)}{n^2}, \frac{A(n,\alpha)}{n+\alpha}\right\}\bar{\sigma}_i\right.$$
$$\left. + \min\left\{\frac{4(n+\alpha)}{n^2}, \frac{B(n,\alpha)}{n+\alpha}\right\} C_0\sqrt{\bar{\sigma}_i} + \min\left\{\frac{n+\alpha}{n^2}, \frac{C(n,\alpha)}{n^2}\right\} C_0^2\right),$$

where

$$A(n,\alpha) = \begin{cases} \frac{8+(n+2)\alpha}{L+1} & \text{if } 0 \leq \alpha < \frac{n+2\sqrt{(n+2)^2+16}}{2} \\ 4+\alpha^2 & \text{if } \alpha \geq \frac{n+2\sqrt{(n+2)^2+16}}{2} \end{cases},$$
$$B(n,\alpha) = \begin{cases} \frac{8+(n+6)\alpha-\alpha^2}{L+1} & \text{if } 0 \leq \alpha < \frac{n+2\sqrt{(n+2)^2+16}}{2} \\ 4(\alpha+1) & \text{if } \alpha \geq \frac{n+2\sqrt{(n+2)^2+16}}{2} \end{cases},$$

$$C(n,\alpha) = \begin{cases} \frac{8+(n+10)\alpha+3\alpha^2}{4(L+1)} & \text{if } 0 \leq \alpha < \frac{n+2\sqrt{(n+2)^2+16}}{2} \\ \alpha^2+2\alpha+1 & \text{if } \alpha \geq \frac{n+2\sqrt{(n+2)^2+16}}{2} \end{cases}$$
$$\text{with } L = \frac{(4+(n+2)\alpha-\alpha^2)n^2}{4(n+\alpha)^2} > 0.$$

Proof Let $x^1, \ldots, x^n$ be standard coordinates functions in $\mathbb{R}^n$. Setting $f = x^p$, in (3) we have

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2\left\{(1-\delta)\int_\Omega |\mathbf{u}_i|^2|\nabla x^p|^2 d\mu - \delta\alpha\int_\Omega(\nabla x^p\cdot\mathbf{u}_i)^2 d\mu\right\}$$
$$\leq \frac{1}{\delta}\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)||\nabla x^p\cdot\nabla(\mathbf{u}_i)+\frac{1}{2}\mathbf{u}_i\mathbb{L}_\phi x^p||^2.$$

Summing over $p = 1, \ldots, n$, we infer

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2\{(1-\delta)n-\delta\alpha\} \leq \frac{1}{\delta}\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)\sum_{p=1}^{n}||\nabla x^p$$
$$\cdot\nabla(\mathbf{u}_i)+\frac{1}{2}\mathbf{u}_i\mathbb{L}_\phi x^p||^2. \tag{13}$$

By a simple calculation, we infer

$$\begin{aligned}
\sum_{p=1}^{n} ||\nabla x^p \cdot \nabla(\mathbf{u}_i) + \frac{1}{2}\mathbf{u}_i \mathbb{L}_\phi x^p||^2 &= \sum_{p=1}^{n} \int_\Omega \{|\nabla x^p \cdot \nabla \mathbf{u}_i|^2 + \mathbf{u}_i \mathbb{L}_\phi x^p (\nabla x^p \cdot \nabla \mathbf{u}_i) \\
&\quad + \frac{1}{4}\mathbf{u}_i^2 (\mathbb{L}_\phi x^p)^2\} d\mu = \int_\Omega |\nabla \mathbf{u}_i|^2 d\mu - \int_\Omega \mathbf{u}_i \nabla\phi \cdot \nabla \mathbf{u}_i d\mu + \frac{1}{4}\int_\Omega \mathbf{u}_i^2 |\nabla\phi|^2 d\mu \\
&\leq \int_\Omega \mathbf{u}_i(-\mathbb{L}_\phi)\mathbf{u}_i d\mu + \left(\int_\Omega \mathbf{u}_i^2 d\mu\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla\phi \cdot \nabla \mathbf{u}_i|^2\right)^{\frac{1}{2}} + \frac{C_0^2}{4} \\
&\leq \int_\Omega \mathbf{u}_i(\bar{\sigma}_i \mathbf{u}_i + \alpha(\nabla \mathrm{div}\mathbf{u}_i - \nabla\phi \mathrm{div}\mathbf{u}_i)) d\mu + C_0 \left(\int_\Omega |\nabla \mathbf{u}_i|^2 d\mu\right)^{\frac{1}{2}} + \frac{C_0^2}{4} \\
&= \bar{\sigma}_i + \alpha \int_\Omega \mathbf{u}_i (\nabla \mathrm{div}\mathbf{u}_i) d\mu - \alpha \int_\Omega \mathbf{u}_i \nabla\phi \mathrm{div}\mathbf{u}_i d\mu \\
&\quad + C_0 \left(\int_\Omega |\nabla \mathbf{u}_i|^2 d\mu\right)^{\frac{1}{2}} + \frac{C_0^2}{4} = \bar{\sigma}_i - \alpha \int_\Omega \mathrm{div}\mathbf{u}_i [\mathrm{div}\mathbf{u}_i - \nabla\phi \cdot \mathbf{u}_i] d\mu \\
&\quad - \alpha \int_\Omega \mathbf{u}_i \mathrm{div}\mathbf{u}_i \nabla\phi d\mu + C_0 \left(\int_\Omega |\nabla \mathbf{u}_i|^2\right)^{\frac{1}{2}} + \frac{C_0^2}{4} \\
&= \bar{\sigma}_i - \alpha||\mathrm{div}\mathbf{u}_i||^2 + C_0\sqrt{\bar{\sigma}_i - \alpha||\mathrm{div}\mathbf{u}_i||^2} + \frac{C_0^2}{4}. \qquad (14)
\end{aligned}$$

Replacing (14) in (13), we have

$$\begin{aligned}
\sum_{i=1}^{k} (\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 \{n - \delta(n+\alpha)\} &\leq \frac{1}{\delta} \sum_{i=1}^{k} (\bar{\sigma}_{k+1} - \bar{\sigma}_i) \\
&\times \left(\bar{\sigma}_i - \alpha||\mathrm{div}\mathbf{u}_i||^2 + C_0\sqrt{\bar{\sigma}_i - \alpha||\mathrm{div}\mathbf{u}_i||^2} + \frac{C_0^2}{4}\right).
\end{aligned}$$

Putting $\delta = \left\{ \frac{\sum_{i=1}^{k} (\bar{\sigma}_{k+1} - \bar{\sigma}_i)\left(\bar{\sigma}_i - \alpha||\mathrm{div}\mathbf{u}_i||^2 + C_0\sqrt{\bar{\sigma}_i - \alpha||\mathrm{div}\mathbf{u}_i||^2} + \frac{C_0^2}{4}\right)}{\sum_{i=1}^{k} (\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 (n+\alpha)} \right\}^{\frac{1}{2}}$, we get

$$\begin{aligned}
\sum_{i=1}^{k} (\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 &\leq \left\{\frac{4(n+\alpha)}{n^2}\right\}^{\frac{1}{2}} \left\{\sum_{i=1}^{k} (\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2\right\}^{\frac{1}{2}} \\
&\times \left\{\sum_{i=1}^{k} (\bar{\sigma}_{k+1} - \bar{\sigma}_i)\left(\bar{\sigma}_i - \alpha||\mathrm{div}\mathbf{u}_i||^2 + C_0\sqrt{\bar{\sigma}_i - \alpha||\mathrm{div}\mathbf{u}_i||^2} + \frac{C_0^2}{4}\right)\right\}^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2 \leq \frac{4(n+\alpha)}{n^2}\left\{\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)\right. \\ \left. \times\left(\bar{\sigma}_i-\alpha||\mathrm{div}\mathbf{u}_i||^2+C_0\sqrt{\bar{\sigma}_i-\alpha||\mathrm{div}\mathbf{u}_i||^2}+\frac{C_0^2}{4}\right)\right\}. \tag{15}$$

On the other hand, taking $f = x^p$ in (2) and summing over $p = 1, \ldots, n$, we have

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2(n+\alpha) \leq \sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)\sum_{p=1}^{n}||P_i(x^p)||^2.$$

A straightforward computation yields

$$\begin{aligned}
\sum_{p=1}^{n}||P_i(x^p)||^2 &= \sum_{p=1}^{n}||2(\nabla x^p\cdot\nabla\mathbf{u}_i)+\mathbf{u}_i\mathbb{L}_\phi x^p+\alpha(\nabla(\nabla x^p\cdot\mathbf{u}_i) \\
&\quad +\nabla x^p\mathrm{div}\mathbf{u}_i-\nabla\phi(\nabla x^p\cdot\mathbf{u}_i))||^2 \\
&= 4\sum_{p=1}^{n}||\nabla x^p\cdot\nabla\mathbf{u}_i||^2+4\sum_{p=1}^{n}\int_\Omega\mathbf{u}_i\mathbb{L}_\phi x^p(\nabla x^p\cdot\nabla\mathbf{u}_i)d\mu+\sum_{p=1}^{n}\int_\Omega(\mathbf{u}_i\mathbb{L}_\phi x^p)^2d\mu \\
&\quad +4\alpha\sum_{p=1}^{n}\int_\Omega(\nabla x^p\cdot\nabla\mathbf{u}_i)(\nabla(\nabla x^p\cdot\mathbf{u}_i))d\mu+4\alpha\sum_{p=1}^{n}\int_\Omega(\nabla x^p\cdot\nabla\mathbf{u}_i)\nabla x^p\mathrm{div}\mathbf{u}_i d\mu \\
&\quad +2\alpha\sum_{p=1}^{n}\int_\Omega\mathbf{u}_i\mathbb{L}_\phi x^p(\nabla(\nabla x^p\cdot\mathbf{u}_i))d\mu+2\alpha\sum_{p=1}^{n}\int_\Omega\mathbf{u}_i\mathbb{L}_\phi x^p\nabla x^p\mathrm{div}\mathbf{u}_i d\mu \\
&\quad +\alpha^2\sum_{p=1}^{n}||\nabla(\nabla x^p\cdot\mathbf{u}_i)||^2+2\alpha^2\sum_{p=1}^{n}\int_\Omega(\nabla(\nabla x^p\cdot\mathbf{u}_i)\nabla x^p)\mathrm{div}\mathbf{u}_i d\mu \\
&\quad +\alpha^2\sum_{p=1}^{n}\int_\Omega(\nabla x^p\mathrm{div}\mathbf{u}_i)^2d\mu-4\alpha\sum_{p=1}^{n}\int_\Omega\nabla\phi(\nabla x^p\cdot\mathbf{u}_i)(\nabla x^p\cdot\nabla\mathbf{u}_i)d\mu \\
&\quad -2\alpha\sum_{p=1}^{n}\int_\Omega\nabla\phi(\nabla x^p\cdot\mathbf{u}_i)\mathbf{u}_i\mathbb{L}_\phi x^p d\mu-2\alpha^2\sum_{p=1}^{n}\int_\Omega\nabla\phi(\nabla x^p\cdot\mathbf{u}_i)(\nabla(\nabla x^p\cdot\mathbf{u}_i))d\mu \\
&\quad -2\alpha^2\sum_{p=1}^{n}\int_\Omega\nabla\phi(\nabla x^p\cdot\mathbf{u}_i)\nabla x^p\mathrm{div}\mathbf{u}_i d\mu+\alpha^2\sum_{p=1}^{n}\int_\Omega|\nabla\phi|^2|\nabla x^p\cdot\mathbf{u}_i|^2d\mu \\
&\leq -(4+\alpha^2)\int_\Omega\mathbf{u}_i\mathbb{L}_\phi\mathbf{u}_i d\mu+(8\alpha+2\alpha^2+\alpha^2 n)||\mathrm{div}\mathbf{u}_i||^2-4(\alpha+1)\int_\Omega\mathbf{u}_i\nabla\phi\nabla\mathbf{u}_i d\mu \\
&\quad +(\alpha^2+2\alpha+1)\int_\Omega|\nabla\phi|^2|\mathbf{u}_i|^2d\mu-(4\alpha+4\alpha^2)\int_\Omega(\nabla\phi\cdot\mathbf{u}_i)\mathrm{div}\mathbf{u}_i d\mu \\
&\leq \bar{\sigma}_i(4+\alpha^2)-\alpha(\alpha^2-(n+2)\alpha-4)||\mathrm{div}\mathbf{u}_i||^2+4(\alpha+1)C_0\sqrt{\bar{\sigma}_i-\alpha||\mathrm{div}\mathbf{u}_i||^2}
\end{aligned}$$

$$+ (\alpha^2 + 2\alpha + 1)C_0^2 + 4\alpha(\alpha + 1)C_0||\mathrm{div}\mathbf{u}_i||.$$

Hence

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 \le \sum_{i=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)\left\{\bar{P}_i\right\},$$

where

$$\bar{P}_i := \frac{(4+\alpha^2)}{n+\alpha}\bar{\sigma}_i - \frac{\alpha(\alpha^2-(n+2)\alpha-4)}{n+\alpha}||\mathrm{div}\mathbf{u}_i||^2 + \frac{4(\alpha+1)}{n+\alpha}C_0\sqrt{\bar{\sigma}_i - \alpha||\mathrm{div}\mathbf{u}_i||^2} + \frac{(\alpha^2+2\alpha+1)}{n+\alpha}C_0^2 + \frac{4\alpha(\alpha+1)}{n+\alpha}C_0||\mathrm{div}\mathbf{u}_i||.$$

If $\alpha^2 - (n+2)\alpha - 4 \ge 0$, namely, $\alpha \ge \frac{n+2+\sqrt{(n+2)^2+16}}{2}$, we have

$$\bar{P}_i \le \frac{(4+\alpha^2)}{n+\alpha}\bar{\sigma}_i + \frac{4(\alpha+1)}{n+\alpha}C_0\sqrt{\bar{\sigma}_i} + \frac{(\alpha^2+2\alpha+1)}{n+\alpha}C_0^2 + \frac{4\alpha(\alpha+1)}{n+\alpha}C_0||\mathrm{div}\mathbf{u}_i||. \tag{16}$$

If $\alpha^2 - (n+2)\alpha - 4 < 0$, namely, $0 \le \alpha < \frac{n+2+\sqrt{(n+2)^2+16}}{2}$, we have

$$\bar{P}_i \le \frac{(4+\alpha^2)}{n+\alpha}\bar{\sigma}_i + \frac{\alpha(4+(n+2)\alpha-\alpha^2)}{n+\alpha}||\mathrm{div}\mathbf{u}_i||^2 + \frac{4(\alpha+1)}{n+\alpha}C_0\sqrt{\bar{\sigma}_i - \alpha||\mathrm{div}\mathbf{u}_i||^2} + \frac{(\alpha^2+2\alpha+1)}{n+\alpha}C_0^2 + \frac{4\alpha(\alpha+1)}{n+\alpha}C_0||\mathrm{div}\mathbf{u}_i||. \tag{17}$$

From (15) and (17), it follows

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)^2 \le \sum_{i=1}^{k}(\bar{\sigma}_{k+1} - \bar{\sigma}_i)\left\{\frac{8+(n+2)\alpha}{(n+\alpha)(L+1)}\bar{\sigma}_i + \frac{8+(n+6)\alpha-\alpha^2}{(n+\alpha)(L+1)}C_0\sqrt{\bar{\sigma}_i} + \frac{8+(n+10)\alpha+3\alpha^2}{4(n+\alpha)(L+1)}C_0^2 + \frac{4\alpha(\alpha+1)}{n+\alpha}C_0||\mathrm{div}\mathbf{u}_i||\right\},$$

where $L = \frac{(4+(n+2)\alpha-\alpha^2)n^2}{4(n+\alpha)^2}$.

Thus, for any $\alpha \geq 0$

$$\sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i)^2 \leq \sum_{i=1}^{k}(\bar{\sigma}_{k+1}-\bar{\sigma}_i) \times \left\{\frac{A(n,\alpha)}{n+\alpha}\bar{\sigma}_i + \frac{B(n,\alpha)}{n+\alpha}C_0\sqrt{\bar{\sigma}_i} + \frac{C(n,\alpha)}{n+\alpha}C_0^2 + \frac{4\alpha(\alpha+1)}{n+\alpha}C_0||\mathrm{div}\mathbf{u}_i||\right\}. \quad (18)$$

Making a comparison between (15) and (18), we finish the proof. □

3 Lower order estimates

In this section, we shall give many informations about lower order estimates.

Theorem 3.1 *Let $\bar{\sigma}_i$ denote the i-th eigenvalue of the problem (1), $i = 1, \ldots, n$. Then, we have*

$$\sum_{i=1}^{n}(\bar{\sigma}_{i+1}-\bar{\sigma}_1) \leq 4(1+\alpha)\left(\bar{\sigma}_1 + C_0\sqrt{\bar{\sigma}_1} + \frac{C_0^2}{4}\right)$$

Proof Let $\{x^i\}_{i=1}^n$ be the standard coordinate functions of $\mathbb{R}^n$. Let us define a $n \times n$ matrix $C := (c_{ij})$ where $c_{ij} = \int_\Omega x^i \mathbf{u}_1\mathbf{u}_{j+1}d\mu$. Using the orthogonalization of Gram and Schmidt, we know that there exists an upper triangle matrix B and an orthogonal matrix T such that $B = TC$, namely

$$b_{ij} = \sum_{k=1}^{n} t_{ik}c_{kj} = \sum_{k=1}^{n} t_{ik}\int_\Omega x^k\mathbf{u}_1\mathbf{u}_{j+1}d\mu = \int_\Omega\left(\sum_{k=1}^{n} t_{ik}x^k\right)\mathbf{u}_1\mathbf{u}_{j+1}d\mu = 0$$

for $1 \leq j < i$

Putting $y_i = \sum_{k=1}^n t_{ik}x^k$, we have $\int_\Omega y_i\mathbf{u}_1\mathbf{u}_{j+1}d\mu = 0$ for $1 \leq j < i$.

We define a vector-valued function $\mathbf{w}_i = (y_i - a_i)\mathbf{u}_1$ where $a_i = \int_\Omega y_i\mathbf{u}_1^2 d\mu$. We infer that $\mathbf{w}_i|_{\partial\Omega} = 0$ and $\int_\Omega \mathbf{w}_i\mathbf{u}_{j+1}d\mu = 0$ for any $j = 1, \ldots, i-1$. From the Rayleight-Ritz inequality we have

$$\bar{\sigma}_{k+1} \leq \frac{\int_\Omega\{-\mathbf{w}_i\mathbb{L}_\phi\mathbf{w}_i + \alpha(\mathrm{div}\mathbf{w}_i)^2\}d\mu}{\int_\Omega|\mathbf{w}_i|^2d\mu}. \quad (19)$$

We derive that

$$
\begin{aligned}
-\int_\Omega \mathbf{w}_i \mathbb{L}_\phi \mathbf{w}_i d\mu &= -\int_\Omega \mathbf{w}_i \big[(y_i - a_i)\mathbb{L}_\phi \mathbf{u}_1 + \mathbf{u}_1 \mathbb{L}_\phi y_i + 2(\nabla y_i \cdot \nabla \mathbf{u}_1)\big] d\mu \\
&= -\int_\Omega \mathbf{w}_i \big[(y_i - a_i)(-\bar{\sigma}_1 \mathbf{u}_1 - \alpha(\nabla \mathrm{div} \mathbf{u}_1 - \nabla\phi \mathrm{div} \mathbf{u}_1)) + \mathbf{u}_1 \mathbb{L}_\phi y_i + 2(\nabla y_i \cdot \nabla \mathbf{u}_1)\big] d\mu \\
&= \bar{\sigma}_1 ||\mathbf{w}_i||^2 - \alpha \int_\Omega \mathrm{div}\mathbf{u}_1 \big[\mathrm{div}(\mathbf{w}_i(y_i - a_i)) - \nabla\phi \cdot (\mathbf{w}_i(y_i - a_i))\big] d\mu \\
&\quad -\alpha \int_\Omega \mathbf{w}_i (y_i - a_i)\nabla\phi \mathrm{div}\mathbf{u}_1 d\mu - \int_\Omega \mathbf{w}_i \big(\mathbf{u}_1 \mathbb{L}_\phi y_i + 2\nabla y_i \cdot \nabla \mathbf{u}_1\big) d\mu \\
&= \bar{\sigma}_1 ||\mathbf{w}_i||^2 - \alpha \int_\Omega (y_i - a_i)\mathrm{div}\mathbf{u}_1 \mathrm{div}\mathbf{w}_i d\mu - \alpha \int_\Omega (\nabla y_i \cdot \mathbf{w}_i)\mathrm{div}\mathbf{u}_1 d\mu \\
&\quad - \int_\Omega \mathbf{w}_i \big(\mathbf{u}_1 \mathbb{L}_\phi y_i + 2\nabla y_i \cdot \nabla \mathbf{u}_1\big) d\mu = \bar{\sigma}_1 ||\mathbf{w}_i||^2 - \alpha \int_\Omega [\mathrm{div}\mathbf{w}_i \\
&\quad -\nabla y_i \cdot \mathbf{u}_1]\mathrm{div}\mathbf{w}_i d\mu - \alpha \int_\Omega (\nabla y_i \cdot \mathbf{w}_i)\mathrm{div}\mathbf{u}_1 d\mu - \int_\Omega \mathbf{w}_i \big(\mathbf{u}_1 \mathbb{L}_\phi y_i + 2\nabla y_i \cdot \nabla \mathbf{u}_1\big) d\mu \\
&= \bar{\sigma}_1 ||\mathbf{w}_i||^2 - \alpha \int_\Omega (\mathrm{div}\mathbf{w}_i)^2 d\mu + \alpha \int_\Omega (\nabla y_i \cdot \mathbf{u}_1)(\nabla\phi \cdot \mathbf{w}_i) d\mu - \alpha \int_\Omega \nabla(\nabla y_i \cdot \mathbf{u}_1) \\
&\quad \cdot \mathbf{w}_i d\mu - \alpha \int_\Omega (\nabla y_i \cdot \mathbf{w}_i)\mathrm{div}\mathbf{u}_1 d\mu - \int_\Omega \mathbf{w}_i \big(\mathbf{u}_1 \mathbb{L}_\phi y_i + 2\nabla y_i \cdot \nabla \mathbf{u}_1\big) d\mu \\
&= \bar{\sigma}_1 ||\mathbf{w}_i||^2 - \alpha \int_\Omega (\mathrm{div}\mathbf{w}_i)^2 d\mu - \alpha \int_\Omega \mathbf{w}_i \left\{\nabla(\nabla y_i \cdot \mathbf{u}_1) + \nabla y_i \mathrm{div}\mathbf{u}_1 - \nabla\phi(\nabla y_i \cdot \mathbf{u}_1)\right\} d\mu \\
&\quad - \int_\Omega \mathbf{w}_i \big(\mathbf{u}_1 \mathbb{L}_\phi y_i + 2\nabla y_i \cdot \nabla \mathbf{u}_1\big) d\mu.
\end{aligned}
$$

Replacing the above identity in (19), we have

$$
(\bar{\sigma}_{i+1} - \bar{\sigma}_1)||\mathbf{w}_i||^2 \leq -\int_\Omega \mathbf{w}_i \left\{P_1(y_i)\right\} d\mu \tag{20}
$$

where $P_1(y_i) := \mathbf{u}_1 \mathbb{L}_\phi y_i + 2\nabla y_i \cdot \nabla \mathbf{u}_1 + \alpha\big[\nabla(\nabla y_i \cdot \mathbf{u}_1) + \nabla y_i \mathrm{div}\mathbf{u}_1 - \nabla\phi(\nabla y_i \cdot \mathbf{u}_1)\big]$.

By a simple computation, we derive that

$$
-\int_\Omega \mathbf{w}_i \big(\mathbf{u}_1 \mathbb{L}_\phi y_i + 2\nabla y_i \cdot \nabla \mathbf{u}_1\big) d\mu = \int_\Omega |\mathbf{u}_1|^2 |\nabla y_i|^2 d\mu \tag{21}
$$

and

$$
-\alpha \int_\Omega \mathbf{w}_i \left\{\nabla(\nabla y_i \cdot \mathbf{u}_1) + \nabla y_i \mathrm{div}\mathbf{u}_1 - \nabla\phi(\nabla y_i \cdot \mathbf{u}_1)\right\} d\mu = \alpha \int_\Omega |\nabla y_i \cdot \mathbf{u}_1|^2 d\mu. \tag{22}
$$

So, from (20), we infer

$$
(\bar{\sigma}_{i+1} - \bar{\sigma}_1)||\mathbf{w}_i||^2 \leq \int_\Omega |\mathbf{u}_1|^2 |\nabla y_i|^2 d\mu + \alpha \int_\Omega |\nabla y_i \cdot \mathbf{u}_1|^2 d\mu.
$$

On the other hand, we have

$$\begin{aligned}(\bar{\sigma}_{i+1}-\bar{\sigma}_1)\int_\Omega |\mathbf{u}_1|^2|\nabla y_i|^2 d\mu &= (\bar{\sigma}_{i+1}-\bar{\sigma}_1)\left\{-2\int_\Omega \mathbf{w}_i\Big(\frac{1}{2}\mathbf{u}_1\mathbb{L}_\phi y_i+\nabla y_i\cdot\nabla\mathbf{u}_1\Big)d\mu\right\}\\
&\leq \delta(\bar{\sigma}_{i+1}-\bar{\sigma}_1)^2||\mathbf{w}_i||^2+\frac{1}{\delta}\left|\left|\frac{1}{2}\mathbf{u}_1\mathbb{L}_\phi y_i+\nabla y_i\cdot\nabla\mathbf{u}_1\right|\right|^2\\
&\leq \delta(\bar{\sigma}_{i+1}-\bar{\sigma}_1)\left(\int_\Omega |\mathbf{u}_1|^2|\nabla y_i|^2 d\mu+\alpha\int_\Omega|\nabla y_i\cdot\mathbf{u}_1|^2 d\mu\right)\\
&\quad+\frac{1}{\delta}\left|\left|\frac{1}{2}\mathbf{u}_1\mathbb{L}_\phi y_i+\nabla y_i\cdot\nabla\mathbf{u}_1\right|\right|^2.\end{aligned}$$

Summing over $i=1,\ldots,n$, we conclude

$$\begin{aligned}\sum_{i=1}^n(\bar{\sigma}_{i+1}-\bar{\sigma}_1)(1-\delta)\int_\Omega|\mathbf{u}_1|^2|\nabla y_i|^2d\mu &\leq \delta\alpha\sum_{i=1}^n(\bar{\sigma}_{i+1}-\bar{\sigma}_1)\int_\Omega|\nabla y_i\cdot\mathbf{u}_1|^2d\mu\\
&\quad+\frac{1}{\delta}\sum_{i=1}^n\left|\left|\frac{1}{2}\mathbf{u}_1\mathbb{L}_\phi y_i+\nabla y_i\cdot\nabla\mathbf{u}_1\right|\right|^2,\end{aligned} \tag{23}$$

Since

$$\sum_{i=1}^n\left|\left|\frac{1}{2}\mathbf{u}_1\mathbb{L}_\phi y_i+\nabla y_i\cdot\nabla\mathbf{u}_1\right|\right|^2=\bar{\sigma}_1-\alpha||\mathrm{div}\mathbf{u}_1||^2+C_0\sqrt{\bar{\sigma}_1-\alpha||\mathrm{div}\mathbf{u}_1||^2}+\frac{C_0^2}{4},$$

putting $\delta=\left\{\frac{\bar{\sigma}_1-\alpha||\mathrm{div}\mathbf{u}_1||^2+C_0\sqrt{\bar{\sigma}_1-\alpha||\mathrm{div}\mathbf{u}_1||^2}+\frac{C_0^2}{4}}{\sum_{i=1}^n(\bar{\sigma}_{i+1}-\bar{\sigma}_1)(1+\alpha)}\right\}^{\frac{1}{2}}$, we infer

$$\sum_{i=1}^n(\bar{\sigma}_{i+1}-\bar{\sigma}_1)\leq 4(1+\alpha)\Big(\bar{\sigma}_1+C_0\sqrt{\bar{\sigma}_1}+\frac{C_0^2}{4}\Big). \tag{24}$$

□

It is not difficult to see that, when $\phi\equiv 0$, the inequality (24) is the same obtained in [5].

The theorem below and the corollary can be found in [11].

Theorem 3.2 (Algebraic) *Let $\mathcal{H}$ and $\mathcal{G}$ be self-adjoint operators with domains $D_\mathcal{H}$ and $D_\mathcal{G}$ respectively, such that $\mathcal{G}(D_\mathcal{H})\subseteq D_\mathcal{H}\subseteq D_\mathcal{G}$. Let $\bar{\sigma}_j$ and $\mathbf{v}_j$ be eigenvalues and eigenvectors of $\mathcal{H}$; then, for each j*

$$-\frac{1}{2}\big\langle[[\mathcal{H},\mathcal{G}],\mathcal{G}]\mathbf{v}_j,\mathbf{v}_j\big\rangle=\sum_k\frac{\big\langle[\mathcal{H},\mathcal{G}]\mathbf{v}_j,\mathbf{v}_k\big\rangle^2}{\bar{\sigma}_k-\bar{\sigma}_j}.$$

Corollary 3.3 *Under conditions of Theorem 3.2*

$$-(\bar{\sigma}_{m+1}-\bar{\sigma}_m)\sum_{j=1}^{m}\left\langle [[\mathcal{H},\mathcal{G}],\mathcal{G}]\,\mathbf{v}_j,\mathbf{v}_j\right\rangle \le 2\sum_{j=1}^{m}\|[\mathcal{H},\mathcal{G}]\mathbf{v}_j\|^2. \tag{25}$$

With the help of the above corollary, we obtain an estimate about the gap from any consecutive eigenvalues.

Theorem 3.4 *Let Ω be a bounded domain in $\mathbb{R}^n$. We consider ϕ a smooth function in Ω with $C_0=\max_{\bar{\Omega}}|\nabla\phi|$. Denote by $\bar{\sigma}_i$ the ith eigenvalue of the problem (1) and $\mathbf{u}_i$ the orthonormal vector-valued eigenfunction corresponding to $\bar{\sigma}_i$. Then*

$$\bar{\sigma}_{k+1}-\bar{\sigma}_k \le \frac{\max\{4+\alpha^2, 8+\alpha(n+2)\}}{n+\alpha}\frac{1}{k}\sum_{j=1}^{k}\bar{\sigma}_j+\frac{4(\alpha+1)}{n+\alpha}C_0\frac{1}{k}\sum_{j=1}^{k}\sqrt{\bar{\sigma}_j}$$
$$+\frac{(\alpha+1)^2}{n+\alpha}C_0^2.$$

Proof We denote $N=(-\mathbb{L}_\phi)$ and $M=\nabla\phi\ \mathrm{div}-\nabla\mathrm{div}$, so $\mathcal{H}=N+\alpha M$. This operator is associated to the eigenvalue problem (1). We consider $G_p=x^p$, then from (25), we have

$$-(\bar{\sigma}_{k+1}-\bar{\sigma}_k)\sum_{j=1}^{k}\sum_{p=1}^{n}\left\langle \left[[\mathcal{H},x^p],x^p\right]\mathbf{u}_j,\mathbf{u}_j\right\rangle \le 2\sum_{j=1}^{k}\sum_{p=1}^{n}\|[\mathcal{H},x^p]\mathbf{u}_j\|^2. \tag{26}$$

We derive

$$-\sum_{p=1}^{n}\left\langle \left[[\mathcal{H},x^p],x^p\right]\mathbf{u}_j,\mathbf{u}_j\right\rangle = 2\sum_{p=1}^{n}\int_\Omega |\nabla x^p|^2|\mathbf{u}_j|^2d\mu+2\alpha\sum_{p=1}^{n}\int_\Omega(\nabla x^p\cdot\mathbf{u}_j)^2d\mu$$
$$=2(n+\alpha).$$

Notice that

$$[\mathcal{H},x^p]\mathbf{u}_j=-P_j(x^p).$$

Hence, it follows from Theorem 2.2

$$
\begin{aligned}
\sum_{p=1}^{n}||[\mathcal{H},x^p]\mathbf{u}_j||^2 &= \sum_{p=1}^{n}||P_j(x^p)||^2 \leq -(4+\alpha^2)\int_\Omega \mathbf{u}_j\mathbb{L}_\phi\mathbf{u}_j d\mu \\
&+(8\alpha+2\alpha^2+\alpha^2 n)||\mathrm{div}\mathbf{u}_j||^2-4(\alpha+1)\int_\Omega \mathbf{u}_j\nabla\phi\nabla\mathbf{u}_j d\mu \\
&+(\alpha^2+2\alpha+1)\int_\Omega|\nabla\phi|^2|\mathbf{u}_j|^2 d\mu-(4\alpha+4\alpha^2)\int_\Omega(\nabla\phi\cdot\mathbf{u}_j)\mathrm{div}\mathbf{u}_j d\mu \\
&\leq -(4+\alpha^2)\int_\Omega \mathbf{u}_j\mathbb{L}_\phi\mathbf{u}_j d\mu+(8\alpha+2\alpha^2+\alpha^2 n) \\
&\times\left[\int_\Omega \mathrm{div}\mathbf{u}_j(\nabla\phi\cdot\mathbf{u}_j)d\mu-\int_\Omega\nabla\mathrm{div}\mathbf{u}_j\cdot\mathbf{u}_j d\mu\right]-4(\alpha+1)\int_\Omega\mathbf{u}_j\nabla\phi\nabla\mathbf{u}_j d\mu \\
&+(\alpha^2+2\alpha+1)C_0^2-4\alpha(1+\alpha)\int_\Omega(\nabla\phi\cdot\mathbf{u}_j)\mathrm{div}\mathbf{u}_j d\mu \\
&=-(4+\alpha^2)\int_\Omega \mathbf{u}_j\mathbb{L}_\phi\mathbf{u}_j d\mu+(4\alpha-2\alpha^2+\alpha^2 n)\int_\Omega\mathbf{u}_j\nabla\phi\mathrm{div}\mathbf{u}_j d\mu \\
&-4(\alpha+1)\int_\Omega\mathbf{u}_j\nabla\phi\nabla\mathbf{u}_j d\mu+(\alpha^2+2\alpha+1)C_0^2-(8\alpha+2\alpha^2+\alpha^2 n) \\
&\times\int_\Omega\mathbf{u}_j\nabla\mathrm{div}\mathbf{u}_j d\mu\leq-(4+\alpha^2)\int_\Omega\mathbf{u}_j\mathbb{L}_\phi\mathbf{u}_j d\mu+(8\alpha+2\alpha^2+\alpha^2 n) \\
&\times\int_\Omega\mathbf{u}_j\nabla\phi\mathrm{div}\mathbf{u}_j d\mu-4(\alpha+1)\int_\Omega\mathbf{u}_j\nabla\phi\nabla\mathbf{u}_j d\mu+(\alpha^2+2\alpha+1)C_0^2 \\
&-(8\alpha+2\alpha^2+\alpha^2 n)\int_\Omega\mathbf{u}_j\nabla\mathrm{div}\mathbf{u}_j d\mu\leq-(4+\alpha^2)\int_\Omega\mathbf{u}_j\mathbb{L}_\phi\mathbf{u}_j d\mu \\
&+(8\alpha+2\alpha^2+\alpha^2 n)\int_\Omega[\mathbf{u}_j\nabla\phi\mathrm{div}\mathbf{u}_j-\mathbf{u}_j\nabla\mathrm{div}\mathbf{u}_j]d\mu \\
&+4(\alpha+1)C_0\left(\int_\Omega|\nabla\mathbf{u}_j|^2 d\mu\right)^{\frac{1}{2}}+(\alpha^2+2\alpha+1)C_0^2 \\
&\leq\max\{4+\alpha^2,8+\alpha(n+2)\}\int_\Omega\mathbf{u}_j[(-\mathbb{L}_\phi)\mathbf{u}_j-\alpha(\nabla\mathrm{div}\mathbf{u}_j-\nabla\phi\mathrm{div}\mathbf{u}_j)]d\mu \\
&+4(\alpha+1)C_0\sqrt{\bar{\sigma}_j-\alpha||\mathrm{div}\mathbf{u}_j||^2}+(\alpha^2+2\alpha+1)C_0^2=\max\{4+\alpha^2,8 \\
&+\alpha(n+2)\}\bar{\sigma}_j+4(\alpha+1)C_0\sqrt{\bar{\sigma}_j-\alpha||\mathrm{div}\mathbf{u}_j||^2}+(\alpha^2+2\alpha+1)C_0^2.
\end{aligned}
$$

Inserting the above inequality in (26), we have

$$
\bar{\sigma}_{k+1}-\bar{\sigma}_k\leq\frac{\max\{4+\alpha^2,8+\alpha(n+2)\}}{n+\alpha}\frac{1}{k}\sum_{j=1}^{k}\bar{\sigma}_j+\frac{4(\alpha+1)}{n+\alpha}C_0\frac{1}{k}\sum_{j=1}^{k}\sqrt{\bar{\sigma}_j}+\frac{(\alpha+1)^2}{n+\alpha}C_0^2. \tag{27}
$$

When $\phi\equiv 0$, we have that (27) is the same inequality obtained in Example 4.4 [11]. □

In the next theorem our aim is to get a result similar to Ashbaugh and Benguria in [1]. This result is very interesting because it doesn't make use the Rayleigh-Ritz inequality.

Theorem 3.5 *Let M^n be a complete Riemannian manifold and let Ω be a bounded domain with smooth boundary in M. Let ϕ be a smooth function in Ω with $C_0 = \max_\Omega |\nabla\phi|$. Denote by λ_j the j-th eigenvalue of the drifting problem, namely*

$$\begin{aligned} (-\mathbb{L}_\phi)u_j &= \lambda_j u_j \quad in \quad \Omega \\ u_j &= 0 \quad in \quad \partial\Omega. \end{aligned} \tag{28}$$

If M is isometrically immersed in $\mathbb{R}^m$ with mean curvature vector $\vec{H}$, then

$$\sum_{k=1}^{n} \lambda_{l+k} \le (n+4)\lambda_l + n^2 H_0^2 + C_0^2 + 4C_0\lambda_l^{\frac{1}{2}},$$

where $H_0 = \sup_\Omega |\vec{H}|$, $l \in \mathbb{N}$.

Proof Let $\{x_\alpha\}_{\alpha=1}^m$ be the standard coordinate functions in $\mathbb{R}^m$. For each j fixed, we considerer a $m \times m$ matrix, C where $c_{\alpha\beta} = \langle [\mathcal{H}, x_\alpha] u_j, u_{j+\beta} \rangle$

Using the orthogonalization process we have that there exist $B := (b_{\alpha\beta})$ an upper triangle matrix and $T := (t_{\alpha\beta})$ an orthogonal matrix such that $B = TC$, that is

$$b_{\alpha\beta} = \sum_{\gamma=1}^{m} t_{\alpha\gamma} c_{\gamma\beta} = \sum_{\gamma=1}^{m} t_{\alpha\gamma} \left\langle [\mathcal{H}, x_\gamma] u_j, u_{j+\beta} \right\rangle = \left\langle \left[\mathcal{H}, \sum_{\gamma=1}^{m} t_{\alpha\gamma} x_\gamma \right] u_j, u_{j+\beta} \right\rangle = 0,$$

for $1 \le \beta < \alpha \le m$.

Defining $g_\alpha = \sum_{\gamma=1}^m t_{\alpha\gamma} x_\gamma$, we have that $\langle [\mathcal{H}, g_\alpha] u_j, u_{j+\beta} \rangle = 0$ for $1 \le \beta < \alpha$.

At this moment, we analyze the term

$$\frac{\langle [\mathcal{H}, g_\alpha] u_j, u_k \rangle^2}{\lambda_k - \lambda_j}.$$

Observe that, for $k = j+1, \ldots, j+(\alpha-1)$, we obtain

$$\frac{\langle [\mathcal{H}, g_\alpha] u_j, u_k \rangle^2}{\lambda_k - \lambda_j} = 0.$$

For $k \ge j + \alpha$, we have that $\lambda_k \ge \lambda_{j+\alpha}$. Hence,

$$\frac{\langle [\mathcal{H}, g_\alpha] u_j, u_k \rangle^2}{\lambda_{j+\alpha} - \lambda_j} \ge \frac{\langle [\mathcal{H}, g_\alpha] u_j, u_k \rangle^2}{\lambda_k - \lambda_j}.$$

When $k = j$, in the algebraic lemma we assume $\frac{0}{0} = 0$. For the case in that $k < j$, we have $\lambda_k - \lambda_j < 0$. So,

$$\frac{\left\langle [\mathcal{H}, g_\alpha]\, u_j, u_k \right\rangle^2}{\lambda_k - \lambda_j} < 0\cdot$$

Therefore, for any positive integer k, we have

$$\frac{\left\langle [\mathcal{H}, g_\alpha]\, u_j, u_k \right\rangle^2}{\lambda_{j+\alpha} - \lambda_j} \geq \frac{\left\langle [\mathcal{H}, g_\alpha]\, u_j, u_k \right\rangle^2}{\lambda_k - \lambda_j}.$$

Summing over k, we get

$$\frac{\sum_{k=1}^{\infty} \left\langle [\mathcal{H}, g_\alpha]\, u_j, u_k \right\rangle^2}{\lambda_{j+\alpha} - \lambda_j} \geq \sum_{k=1}^{\infty} \frac{\left\langle [\mathcal{H}, g_\alpha]\, u_j, u_k \right\rangle^2}{\lambda_k - \lambda_j}. \tag{29}$$

In the other hand, the Parseval's Identity gives us

$$\sum_{k=1}^{\infty} \left\langle [\mathcal{H}, g_\alpha]\, u_j, u_k \right\rangle^2 = \| \, [\mathcal{H}, g_\alpha] \, \|^2 . \tag{30}$$

In Algebraic Lemma 3.2, we assume $\mathcal{G} = g_\alpha$ and we use the inequalities (29), (30). Thus, we can conclude that

$$\begin{aligned} -\frac{1}{2}\left\langle [[\mathcal{H}, g_\alpha]\,, g_\alpha]\, u_j, u_j \right\rangle &= \sum_{k=1}^{\infty} \frac{\left\langle [\mathcal{H}, g_\alpha]\, u_j, u_k \right\rangle^2}{\lambda_k - \lambda_j} \\ &\leq \frac{\sum_{k=1}^{\infty} \left\langle [\mathcal{H}, g_\alpha]\, u_j, u_k \right\rangle^2}{\lambda_{j+\alpha} - \lambda_j} \\ &= \frac{\| \, [\mathcal{H}, g_\alpha] \, \|^2}{\lambda_{j+\alpha} - \lambda_j}. \end{aligned} \tag{31}$$

Here, we assume that $\mathcal{H} = (-\mathbb{L}_\phi)$ and we obtain

$$\begin{aligned} &-\frac{1}{2}\left\langle [[\mathcal{H}, g_\alpha]\,, g_\alpha]\, u_j, u_j \right\rangle \\ &= -\frac{1}{2}\int_\Omega u_j \times \left[(-\mathbb{L}_\phi)(g_\alpha^2 u_j) - 2g_\alpha(-\mathbb{L}_\phi)(g_\alpha u_j) + g_\alpha^2(-\mathbb{L}_\phi)u_j \right] d\mu \\ &= \int_\Omega |\, u_j \nabla g_\alpha \,|^2 \, d\mu. \end{aligned}$$

Summing over α in (31)

$$\sum_{\alpha=1}^{m}\int_{\Omega}\mid u_j\nabla g_\alpha\mid^2(\lambda_{j+\alpha}-\lambda_j)d\mu \;\le\; \sum_{\alpha=1}^{m}\|\,[(-\mathbb{L}_\phi),g_\alpha]u_j\,\|^2\cdot \tag{32}$$

We remember that M is isometrically immersed in $\mathbb{R}^n$, namely,

$$\mathbb{L}_\phi g_\alpha = n\overrightarrow{H}g_\alpha - \langle\nabla\phi,\nabla g_\alpha\rangle\,,\ \alpha=1,\ldots,m.$$

Consequently, we have

$$\begin{aligned}
\sum_{\alpha=1}^{m}\|\,[(-\mathbb{L}_\phi),g_\alpha]u_j\,\|^2 &= \sum_{\alpha=1}^{m}\int_{\Omega}\mid u_j\mathbb{L}_\phi g_\alpha+2\left\langle\nabla g_\alpha,\nabla u_j\right\rangle\mid^2 d\mu\\
&= \sum_{\alpha=1}^{m}\int_{\Omega}u_j^2(\mathbb{L}_\phi g_\alpha)^2 d\mu + 4\sum_{\alpha=1}^{m}\int_{\Omega}u_j\mathbb{L}_\phi g_\alpha\left\langle\nabla g_\alpha,\nabla u_j\right\rangle d\mu\\
&\quad +4\sum_{\alpha=1}^{m}\int_{\Omega}\left\langle\nabla g_\alpha,\nabla u_j\right\rangle^2 d\mu\\
&= \int_{\Omega}u_j^2(n^2\mid\overrightarrow{H}\mid^2+\mid\phi\mid^2)d\mu - 4\int_{\Omega}u_j\left\langle\nabla\phi,\nabla u_j\right\rangle d\mu\\
&\quad +4\int_{\Omega}\mid\nabla u_j\mid^2 d\mu\\
&\le n^2H_0^2+C_0^2+4C_0\left(\int_{\Omega}\mid\nabla u_j\mid^2\right)^{\frac{1}{2}}+4\int_{\Omega}u_j(-\mathbb{L}_\phi)u_j d\mu\\
&\le n^2H_0^2+C_0^2\\
&\quad +4C_0\lambda_j^{\frac{1}{2}}+4\lambda_j.
\end{aligned}$$

Furthermore, from (32), we obtain

$$\sum_{\alpha=1}^{m}\int_{\Omega}\mid u_j\nabla g_\alpha\mid^2(\lambda_{j+\alpha}-\lambda_j)d\mu \;\le\; n^2H_0^2+C_0^2+4C_0\lambda_j^{\frac{1}{2}}+4\lambda_j.$$

Since $\mid\nabla g_\alpha\mid^2\le 1$ then

$$\begin{aligned}
&\sum_{\alpha=1}^{m}\int_{\Omega}\mid u_j\nabla g_\alpha\mid^2(\lambda_{j+\alpha}-\lambda_j)d\mu = \sum_{\alpha=1}^{m}\mid\nabla g_\alpha\mid^2(\lambda_{j+\alpha}-\lambda_j)\\
&\quad = \sum_{\alpha=1}^{m}\mid\nabla g_\alpha\mid^2\lambda_{j+\alpha}-n\lambda_j = \sum_{\alpha=1}^{n}\mid\nabla g_\alpha\mid^2\lambda_{j+\alpha}+\sum_{\alpha=n+1}^{m}\mid\nabla g_\alpha\mid^2\lambda_{j+\alpha}-n\lambda_j
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{\alpha=1}^{n} \mid \nabla g_\alpha \mid^2 \lambda_{j+\alpha} + \lambda_{j+n} \sum_{\alpha=n+1}^{m} \mid \nabla g_\alpha \mid^2 - n\lambda_j \\
&= \sum_{\alpha=1}^{n} \mid \nabla g_\alpha \mid^2 \lambda_{j+\alpha} + \lambda_{j+n} \left(n - \sum_{\alpha=1}^{n} \mid \nabla g_\alpha \mid^2 \right) - n\lambda_j \\
&= \sum_{\alpha=1}^{n} \mid \nabla g_\alpha \mid^2 \lambda_{j+\alpha} + \lambda_{j+n} \sum_{\alpha=1}^{n} \left(1 - \mid \nabla g_\alpha \mid^2 \right) - n\lambda_j \\
&\geq \sum_{\alpha=1}^{n} \mid \nabla g_\alpha \mid^2 \lambda_{j+\alpha} + \sum_{\alpha=1}^{n} \lambda_{j+\alpha} \left(1 - \mid \nabla g_\alpha \mid^2 \right) - n\lambda_j \\
&= \sum_{\alpha=1}^{n} \lambda_{j+\alpha} - n\lambda_j .
\end{aligned}
$$

Therefore,

$$
\sum_{\alpha=1}^{n} \lambda_{j+\alpha} \ \leq \ n^2 H_0^2 + C_0^2 + 4C_0 \lambda_j^{\frac{1}{2}} + (n+4)\lambda_j .
$$

□

For $j = 1$ and ϕ constant, the inequality derived covers Ashbaugh and Benguria in [1].

Corollary 3.6 *Under the same assumptions as in Theorem 3.5. We have*

i) If M is isometrically immersed in the unit sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ with mean curvature vector $\overrightarrow{H}$ then

$$
\sum_{k=1}^{n} \lambda_{l+k} \ \leq \ (n+4)\lambda_l + n^2 \left(H_1^2 + 1 \right) + C_0^2 + 4C_0 \lambda_l^{\frac{1}{2}} ,
$$

where $H_1 = \sup_\Omega \mid \overrightarrow{H} \mid$.

ii) If M is isometrically immersed in a projective space $\mathbb{F}P^m$ with mean curvature $\overrightarrow{H}$, then

$$
\sum_{k=1}^{n} \lambda_{l+k} \ \leq \ (n+4)\lambda_l + n^2 \left(H_2^2 + \frac{2(n+d)}{n} \right) + C_0^2 + 4C_0 \lambda_l^{\frac{1}{2}} ,
$$

where $H_2 = \sup_\Omega \mid \overrightarrow{H} \mid$ and $d = \dim_{\mathbb{F}} = \begin{cases} 1, & \mathbb{F} = \mathbb{R} \\ 2, & \mathbb{F} = \mathbb{C} \\ 4, & \mathbb{F} = \mathcal{Q} \end{cases}$

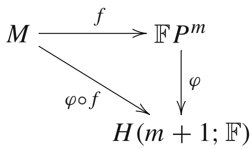
Proof i) Let f be standard embedding from M into an unit sphere $\mathbb{S}^{m-1}$ and let j be the inclusion map. Denote by $\overrightarrow{H}$ the mean curvature vector of f and by $\overrightarrow{H'}$ the mean curvature vector of $j \circ f$. Since $j : \mathbb{S}^{m-1} \hookrightarrow \mathbb{R}^m$ has $-I$ like the Gauss normal map then the shape operator $S_N = I$. Hence,

$$| H' |^2 = | H |^2 + 1.$$

Using the Theorem 3.5, we obtain

$$\sum_{k=1}^{n} \lambda_{l+k} \leq (n+4)\lambda_l + n^2 \left(H_1^2 + 1\right) + C_0^2 + 4C_0\lambda_l^{\frac{1}{2}}$$

ii)



Let $\overrightarrow{H}$ be the mean curvature vector of isometric immersion f and let $\overrightarrow{H'}$be the mean curvature vector of $\varphi \circ f$. Let φ be the standard immersion of $\mathbb{F}P^m$ into $H(m+1;\mathbb{F})$, where $H(m+1;\mathbb{F})$ is the vector space of $(m+1) \times (m+1)$ Hermitian matrices with coefficients in the field $\mathbb{F}$. We're going to use the Lemma 2.2 that is shown us in [6], namely

$$| H' |^2 \leq | H |^2 + \frac{2(n+d)}{n}.$$

Replacing in Theorem 3.5 we obtain the result required.

□

See [3] for more informations about the standard imbeddings of projective spaces.

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