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A multi-variable Rankin–Selberg integral for a product of GL_2-twisted Spinor L-functions

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Abstract We consider a new integral representation for $L(s_1, \Pi \times \tau_1)L(s_2, \Pi \times \tau_2)$, where Π is a globally generic cuspidal representation of GSp_4, and τ_1 and τ_2 are two cuspidal representations of GL_2 having the same central character. As and application, we find a new period condition for two such L functions to have a pole simultaneously. This points to an intriguing connection between a Fourier coefficient of a residual representation on GSO(12) and a theta function on Sp(16). A similar integral on GSO(18) fails to unfold completely, but in a way that provides further evidence of a connection.

Keywords Rankin–Selberg · Integral representation · Spinor L-function · Theta correspondence · Fourier coefficient

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1 Introduction

An important problem in the theory of automorphic forms is to understand periods, and how they are related with L-functions and their special values, as well as with functorial liftings. A prototypical example for this is the connection between symmetric and exterior square L-functions, functorial liftings from classical groups to GL_n, and certain periods, which dates back at least to [22], and is more fully explicated in [15,16]. Some more exotic examples are found, for example, in [9,10], and a general framework which extends beyond classical groups is discussed in [13].

The connection between poles and liftings is well- understood, at least philosophically: one expects that the L function attached to a generic cuspidal representation π and a finite dimensional representation r of the relevant L-group will have a pole at 1 if and only if the stabilizer of a point in general position for the representation r is reductive and π is in the image of the functorial lifting attached to the inclusion of the stabilizer of such a point. For example, in the exterior square representation of $GL_{2n}(\mathbb{C})$, the stabilizer of a point in general position is $Sp_{2n}(\mathbb{C})$, so one expects that a pole of the exterior square L function indicates cuspidal representations which are lifts from SO_{2n+1}. The connection with periods is up to now less well understood.

In order to prove the expected relationship between poles and liftings in specific examples, and in order to draw periods into the picture, it is useful, perhaps essential, to have some sort of an analytic handle on both the L-function and the lifting. An analytic handle on the L-function may be provided by an integral representation, either of Langlands–Shahidi type or otherwise (integral representations which are not of Langlands–Shahidi type are often termed "Rankin–Selberg"). An analytic handle on the lifting may be provided by an explicit construction.

Integral representation of L functions and explicit construction of liftings between automorphic forms on different groups are important subjects in their own right as well. For example, integral representations are, as far as we know, the only way to establish analytic properties of L functions in new cases. When an integral representation produces L functions whose analytic properties are already well understood, it nevertheless provides a new insight into the connection with periods, and identities among periods which can be otherwise quite surprising. This is the case in the present paper.

For explicit construction of liftings, there are two main ideas we know of. Each is related to the other and both are related to the theory of Fourier coefficients attached to nilpotent orbits [8,14].

The first main idea is to use a "small" representation as a kernel function. The prototypical example of this type is the classical theta correspondence [19]. In this type of construction, an automorphic form, which is defined on a large reductive group H is restricted to a pair of commuting reductive subgroups, and integrated against automorphic forms on one member of the pair to produce automorphic forms on the other. In general, there is no reason such a construction should preserve irreducibility, much less be functorial. The right approach seems to be to take automorphic forms on H which only support Fourier coefficients attached to very small nilpotent orbits. For example, a theta function, defined on the group $\widetilde{Sp}_{4mn}(\mathbb{A})$ only supports Fourier coefficients attached to the minimal nilpotent orbit of this group. Its restriction to $Sp_{2n}(\mathbb{A}) \times O_{2m}(\mathbb{A})$ provides a kernel for the theta lifting between these groups. Functoriality of this lifting was established in [29]. This method has enjoyed brilliant success, but also has significant limitations. For example, it is not at all clear how the classical theta correspondence could be extended to other groups of type $C_n \times D_m$: the embedding into Sp_{4mn} is specific to $Sp_{2n} \times O_{2m}$. The results of this paper hint at a possible way around this difficulty.

The second main idea in explicit construction of correspondences is the descent method of Ginzburg, Rallis, and Soudry ([16], see also [20]). This construction treats the Fourier coefficients themselves essentially as global twisted Jacquet modules, mapping representations of larger reductive groups to representations of smaller reductive groups. As before, in general there is no reason for this construction to respect irreducibility, much less be functorial, and a delicate calculus involving Fourier coefficients seems to govern when it is.

In this paper we define and study two new multi-variable Rankin–Selberg integrals, which are defined on the similitude orthogonal groups GSO_{12} and GSO_{18}. These integrals are similar to those considered in [3,6,7,11,12], in that each involves applying a Fourier–Jacobi coefficient to a degenerate Eisenstein series and then pairing the result with a cusp form defined on a suitable reductive subgroup. To be precise, GSO_{6n} has a standard parabolic subgroup Q whose Levi is isomorphic to $GL_{2n} \times GSO_{2n}$. The unipotent radical is a two step nilpotent group and the set of characters of the center may be thought of as the exterior square representation of GL_{2n} twisted by the similitude factor of GSO_{2n}. The stabilizer of a character in general position is isomorphic to

$$C := \{(g_1, g_2) \in GSp_{2n} \times GSO_{2n} : \lambda(g_1) = \lambda(g_2^{-1})\}.$$

Here, λ denotes the similitude factor. The choice of a character in general position as above also determines a projection of the unipotent radical onto a Heisenberg group in $4n^2 + 1$ variables, and a compatible embedding of C into Sp_{4n^2}.

Our Fourier–Jacobi coefficient defines a map from automorphic functions on $GSO_{6n}(\mathbb{A})$ to automorphic functions on $C(\mathbb{A})$. In the case $n = 2$ and 3 we apply this coefficient to a degenerate Eisenstein series on $GSO_{6n}(\mathbb{A})$ induced from a character of the parabolic subgroup P whose Levi factor is isomorphic to $GL_3 \times GL_{3n-3} \times GL_1$. We then pair the result with a pair of cusp forms defined on $GSp_{2n}(\mathbb{A})$ and $GSO_{2n}(\mathbb{A})$ respectively. The results suggest an intriguing connection with the theta correspondence for similitude groups.

Indeed, in the case $n = 2$, the global integral turns out to be Eulerian, and to give an integral representation of

$$L^S(s_1, \widetilde{\Pi} \times \tau_1) L^S(s_2, \widetilde{\Pi} \times \tau_2),$$

where Π is a generic cuspidal automorphic representation of $GSp_4(\mathbb{A})$ and τ_1, τ_2 are two (generic) cuspidal automorphic representations of $GL_2(\mathbb{A})$ having the same central character, so that $\tau_1 \otimes \tau_2$ is a (generic) cuspidal automorphic representation of $GSO_4(\mathbb{A})$. It follows that the original integral has poles along both the plane $s_1 = 1$ and the plane $s_2 = 1$ if and only if Π is the weak lift of $\tau_1 \otimes \tau_2$ corresponding to the embedding

$$GSpin_4(\mathbb{C}) = \{(g_1, g_2) \in GL(2, \mathbb{C})^2 \mid \det g_1 = \det g_2\} \hookrightarrow GSpin_5(\mathbb{C}) = GSp_4(\mathbb{C}).$$

It is known that the functorial lift corresponding to the embedding

$$SO_4(\mathbb{C}) \hookrightarrow SO_5(\mathbb{C})$$

is realized via the theta correspondence. Our Eulerian integral suggests that the Fourier–Jacobi coefficient of the iterated residue of our Eisenstein series provides a kernel for the theta correspondence for similitude groups. This is particularly intriguing since the Fourier–Jacobi coefficient construction extends directly to any group of type D_{3n}, whereas there seems to be no hope of extending the theta correspondence to any representations of such groups which do not factor through the orthogonal quotient in any direct way.

The result is also intriguing in that it points to a possible identity relating our Fourier–Jacobi coefficient with a theta series on $\widetilde{Sp}_{16}(\mathbb{A})$. We are not aware of any way to see such an identity directly.

The integral corresponding to $n = 3$ provides some more evidence for a connection with the theta correspondence, in that the global integral unfolds to a period of GSp_6 which is known to be nonvanishing precisely on the image of the theta lift from GSO_6 [15].

We now describe the contents of this paper. In Sect. 2 we fix notation and describe a family of global integrals, indexed by positive integers n. In Sect. 3 we unfold the global integral corresponding to the case $n = 2$, obtaining a global integral involving the Whittaker function of the cusp form involve which, formally, factors as a product of local zeta integrals. These local zeta integrals are studied in Sects. 5, 6, 7, after certain algebraic results required for the unramified case are established in Sect. 4. Once the local zeta integrals have been studied we return to the global setting for Sects. 8 and 9, where we record the global identity relating the original zeta integral and $L^S(s_1, \widetilde{\Pi} \times \tau_1) L^S(s_2, \widetilde{\Pi} \times \tau_2)$, and deduce a new identity relating poles of these L functions and periods. Finally, in Sect. 10, we briefly describe what happens in the case $n = 3$, omitting details. We remark that the case $n = 1$ is somewhat degenerate, as the split form of GSO_2 is a torus; our global integral appears to vanish identically in this case.

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2 Notation

Write J_n for the matrix

$$\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

If g is an $n \times m$ matrix, write ${}^t g$ for the transpose of g and ${}_t g$ for the "other transpose," $J_m\, {}^t g J_n$. Let $g^* = {}_t g^{-1}$. Let $G = GSO_n$ denote the identity component of $GO_n := \{g \in GL_n : gJ_n\, {}^t g \in GL_1 \cdot J_n\}$. If n is odd, then GO_n is the product of SO_n and the center of GL_n. If n is even, then GSO_n is the semidirect product of SO_n and $\{\text{diag}(\lambda I_{\frac{n}{2}}, I_{\frac{n}{2}}) : \lambda \in GL_1\}$. Here I_k is the $k \times k$ identity matrix. The group GSO_n has a rational character $\lambda : GSO_n \to GL_1$, called the similitude factor, such that

$$gJ_n\, {}^t g = \lambda(g) \cdot J_n, \qquad (g \in GSO_n).$$

The set of upper triangular (resp. diagonal) elements of GSO_n is a Borel subgroup (resp. split maximal torus) which we denote B_{GSO_n} (resp. T_{GSO_n}). A parabolic (resp. Levi) subgroup will be said to be standard if it contains B_{GSO_n} (resp. T_{GSO_n}). The unipotent radical of B_{GSO_n} will be denoted U. We number the simple (relative to $B_{GSO_{2n}}$) roots of $T_{GSO_{2n}}$ in G $\alpha_1, \dots, \alpha_n$ so that $t^{\alpha_i} = t_{ii}/t_{i+1,i+1}$ for $1 \le i \le n-1$, and $t^{\alpha_n} = t_{n-1,n-1}/t_{n+1,n+1}$. Here, we have used the exponential notation for rational characters, i.e., written t^α instead of $\alpha(t)$ for the value of the root α on the torus element t.

Define $m_P : GL_3 \times GL_{3(n-1)} \times GL_1$ into GSO_{6n} by

$$m_P(g_1, g_2, \lambda) \mapsto diag(\lambda g_1, \lambda g_2, g_2^*, g_1^*). \tag{1}$$

Denote the image by M_P. It is a standard Levi subgroup. Let P be the corresponding standard parabolic subgroup. Thus, $P = M_P \ltimes U_P$, where U_P is the unipotent radical. We use (1) to identify M_P with $GL_3 \times GL_{3n-3} \times GL_1$.

Recall that a character of $F^\times\backslash\mathbb{A}^\times$ (i.e., a character of $\mathbb{A}^\times$ trivial on $F^\times$) is normalized if it is trivial on the positive real numbers (embedded into $\mathbb{A}^\times$ diagonally at the finite places). An arbitrary quasicharacter of $F^\times\backslash\mathbb{A}^\times$ may be expressed uniquely as the product of a normalized character and a complex power of the absolute value. If $\chi = (\chi_1, \chi_2, \chi_3)$ is a triple of normalized characters of $F^\times\backslash\mathbb{A}^\times$ and $s = (s_1, s_2, s_3) \in \mathbb{C}^3$, write $(\chi; s)$ for the quasicharacter $M_P(\mathbb{A}) \to \mathbb{C}$ by

$$(\chi; s)(g_1, g_2, \lambda) := \chi_1(\det g_1)|\det g_1|^{s_1}\chi_2(\det g_2)|\det g_2|^{s_2}\chi_3(\lambda)|\lambda|^{s_3}. \tag{2}$$

Then $(\chi; s)(aI_{6n}) = (\chi; s)(m_P(a^{-1}I_3, a^{-1}I_{3(n-1)}, a^2)) = \chi_1^{-3}\chi_2^{3-3n}\chi_3^2(a)|a|^{2s_3-3s_1+(3-3n)s_2}$. The pullback of $(\chi; s)$ to a quasicharacter of $P(\mathbb{A})$ will also be denoted $(\chi; s)$.

Consider the family of induced representations $Ind_{P(\mathbb{A})}^{G(\mathbb{A})}(\chi; s)$ (non-normalized induction), for fixed χ and s varying. Here, we fix a maximal compact subgroup K of $G(\mathbb{A})$ and consider K-finite vectors. The map $Ind_{P(\mathbb{A})}^{G(\mathbb{A})}(\chi; s) \mapsto s$ gives this family the structure of a fiber bundle over $\mathbb{C}^3$. By a section we mean a function $\mathbb{C}^3 \times G(\mathbb{A}) \to \mathbb{C}$, written $(s, g) \mapsto f_{\chi;s}(g)$, such that $f_{\chi;s} \in Ind_{P(\mathbb{A})}^{G(\mathbb{A})}(\chi; s)$ for each $s \in \mathbb{C}^3$. A section $f_{\chi;s}$ is flat if the restriction of $f_{\chi;s}$ to K is independent of s. Write Flat(χ) for the space of flat sections.

For such a function $f_{\chi;s}$, let $E(f_{\chi;s}, g)$ be the corresponding Eisenstein series, defined by

$$E(f_{\chi;s}, g) = \sum_{\gamma \in P(F)\backslash G(F)} f_{\chi;s}(\gamma g)$$

when this sum is convergent and by meromorphic continuation elsewhere. The sum is convergent for $\mathrm{Re}(s_1 - s_2)$ and $\mathrm{Re}(s_2)$ both sufficiently large (Cf. [27], §II.1.5).

Let $Q = M_Q \ltimes U_Q$ be the unique standard parabolic subgroup of G, such that $M_Q \cong GL_{2n} \times GSO_{2n}$. We identify M_Q with $GL_{2n} \times GSO_{2n}$ via the isomorphism

$$m_Q(g_1, g_2) := diag(\lambda(g_2)g_1, g_2, g_1^*), \qquad (g_1 \in GL_{2n},\ g_2 \in GSO_{2n}). \tag{3}$$

The unipotent radical, U_Q, of Q can be described as

$$\left\{ \begin{pmatrix} I_{2n} & X & Y & Z' \\ 0 & I_n & 0 & -{}_tY \\ 0 & 0 & I_n & -{}_tX \\ 0 & 0 & 0 & I_{2n} \end{pmatrix} : Z' + X_tY + Y_tX + {}_tZ' = 0 \right\}.$$

Let ${}^2\wedge_{2n} := \{Z \in \mathrm{Mat}_{2n\times 2n} : {}_tZ = -Z\}$. Then we can define a bijection (which is not a homomorphism) $u_Q : \mathrm{Mat}_{2n\times n} \times \mathrm{Mat}_{2n\times n} \times {}^2\wedge_{2n} \to U_Q$ by

$$u_Q(X, Y, Z) = \begin{pmatrix} I_{2n} & X & Y & Z - \frac{1}{2}(X_tY + Y_tX) \\ 0 & I_n & 0 & -{}_tY \\ 0 & 0 & I_n & -{}_tX \\ 0 & 0 & 0 & I_{2n} \end{pmatrix}, \quad X, Y \in \mathrm{Mat}_{2n\times n}, \quad Z \in {}^2\wedge_{2n}.$$

Then

$$u_Q(X, Y, Z)u_Q(U, V, W) = u_Q(X + U, Y + V, Z + W - \langle X, V\rangle + \langle U, Y\rangle),$$

$(X, Y, U, V \in \mathrm{Mat}_{n\times 2n}, Z, W \in {}^2\wedge_{2n})$, where

$$\langle A, B\rangle := A_tB - B_tA, \qquad (A, B \in \mathrm{Mat}_{n\times 2n}).$$

It follows that $u_Q(X, Y, Z)^{-1} = u_Q(-X, -Y, -Z)$, and that if $[x, y] = xyx^{-1}y^{-1}$ denotes the commutator, then $[u_Q(X, 0, 0),\ u_Q(0, Y, 0)] = u_Q(0, 0, -X_tY + Y_tX) = u_Q(0, 0, \langle Y, X\rangle)$. Define $l(Z) = Tr(Z \cdot diag(I_n, 0)) = \sum_{i=1}^n Z_{i,i}$. For $n \in \mathbb{Z}$ define $\mathcal{H}_{2n+1}$ to be $\mathbb{G}_a^n \times \mathbb{G}_a^n \times \mathbb{G}_a$ equipped with the product

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_{1t}y_2 - y_{1t}x_2).$$

Write r for the map from $\mathrm{Mat}_{2n\times n}$ to row vectors corresponding to unwinding the rows: $r(X) := x_{1,1}, \dots x_{1,n}, x_{2,1}, \dots x_{2n,n}$, and write r' for the similar map which unwinds the rows and negates the last n. Explicitly:

$$r'(Y) = r\begin{pmatrix} Y_1 \\ -Y_2 \end{pmatrix} = (y_{1,1}, \dots y_{1,n}, y_{1,1}, \dots y_{n,n}, -y_{n+1,1}, \dots, -y_{2n,n})$$

for $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \mathrm{Mat}_{2n\times n}$, $Y_1, Y_2 \in \mathrm{Mat}_{n\times n}$. Then we can define a homomorphism from U_Q to $\mathcal{H}_{4n^2+1}$, the Heisenberg group with $4n^2+1$ variables, by

$$j(u_Q(X, Y, Z)) = (r(X), r'(Y), l(Z)).$$

The stabilizer of l in M_Q is

$$C_Q := (GSp_{2n} \times GSO_{2n})^\circ = \{(g_1, g_2) \in GSp_{2n} \times GSO_{2n} \mid \lambda(g_1) = \lambda(g_2)^{-1}\},$$

where $\lambda(g_i)$ is the similitude of g_i. For any subgroup H of $GSp_{2n} \times GSO_{2n}$, let $H^\circ := H \cap C_Q$. The kernel of l is a C_Q-stable subgroup of U_Q, and is also equal to the kernel of j. Note that $\lambda(g_2)$ is also the similitude factor of (g_1, g_2) as an element of GSO_{6n}, and that the center of C_Q is equal to that of GSO_{6n}. Define $T = T_{GSO_n} \cap C_Q$, $B = B_{GSO_n} \cap C_Q$ and $N = U \cap C_Q$. They are a split maximal torus, Borel subgroup, and maximal unipotent subgroup of C_Q, respectively.

The group of automorphisms of $\mathcal{H}_{4n^2+1}$ whose restrictions to the center of $\mathcal{H}_{4n^2+1}$ are the identity is isomorphic to Sp_{4n^2}. Identifying the two groups defines a semidirect product $Sp_{4n^2} \ltimes \mathcal{H}_{4n^2+1}$. Let $R_Q = C_Q \ltimes U_Q$. The homomorphism $j : U_Q \to \mathcal{H}_{4n^2+1}$ extends to a homomorphism $R_Q \to Sp_{4n^2} \ltimes \mathcal{H}_{4n^2+1}$. Indeed, for each $c \in C_Q$, the automorphism of U_Q defined by conjugation by c preserves the kernel of j, and therefore induces an automorphism of $\mathcal{H}_{4n^2+1}$. Moreover, this automorphism is identity on the center of $\mathcal{H}_{4n^2+1}$ because c fixes l. This induces a homomorphism $C_Q \to Sp_{4n^2}$, which we denote by the same symbol j, and which has the defining property that $j(cuc^{-1}) = j(c)j(u)j(c)^{-1}$ for all $c \in C_Q$ and $u \in U_Q$. We may then regard the two homomorphisms together as a single homomorphism (still denoted j) from R_Q to $Sp_{4n^2} \ltimes \mathcal{H}_{4n^2+1}$.

For a positive integer M, identify the Siegel Levi of Sp_{2M} with GL_M via the map $\begin{pmatrix} g & \\ & g^* \end{pmatrix} \mapsto g$. It acts on $\mathcal{H}_{2M+1}$ by $g(x, y, z)g^{-1} = (xg^{-1}, yg, z)$. Note that for $g_1 \in GSp_{2n}$ and $g_2 \in GSO_{2n}$, the matrix $m_Q(g_1, g_2) \in M_Q$ maps into GL_{2n^2} if and only if it normalizes $\{u_Q(X, 0, 0) : X \in \mathrm{Mat}_{2n\times n}\}$, i.e., if and only if g_2 is of the form $\begin{pmatrix} \lambda(g_1^{-1})g_3 & \\ & g_3^* \end{pmatrix}$ for $g_3 \in GL_n$. Write

$$m_Q^1(g_1, g_2) := m_Q\left(g_1, \begin{pmatrix} \lambda(g_1)^{-1}g_2 & \\ & g_2^* \end{pmatrix}\right), \qquad (g_1 \in GSp_{2n},\ g_2 \in GL_n). \quad (4)$$

Then

$$m_Q^1(g_1, g_2)u_Q(X, 0, 0)m_Q^1(g_1^{-1}, g_2^{-1}) = u_Q(g_1Xg_2^{-1}, 0, 0), \qquad (\forall g_1 \in GSp_{2n},\ g_2 \in GL_n),$$

and $j(m_Q^1(g_1, g_2)) \in GL_{2n^2} \subset Sp_{4n^2}$ is the matrix satisfying

$$r(X)j(m_Q^1(g_1, g_2)) = r(g_1^{-1}Xg_2).$$

The determinant map $GL_{2n^2} \to GL_1$ pulls back to a rational character of this subgroup of C_Q which we denote by det. Thus $\det(m_Q^1(g_1, g_2)) = \det g_1^{-n} \det g_2^{2n} = \lambda(g_1)^{-n^2} \det g_2^{2n}$ for $g_1 \in GSp_{2n}, g_2 \in GL_n$. On $T \subset C_Q$, the rational character det coincides with the restriction of the sum of the roots of $T_{GSO_{6n}}$ in $\{u_Q(0, Y, 0) : Y \in \mathrm{Mat}_{2n\times n}\}$.

Let ψ be a additive character on $F\backslash\mathbb{A}$ and $\psi_l(Z) := \psi \circ l$. The group $\mathcal{H}_{4n^2+1}(\mathbb{A})$ has a unique (up to isomorphism) unitary representation, ω_ψ, with central character ψ, which extends to a projective representation of $\mathcal{H}_{4n^2+1}(\mathbb{A}) \rtimes Sp_{4n^2}(\mathbb{A})$ or a genuine representation of $\mathcal{H}_{4n^2+1}(\mathbb{A}) \rtimes \widetilde{Sp}_{4n^2}(\mathbb{A})$, where $\widetilde{Sp}_{4n^2}(\mathbb{A})$ denotes the metaplectic double cover.

Lemma 2.1 *The homomorphism $j : C_Q(\mathbb{A}) \to Sp_{4n^2}(\mathbb{A})$ lifts to a homomorphism $C_Q(\mathbb{A}) \to \widetilde{Sp}_{4n^2}(\mathbb{A})$.*

Proof Write pr for the canonical projection $\widetilde{Sp}_{4n^2}(\mathbb{A}) \to Sp_{4n^2}(\mathbb{A})$. We must show that the exact sequence

$$1 \to \{\pm 1\} \to \mathrm{pr}^{-1}(j(C_Q(\mathbb{A}))) \to j(C_Q(\mathbb{A})) \to 1$$

splits, i.e., that the cocycle determined by any choice of section is a coboundary. The analogous result for $Sp_{2n} \times SO_{2n}$, over a local field is proved in [24], corollary 3.3, p. 36, or [28], lemma 4.4, p. 12. The extension to C_Q follows from section 5.1 of [18]. The global statement then follows from the corresponding local ones. □

Thus we obtain a homomorphism $R_Q(\mathbb{A}) \to \widetilde{Sp}_{4n^2}(\mathbb{A}) \ltimes \mathcal{H}_{4n^2+1}(\mathbb{A})$ which we still denote j. Pulling ω_ψ back through j produces a representation of $R_Q(\mathbb{A})$ which we denote $\omega_{\psi,l}$. This representation can be realized on the space of Schwartz functions on $\mathrm{Mat}_{2n\times n}(\mathbb{A})$ with action by

$$[\omega_{\psi,l}(u_Q(0,0,Z)).\phi] = \psi_l(Z)\phi \qquad [\omega_{\psi,l}(u_Q(X,0,0)).\phi](\xi) = \phi(\xi + X)$$

$$[\omega_{\psi,l}(u_Q(0,Y,0)).\phi](\xi) = \psi_l(\langle Y, \xi\rangle)\phi(\xi) = \psi_l(Y\,{}_t\xi - \xi_t Y)\phi(\xi), \tag{5}$$

$$[\omega_{\psi,l}(m_Q^1(g_1,g_2)).\phi](\xi) = \gamma_{\psi,\det m_Q^1(g_1,g_2)}|\det m_Q^1(g_1,g_2)|^{\frac{1}{2}}\phi(g_1^{-1}\xi g_2).$$

(Cf. [16], p. 8). Here $\gamma_{\psi,a}$ denotes the Weil index. The representation ω_ψ has an automorphic realization via theta functions

$$\theta(\phi, u\widetilde{g}) := \sum_{\xi\in \mathrm{Mat}_{2n\times n}(F)} [\omega_\psi(u\widetilde{g}).\phi](\xi), \qquad (u \in \mathcal{H}_{4n^2+1}(\mathbb{A}),\ \widetilde{g} \in \widetilde{Sp}_{4n^2}(\mathbb{A})).$$

Here $\phi \in \mathcal{S}(\mathrm{Mat}_{2n\times n}(\mathbb{A}))$ (the Schwartz space of $\mathrm{Mat}_{2n\times n}(\mathbb{A})$), $\mathcal{H}_{4n^2+1}(\mathbb{A})$ is identified with the quotient of U_Q by the kernel of l, and Sp_{4n^2} is identified with the subgroup of its automorphism group consisting of all elements which act trivially on the center.

Lemma 2.2 *Consider the Weil representation of* $\mathcal{H}_{2n+1}(\mathbb{A}) \rtimes \widetilde{Sp}_{2n}(\mathbb{A})$ *and its automorphic realization by theta functions. Let* V *be a subgroup of* $\mathcal{H}_{2n+1}$ *which intersects the center* Z *trivially. Thus* V *corresponds to an isotropic subspace of the symplectic space* $\mathcal{H}_{2n+1}/Z$*. Let* $V^{\perp} = \{v' \in \mathcal{H}_{2n+1}/Z : \langle v', V\rangle = 0\} \supset V$*, and let* P_V *be the parabolic subgroup of* Sp_{2n} *which preserves the flag* $0 \subset V \subset V^{\perp} \subset \mathcal{H}_{2n+1}/Z$*. Note that the Levi quotient of* P_V *is canonically isomorphic to* $GL(V) \times Sp(V^{\perp}/V)$*. Thus* P_V *has a projection onto the group* $GL_1 \times Sp(V^{\perp}/V)$ *induced by the canonical map onto the Levi quotient and the determinant map* $\det : GL(V) \to GL_1$*. The function*

$$\widetilde{g} \mapsto \int_{[V]} \theta(\phi; v\widetilde{g})\, dv$$

is invariant by the $\mathbb{A}$*-points of the kernel of this map on the left. (Throughout this paper, if* H *is an algebraic group defined over a global field* F*, then* $[H] := H(F)\backslash H(\mathbb{A})$*.)*

Proof First assume that V is the span of the last k standard basis vectors for some $k \leq n$. Then

$$\int_{(F\backslash\mathbb{A})^k} \sum_{\xi\in F^n} [\omega_\psi((0,\ldots,0,v,0)\widetilde{g}).\phi](\xi)\, dv = \sum_{\xi'\in F^{n-k}} [\omega_\psi(\widetilde{g}).\phi](0,\ldots,0,\xi'),$$

and invariance follows easily from the explicit formulae for ω_ψ given, for example on p. 8 of [16]. The general case follows from this special case, since any isotropic subspace can be mapped to the span of the last k standard basis vectors, for the appropriate value of k, by using an element of $Sp_{2n}(F)$. □

For $f_{\chi;s} \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\chi; s)$, and $\phi \in \mathcal{S}(\mathrm{Mat}_{2n\times n}(\mathbb{A}))$, let

$$E^{\theta(\phi)}(f_{\chi;s}, g) = \int_{[U_Q]} du\ E(f_{\chi;s}, ug)\theta(\phi, j(ug)), \qquad (g \in C_Q(\mathbb{A})). \tag{6}$$

Recall that C_Q was identified above with a subgroup of $GSp_{2n} \times GSO_{2n}$. If $g \in C_Q$ then g_1 will denote its GSp_{2n} component and g_2 will denote its GSO_{2n} component. Now take two characters $\omega_1, \omega_2 : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$, and two cuspforms φ_1, defined on $GSp_{2n}(\mathbb{A})$ and φ_1, defined on $GSO_{2n}(\mathbb{A})$, such that $\varphi_i(a\cdot g) = \omega_i(a)\varphi_i(g)$, for $i = 1$ or 2, $a \in \mathbb{A}^\times$, and $g \in GSp_{2n}(\mathbb{A})$ or $GSO_{2n}(\mathbb{A})$ as appropriate. Choose χ_1, χ_2, χ_3 so that $\chi_1^{-3}\chi_2^{-3}\chi_3^2\omega_1^{-1}\omega_2$ is trivial, and consider

$$I(f_{\chi;s}, \varphi_1, \varphi_2, \phi) = \int_{Z(\mathbb{A})C_Q(F)\backslash C_Q(\mathbb{A})} E^{\theta(\phi)}(f_{\chi;s}, g)\varphi_1(g_1)\varphi(g_2)dg. \tag{7}$$

To simplify the notation, we may also treat the product $\varphi_1\varphi_2$ as a single cuspform defined on the group C_Q, and write $\varphi(g) = \varphi_1(g_1)\varphi_2(g_2)$, and $I(f_{\chi;s}, \varphi, \phi)$, etc. Note that the integral converges absolutely and uniformly as s varies in a compact set, simply because $E^{\theta(\phi)}(f_{\chi;s})$ is of moderate growth, while φ_1 and φ_2 are of rapid decay.

3 Global integral for GSO_{12}

In this section we consider a global integral (7) in the case $n = 2$. Thus $G = GSO_{12}$. If $u_Q(X, Y, Z)$ is an element of U_Q, we fix individual coordinates as follows:

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \\ x_7 & x_8 \end{pmatrix}, \quad Y = \begin{pmatrix} y_8 & y_7 \\ y_6 & y_5 \\ y_4 & y_3 \\ y_2 & y_1 \end{pmatrix} \quad Z = \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_4 & z_5 & 0 & -z_3 \\ z_6 & 0 & -z_5 & -z_2 \\ 0 & -z_6 & -z_4 & -z_1 \end{pmatrix}. \tag{8}$$

Theorem 3.1 *For n in the maximal unipotent subgroup N let $\psi_N(n) = \psi(n_{12} + n_{23} - n_{56} + n_{57})$, and let*

$$W_\varphi(g) = \int_{[N]} \varphi(ng)\psi_N(n)\, dn. \tag{9}$$

Let U_4 be the codimension one subgroup of N defined by the condition $n_{23} = n_{56}$. For $\phi \in \mathcal{S}(\mathrm{Mat}_{4\times 2}(\mathbb{A}))$, $g \in R_Q(\mathbb{A})$, write

$$I_0(\phi, g) = \int_{\mathbb{A}^2} da\, db\, [\omega_\psi(g)\phi] \begin{pmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \psi(-a). \tag{10}$$

Finally, let w be the permutation matrix attached to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 10 & 11 & 12 & 4 & 5 & 8 & 9 & 1 & 2 & 3 & 6 \end{pmatrix}, \tag{11}$$

and let $U_Q^w = U_Q \cap w^{-1}Pw$. Then the global integral (7) is equal to

$$\int_{Z(\mathbb{A})U_4(\mathbb{A})\backslash C_Q(\mathbb{A})} W_\varphi(g) \int_{U_Q^w(\mathbb{A})\backslash U_Q(\mathbb{A})} f_{\chi;s}(wug) I_0(\phi, ug)\, du\, dg. \tag{12}$$

Remark 3.2 The permutation matrix w represents an element of the Weyl group of G relative to T_G. We also record an expression for w as reduced product of simple reflections. We also introduce some notation for elements of the Weyl group. We write $w[i]$ for the simple reflection attached to the simple root α_i, and $w[i_1 i_2 \dots i_k]$ for the product $w[i_1]w[i_2]\dots w[i_k]$. Then $w = w[64321465432465434654]$.

Before proceeding to the proof, we need to know the structure of the set $P\backslash G/R_Q$.

3.1 Description of the double coset space $P\backslash G/R_Q$

Clearly, the identity map $G \to G$ induces a map $\mathrm{pr} : P\backslash G/R_Q \to P\backslash G/Q$. Each element of $P\backslash G/Q$ contains a unique element of the Weyl group which is of minimal length. Recall that the group of permutation matrices which are contained in G maps isomorphically onto the Weyl group of G. A Weyl element of minimal length in its P, Q double coset corresponds to a permutation $\sigma : \{1, \ldots, 12\} \to \{1, \ldots, 12\}$ such that

- $\sigma(13-i) = 13 - \sigma(i), \forall i$,
- σ is an even permutation.
- If $1 \le i < j \le 4,\ 5 \le i < j \le 8$, or $9 \le i < j \le 12$, and if $\{i, j\} \ne \{6, 7\}$, then $\sigma(i) < \sigma(j)$.
- If $1 \le i < j \le 3,\ 4 \le i < j \le 6,\ 7 \le i < j \le 9$, or $10 \le i < j \le 12$, then $\sigma^{-1}(i) < \sigma^{-1}(j)$.

Such a permutation σ is determined by the quadruple

$$(\#(\{1, 2, 3, 4\} \cap \sigma^{-1}(\{3i-2, 3i-1, 3i\})))_{i=1}^{4}.$$

Deleting any zeros in this tuple gives the ordered partition of 4 corresponding to the standard parabolic subgroup $P_\sigma := GL_4 \cap \sigma^{-1} P \sigma$ (Here we identify the permutation σ with the corresponding permutation matrix, which is in GSO_{12}, and identify $g \in GL_4$ with $\mathrm{diag}(g, I_4, g^*) \in GSO_{12}$). Now, for any parabolic subgroup P_o of GSO_4, we have $GSO_4 = P_o SO_4$. It follows that $g \mapsto \sigma\,\mathrm{diag}(g, I_4, g^*)$ induces a bijection $P_\sigma \backslash GL_4 / GSp_4 \leftrightarrow \mathrm{pr}^{-1}(P \cdot \sigma \cdot Q) \subset P \backslash G / R_Q$. Therefore we must study the space $P' \backslash GL_4 / GSp_4$, where P' is an arbitrary parabolic subgroup of GL_4.

Lemma 3.3 *Let S be a subset of the set of simple roots in the root system of type A_3. Let P_S, P'_S denote the standard parabolic subgroups GL_4, and SO_6, respectively, corresponding to S. Then $P_S \backslash GL_4 / GSp_4$ and $P'_S \backslash SO_6 / SO_5$ are in canonical bijection.*

Proof This follows from considering the coverings of SO_6 and GL_4 by the group $GSpin_6$ which are described in [20] and section 2.3 of [1], respectively. The preimage of SO_5 in $GSpin_6$ is $GSpin_5 = GSp_4$. Since the kernels of both projections are contained in the central torus of $GSpin_6$, which is contained in any parabolic subgroup of $GSpin_6$ it follows that both $P_S \backslash GL_4 / GSp_4$ and $P'_S \backslash SO_6 / SO_5$ are in canonical bijection with $P''_S \backslash GSpin_6 / GSpin_5$, where P''_S is the parabolic subgroup of $GSpin_6$ determined by S. □

Now, in considering SO_6/SO_5, we embed SO_5 into SO_6 as the stabilizer of a fixed anisotropic element v_0 of the standard representation of SO_6. Then $P''_S \backslash SO_6 / SO_5$ may be identified with the set of P'_S-orbits in $SO_6 \cdot v_0$. For concreteness, take SO_6 to be defined using the quadratic form associated to the matrix J_6, and take $v_0 = {}^t[0, 0, 1, 1, 0, 0]$. The SO_6 orbit of v_0 is the set of vectors satisfying ${}^t v \cdot J_6 \cdot v = {}^t v_0 \cdot J_6 \cdot v = 2$. Note that each of the permutation matrices representing a simple reflection attached to an outer node in the Dynkin diagram maps v_0 to $v_1 := {}^t[0, 1, 0, 0, 1, 0]$, and that a permutation matrix representing the simple reflection attached to the middle node of the Dynkin diagram maps v_1 to $v_2 := {}^t[1, 0, 0, 0, 0, 1]$.

Lemma 3.4 *Number the roots of SO_6 so that α_2 is the middle root. (This is not the standard numbering for SO_6, but it matches the standard numbering for GL_4, and the*

S	*Orbit reps in V*	*Double coset reps*
$\emptyset$	v_0, v_1, v_2	$e, w[1], w[2]w[1]$
$\{1\}, \{3\}$, *or* $\{1, 3\}$	v_0, v_2	$e, w[1]$
$\{2\}$	v_0, v_1	$e, w[2]w[1]$
$\{1, 2\}$ $\{2, 3\}$ *or* $\{1, 2, 3\}$	v_0	e

numbering inherited as a subgroup of GSO_{12}). *Write* V *for the standard representation of* SO_6. *The decomposition of* $SO_6 \cdot v_0$ *into* P'_S *orbits is as follows*:

Proof Direct calculation. □

Remark 3.5 As elements of GSO_{12}, the double coset representatives are identified with permutations of $\{1, \ldots, 12\}$. Writing these permutations in cycle notation, we have $w[1] = (1, 2)(11, 12)$, $w[2]w[1] = (1, 3, 2)(10, 11, 12)$. Replacing $w[1]$ by $w[3]$ in any of the representatives above produces a different element of the same double coset.

3.2 Proof of Theorem 3.1

We now apply this description of $P\backslash G/R_Q$, to the study of $I(f_{\chi;s}, \varphi, \phi)$. For this section only, let w_0 be the permutation matrix attached to (11), and let w be an arbitrary representative for $P(F)\backslash G(F)/R_Q(F)$.

The global integral (7) is equal to

$$\sum_{w \in P(F)\backslash G(F)/R_Q(F)} I_w(f_{\chi;s}, \varphi, \phi),$$

$$\text{where} \quad I_w(f_{\chi;s}, \varphi, \phi) = \int_{Z(\mathbb{A})U_Q^w(F)C_Q^w(F)\backslash C_Q(\mathbb{A})U_Q(\mathbb{A})} f_{\chi;s}(wug)\theta(\phi, j(ug))\varphi(g)dg,$$

where $C_Q^w = C_Q \cap w^{-1}Pw$, and $U_Q^w = U_Q \cap w^{-1}Pw$.

Proposition 3.6 *If* w *does not lie in the double coset containing* w_0, *then* $I_w(f_{\chi;s}, \varphi_1, \varphi_2) = 0$. *Consequently*, $I(f_{\chi;s}, \varphi_1, \varphi_2) = I_{w_0}(f_{\chi;s}, \varphi_1, \varphi_2)$.

Proof Write $w = \sigma\nu$ where w is a permutation of $\{1, \ldots 12\}$ satisfying the four conditions listed at the beginning of Sect. 3.1, and ν is one of the representatives for $P_\sigma\backslash M_Q/C_Q$ given in the table in Lemma 3.4. The integral $I_w(f_{\chi;s}, \varphi, \phi)$ vanishes if ψ_l is nontrivial on $U_Q^w := U_Q \cap w^{-1}Pw$, or equivalently, if the character $\nu \cdot \psi_l$ obtained by composing ψ_l with conjugation by ν is nontrivial on $U_Q \cap \sigma^{-1}P\sigma$. For our representatives ν, we have

$$\nu \cdot \psi_l(u_Q(0, 0, Z)) = \begin{cases} \psi(Z_{1,9} + Z_{2,10}), & \nu = e, \\ \psi(Z_{1,10} + Z_{2,9}), & \nu = w[1], \\ \psi(Z_{1,11} + Z_{3,9}), & \nu = w[2]w[1]. \end{cases}$$

There are 25 possibilities for σ. However, it's clear that $I_w(f_{\chi;s}, \varphi, \phi)$ vanishes, regardless of ν, if $\sigma(1) < \sigma(9)$, or if $\sigma(2) < \sigma(10)$. This eliminates all but seven possibilities for σ. For the remaining seven, the above criterion shows that $I_w(f_{\chi;s}, \varphi, \phi)$ vanishes unless ν is trivial.

Assume now that ψ_l is trivial on U_Q^w. This means that the image of U_Q^w in the Heisenberg group intersects the center trivially, and maps to an isotropic subspace of the quotient $\mathcal{H}_{17}/Z(\mathcal{H}_{17})$ (which has the structure of a symplectic vector group). Write V for this subspace and $V^\perp$ for its perp space. Define $P_V \subset Sp_{16}$ as in Lemma 2.2, and let P_V^1 denote the kernel of the canonical projection $P_V \to GL_1 \times Sp(V^\perp/V)$. It follows immediately from Lemma 2.2 and the cuspidality of φ that $I_w(f_{\chi;s}, \varphi, \phi)$ vanishes whenever $P_V^1 \cap C_Q$ contains the unipotent radical of a proper parabolic subgroup of C_Q. This applies to each of the remaining double coset representatives, except for w_0. □

The following lemma is useful in our calculation.

Lemma 3.7 *Let f_1, f_2 be two continuous functions on $(F\backslash\mathbb{A})^n$, and ψ a nontrivial additive character on $F\backslash\mathbb{A}$. Then*

$$\int_{(F\backslash\mathbb{A})^n} dx\, f_1(x) f_2(x) = \sum_{\alpha\in F^n} \int_{(F\backslash\mathbb{A})^n} dx\, f_1(x)\psi(\alpha\cdot x) \int_{(F\backslash\mathbb{A})^n} dy\, f_2(y)\psi^{-1}(\alpha\cdot y). \tag{13}$$

Moreover, if $\int_{(F\backslash\mathbb{A})^n} dx\, f_1(x) = 0$, then one can replace $\sum_{\alpha\in F^n}$ by $\sum_{\alpha\in F^n-\{0\}}$ in the formula above.

Proof By Fourier theory on $F\backslash\mathbb{A}$,

$$f_i(x) = \sum_{\alpha\in F^n} \psi(-\alpha\cdot x)\hat{f}_i(\alpha),$$

where $\hat{f}_i(\alpha) = \int_{(F\backslash\mathbb{A})^n} dx\, f_1(x)\psi(\alpha x)$ for $i = 1, 2$. So the left hand side of (13) is equal to

$$\sum_{\alpha,\beta\in F^n} \hat{f}_1(\alpha)\hat{f}_2(\beta) \int_{(F\backslash\mathbb{A})^n} dx\, \psi(-(\alpha+\beta)\cdot x). \tag{14}$$

The integral on x vanishes when $\alpha+\beta \neq 0$, and equals 1 if $\alpha+\beta = 0$, so (14) equals

$$\sum_{\alpha\in F^n} \hat{f}_1(\alpha)\hat{f}_2(-\alpha),$$

which is the right hand side of (13). When $\int_{(F\backslash\mathbb{A})^n} dx\, f_1(x) = 0$, we have $\hat{f}_1(0) = 0$, so we can replace $\sum_{\alpha\in F^n}$ by $\sum_{\alpha\in F^n-0}$. □

From now on, let $w = w[64321465432465434654]$. Then

$$U_Q^w := U_Q \cap w^{-1}Pw = \left\{ u_Q^w(y_7, y_8) = u_Q\left(0, \begin{pmatrix} y_8 & y_7 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right) : y_7, y_8 \in F \right\}, \tag{15}$$

$$C_Q^w := C_Q \cap w^{-1}Pw = (P_1 \times P_2)^\circ$$

where P_1 is the Klingen parabolic subgroup of GSp_4 and P_2 is the Siegel parabolic subgroup of GSO_4. Let $P_1 = M_1 \ltimes U_1$ and $P_2 = M_2 \ltimes U_2$ be their Levi decompositions. Note that $f_{\chi;s}(wug) = f_{\chi;s}(wg)$ for all $u \in U_Q^w$. So, by Proposition 3.6, $I(f_{\chi;s}, \varphi, \phi)$ is equal to

$$\int_{Z(\mathbb{A})C_Q^w(F)\backslash C_Q(\mathbb{A})} \varphi(g) \int_{U_Q^w(\mathbb{A})\backslash U(\mathbb{A})} f_{\chi;s}(wu_2g) \int_{[U_Q^w]} \theta(\phi, j(u_1u_2g))\, du_1\, du_2\, dg. \tag{16}$$

But, for $u = u_Q^w(y_7, y_8)$ (defined in (15)),

$$[\omega_\psi(j(u))\phi_1](\xi) = \phi_1(\xi)\psi(\xi_7 y_7 + \xi_8 y_8), \qquad \xi = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \\ \xi_5 & \xi_6 \\ \xi_7 & \xi_8 \end{pmatrix}, \tag{17}$$

for any $\phi_1 \in \mathcal{S}(\mathrm{Mat}_{4\times 2}(\mathbb{A}))$. It follows that (16) is equal to

$$\int_{Z(\mathbb{A})C_Q^w(F)\backslash C_Q(\mathbb{A})} \varphi(g) \int_{U_Q^w(\mathbb{A})\backslash U(\mathbb{A})} f_{\chi;s}(wug)\theta_0(\phi, j(ug))\, du\, dg, \tag{18}$$

where

$$\theta_0(\phi, u\widetilde{g}) := \sum_{\xi \in \mathrm{Mat}_{3\times 2}(F)} [\omega_\psi(u\widetilde{g}).\phi]\begin{pmatrix} \xi \\ 0 \end{pmatrix}, \qquad (u \in \mathcal{H}_{17}(\mathbb{A}),\ \widetilde{g} \in \widetilde{Sp}_{16}(\mathbb{A}))$$

Now, $C_Q^w = (M_1 \times M_2)^\circ \ltimes (U_1 \times U_2)$, and $f_{\chi;s}(wu_1u_2g) = f_{\chi;s}(wg)$, for any $u_1 \in U_1$, $u_2 \in U_2$, and $g \in G$. Moreover, if

$$U_2(a) = \begin{pmatrix} 1 & & a & \\ & 1 & & -a \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{19}$$

then $[\omega_\psi(U_2(a)u_1g)\phi]\begin{pmatrix}\xi_1 & \xi_2\\ \xi_3 & \xi_4\\ \xi_5 & \xi_6\\ 0 & 0\end{pmatrix} = \psi(a(\xi_3\xi_6-\xi_4\xi_5))[\omega_\psi(u_1g)\phi]\begin{pmatrix}\xi_1 & \xi_2\\ \xi_3 & \xi_4\\ \xi_5 & \xi_6\\ 0 & 0\end{pmatrix}$. It then follows from the cuspidality of φ that (18) is equal to

$$\int_{Z(\mathbb{A})C_Q^w(F)\backslash C_Q(\mathbb{A})}\varphi(g)\int_{U_Q^w(\mathbb{A})\backslash U(\mathbb{A})} f_{\chi;s}(wug)\theta_1(\phi, j(ug))\,du\,dg, \tag{20}$$

where

$$\theta_1(\phi, u\widetilde{g}) := \sum_{\xi\in \mathrm{Mat}_{3\times 2}(F):\ (\xi_3\xi_6-\xi_4\xi_5)\neq 0}[\omega_\psi(u\widetilde{g}).\phi]\begin{pmatrix}\xi\\ 0\end{pmatrix}, \qquad (u\in\mathcal{H}_{17}(\mathbb{A}),\ \widetilde{g}\in\widetilde{Sp}_{16}(\mathbb{A})).$$

The group $(M_1\times M_2)^\circ$ is the set of all

$$m(g_3,g_4,t) := diag(t\det g_3, g_3, t^{-1}; \det g_3 g_4, g_4^*; t\det g_3, \det g_3\cdot g_3^*, t^{-1}) \tag{21}$$

where $g_3\in GL_2$, $g_4\in GL_2$ and $t\in GL_1$. Note that the summation over (ξ_1,ξ_2) is invariant under the action of $(M_1\times M_2)^\circ$. Consider the action of $(M_1\times M_2)^\circ$ on $\{(\xi_3,\xi_4,\xi_5,\xi_6)\mid \det\begin{pmatrix}\xi_3 & \xi_4\\ \xi_5 & \xi_6\end{pmatrix}\neq 0\}$. It is not hard to see that it is transitive, and the stabilizer of $(1,0,0,1)$ is $\{m(t,g_3,g_4)\mid g_4 = g_3\cdot\det(g_3)^{-1}\}$, which is the same as $\{M_5(t,g_3) = diag(t\det g_3, g_3, t^{-1}; g_3, g_3^*\det g_3; t\det g_3, g_3^*\det g_3, t^{-1}) : g_3\in GL_2,\ t\in GL_1\}$. We denote this group by M_5. Let ψ_{U_2} be a character on U_2 defined by $\psi_{U_2}(U_2(a)) = \psi(a)$, then Eq. (20) is equal to

$$\int_{Z(\mathbb{A})M_5(F)U_1(F)U_2(\mathbb{A})\backslash C_Q(\mathbb{A})}\varphi^{(U_2,\psi_{U_2})}(g)\int_{U_Q^w(\mathbb{A})\backslash U(\mathbb{A})} f_{\chi;s}(wug)\theta_2(j(ug))\,du\,dg, \tag{22}$$

where

$$\theta_2(j(ug)) := \sum_{(\xi_1,\xi_2)\in F^2}[\omega_\psi(j(ug)).\phi]\begin{pmatrix}\xi_1 & \xi_2\\ 1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix},$$

and the notation $\varphi^{(U_2,\psi_{U_2})}$ is defined as follows. For any unipotent subgroup V of an F-group H, character ϑ of V, and smooth left $V(F)$-invariant function Φ on $H(\mathbb{A})$, we define

$$\Phi^{(V,\vartheta)}(h) := \int_{[V]}\Phi(vh)\vartheta(v)\,dv.$$

Now, U_1 consists of elements

$$U_1(a,b,c) = \begin{pmatrix} 1 & a & b & c \\ & 1 & & b \\ & & 1 & -a \\ & & & 1 \end{pmatrix} \in GSp_4, \tag{23}$$

and for any $g \in R_Q$,

$$[\omega_\psi(U_1(0,0,c)g)\phi]\begin{pmatrix} \xi_1 & \xi_2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = [\omega_\psi(g)\phi]\begin{pmatrix} \xi_1 & \xi_2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Factoring the integration over U_1 and applying Lemma 3.7 to functions

$$(a,b) \mapsto \omega_\psi(U_1(a,b,0)g)\phi\begin{pmatrix} \xi_1 & \xi_2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (a,b) \mapsto \int_{F\backslash\mathbb{A}} dc\, \varphi_2(U_1(a,b,c)g),$$

we deduce that (22) is equal to

$$\int\limits_{Z(\mathbb{A})M_5(F)U_1(\mathbb{A})U_2(\mathbb{A})\backslash C_Q(\mathbb{A})} \varphi^{(U_3,\psi_{U_3}^{\alpha,\beta})}(g) \int\limits_{U_Q^w(\mathbb{A})\backslash U(\mathbb{A})} f_{\chi;s}(wug)\theta_2^{(U_1,\psi_{U_1}^{-\alpha,-\beta})}(j(ug))\, du\, dg, \tag{24}$$

where $\psi_{U_1}^{\alpha,\beta}(U_1(a,b,c)) = \psi(\alpha a + \beta b)$, $U_3 = U_1U_2$, and $\psi_3^{\alpha,\beta} = \psi_{U_2}\psi_{U_1}^{\alpha,\beta}$. The group $M_5(F)$ acts on $U_1(\mathbb{A})$ and permutes the nontrivial characters $\psi_{U_1}^{\alpha,\beta}$ transitively. The stabilizer of $\psi_{U_1} := \psi_{U_1}^{1,0}$ is

$$M_6 := \{M_6(a_1,a_2,a_4)\}, \qquad \text{where} \quad M_6(a_1,a_2,a_4) = M_5\left(a_4^{-1}, \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}\right). \tag{25}$$

Hence Eq. (24) is equal to

$$\int\limits_{Z(\mathbb{A})M_6(F)U_1(\mathbb{A})U_2(\mathbb{A})\backslash C_Q(\mathbb{A})} \varphi^{(U_3,\psi_{U_3})}(g) \int\limits_{U_Q^w(\mathbb{A})\backslash U(\mathbb{A})} f_{\chi;s}(wug)\theta_2^{(U_1,\overline{\psi}_{U_1})}(j(ug))\, du\, dg, \tag{26}$$

where $\psi_{U_3} = \psi_{U_1}\psi_{U_2}$.

Note that

$$[\omega_\psi(U_1(a,b,0)g)\phi]\begin{pmatrix} \xi_1 & \xi_2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = [\omega_\psi(g)\phi]\begin{pmatrix} \xi_1 + a & \xi_2 + b \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and that for $\xi_1, \xi_2 \in F$, $\psi(\alpha \cdot (a+\xi_1) + \beta \cdot (b+\xi_2)) = \psi(\alpha \cdot a + \beta \cdot b)$. We can combine the summation on (ξ_1, ξ_2) with the integral over (a, b). It follows that $\theta_2^{(U_1, \overline{\psi}_{U_1})}(g) = I_0(\phi, g)$, defined in (10). Let $M_6 = U_6 T_6$ be the Levi decomposition. It is not hard to see that both $I_0(\phi, g)$ and the function $g \mapsto f(wg)$ are invariant on the left by $U_6(\mathbb{A})$. So, (26) is equal to

$$\int\limits_{Z(\mathbb{A})T_6(F)U_1(\mathbb{A})U_2(\mathbb{A})\backslash C_Q(\mathbb{A})} \varphi^{(U_4, \psi_{U_4})}(g) \int\limits_{U_Q^w(\mathbb{A})\backslash U(\mathbb{A})} f_{\chi;s}(wug)\theta_2^{(U_1, \overline{\psi}_{U_1})}(j(ug))\, du\, dg, \tag{27}$$

where $U_4 = U_3 U_6$, and ψ_{U_4} is the extension of ψ_{U_3} to a character of U_4 which is trivial on U_6. Now,

$$\varphi^{(U_4, \psi_{U_4})}(g) = \int_{F\backslash\mathbb{A}} \varphi_1^{(U_1, \psi_1)}\left(\begin{pmatrix} 1 &&& \\ & 1 & r & \\ && 1 & \\ &&& 1 \end{pmatrix} g_1\right) \varphi_2^{(U_2, \psi_2)}\left(\begin{pmatrix} 1 & r && \\ & 1 && \\ && 1 & -r \\ &&& 1 \end{pmatrix} g_2\right).$$

Let N_1 denote the standard maximal unipotent subgroup of GSp_4 and N_2 that of GSO_4. Let $\psi_{N_1}^{\gamma}$ and $\psi_{N_2}^{\gamma}$ be the extensions of ψ_{U_1} and ψ_{U_2} to characters of $N_1(\mathbb{A})$ and $N_2(\mathbb{A})$ respectively, such that

$$\psi_{N_1}^{\gamma}\begin{pmatrix} 1 &&& \\ & 1 & r & \\ && 1 & \\ &&& 1 \end{pmatrix} = \psi_{N_2}^{\gamma}\begin{pmatrix} 1 & r && \\ & 1 && \\ && 1 & -r \\ &&& 1 \end{pmatrix} = \psi(\gamma r).$$

Then it follows from Lemma 3.7 (and the cuspidality of φ_1, φ_2,) that

$$\varphi^{(U_4, \psi_{U_4})}(g) = \sum_{\gamma \in F^\times} \varphi_1^{(N_1, \psi_{N_1}^{\gamma})}(g_1)\varphi_2^{(N_2, \psi_{N_2}^{-\gamma})}(g_2) = \varphi^{(N, \psi_N^{\gamma})}(g),$$

where $N = N_1 N_2$, a maximal unipotent subgroup of C_Q, and for $\gamma \in F^\times$, $\psi_N^{\gamma} = \psi_{N_1}^{\gamma}\psi_{N_2}^{-\gamma}$. We plug this in to (27). The group T_6 acts on the characters ψ_N^{γ} transitively, and the stabilizer of $\psi_N := \psi_N^1$ is the center of C_Q. Since $\varphi^{(N, \psi_N)}(g) = W_{\varphi}(g)$, this completes the Proof of Theorem 3.1.

4 Preparation for the unramified calculation

In this section, we establish some results which describe the structure of the symmetric algebras of some representations of $Sp_4 \times SL_2$, and $Sp_4 \times SL_2 \times SL_2$, which will be used to relate our local zeta integrals to products of Langlands L-functions.

We first consider some representations of $Sp_4 \times SL_2$. Let ϖ_1 and ϖ_2 denote the fundamental weights of Sp_4 and ϖ that of SL_2. Write $V_{(n_1, n_2; m)}$ for the irreducible

$Sp_4 \times SL_2$-module with highest weight $n_1\varpi_1 + n_2\varpi_2 + m\varpi$, and let $[n_1, n_2; m]$ denote its trace.

Proposition 4.1 *For i, j, n_1, n_2 and m all non-negative integers, let $\mu_{i,j}(n_1, n_2; m)$ denote the multiplicity of $V_{(n_1,n_2;m)}$ in* $\operatorname{sym}^i V_{(1,0;1)} \otimes \operatorname{sym}^j V_{(1,0;0)}$. *Then*

$$\sum_{i,j,n_1,n_2,m=0}^{\infty} \mu_{i,j}(n_1, n_2; m) t_1^{n_1} t_2^{n_2} t_3^m x^i y^j = \frac{1 - t_1t_2t_3x^3y^2}{(1-t_1t_3x)(1-x^2)(1-t_2x^2)(1-t_1y)(1-t_3xy)(1-t_2t_3xy)(1-t_1x^2y)(1-t_2x^2y^2)}.$$

Proof We first describe $\operatorname{sym}^j V_{(1,0;1)}$. Write $V_{(n_1,n_2)}$ for the irreducible Sp_4-module with highest weight $n_1\varpi_1 + n_2\varpi_2$. Then we may regard $V_{(1,0;1)}$ as two copies of $V_{(1,0)}$ with the standard torus of SL_2 acting on them by eigenvalues, say, η and η^{-1}. Then, using the well known fact that $\operatorname{sym}^k V_{(1,0)} = V_{(j,0)}$, and the decomposition of $V_{(m,0)} \otimes V_{(n,0)}$ described in [23],

$$\operatorname{Tr} \operatorname{sym}^n V_{(1,0;1)} = \sum_{n_1=0}^{n} \eta^{n-2n_1} \sum_{\ell=0}^{\min(n_1,n-n_1)} \sum_{j=0}^{\ell} [n-2\ell, j] = \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\ell} [n-2\ell, j; n-2\ell],$$

whence

$$\sum_{n=0}^{\infty} x^n \operatorname{Tr} \operatorname{sym}^n V_{(1,0;1)} = \sum_{j,k,n=0}^{\infty} x^{n+2j+2k}[n, j; n] = \frac{1}{1-x^2} \sum_{j,n=0}^{\infty} [n, j; n] x^{n+2j}.$$

Using [23] again to compute $V_{(n,j)} \otimes V_{(m,0)}$ one obtains

$$\begin{aligned} &\sum_{n,m,j=0}^{\infty} [n, j][m, 0] t^n x^{n+2j} y^m \\ &= \frac{1}{1-txy} \sum_{\substack{n,j,i_2,k=0 \\ n+i_2 \ge k}}^{\infty} [n+m+i_2-k_2, j+k_2] t^n x^{n+2j+2i_2} y^{m+i_2+k}. \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{i,j,n_1,n_2,m=0}^{\infty} \mu_{i,j}(n_1, n_2; m) t_1^{n_1} t_2^{n_2} t_3^m x^i y^j \\ &= \frac{1}{1-x^2} \frac{1}{1-t_3xy} \sum_{\substack{n_1,n_2,i_2,k,m=0 \\ n+i_2 \ge k}}^{\infty} t_1^{n_1+m+i_2-k} t_2^{n_2+k} t_3^{n_1} x^{n_1+2n_2+2i_2} y^{m+i_2+k} \\ &= \frac{1}{1-x^2} \frac{1}{1-t_3xy} \frac{1}{1-t_2x^2} \frac{1}{1-t_1y} \sum_{\substack{n_1,i_2,k=0 \\ n+i_2 \ge k}}^{\infty} t_1^{n_1+i_2-k} t_2^k t_3^{n_1} x^{n_1+2n_2+2i_2} y^{i_2+k}, \end{aligned}$$

and the result then follows from the identity

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{j,k=0}^{n_2} u^{n_2}v^k w^j = \frac{1-u^2vw}{(1-u)(1-uv)(1-uw)(1-uvw)}.$$

□

Corollary 4.2 *For* $n=(n_1,n_2,n_3)$ *let* $[n]=[n_1,n_2;n_3]$, *and, let*

$$\begin{aligned} a &= {}^t\begin{bmatrix}1 & 0 & 1 & 1 & 2 & 2 & 2\end{bmatrix} \\ b &= {}^t\begin{bmatrix}0 & 1 & 1 & 1 & 0 & 1 & 2\end{bmatrix}\end{aligned}, \qquad g = {}^t\begin{bmatrix}1 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 1 & 0 & 1\\ 1 & 0 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$$

Then $\sum_{i=0}^{\infty} x^i \operatorname{Tr}\operatorname{sym}^i V_{(1,0;1)} \sum_{j=0}^{\infty} y^j \operatorname{Tr}\operatorname{sym}^j V_{(1,0;0)}$ *equals*

$$\frac{1}{1-x^2}\left[\sum_n [n\cdot g]x^{n\cdot a}y^{n\cdot b} - \sum_n [n\cdot g+(1,1,1)]x^{n\cdot a+3}y^{n\cdot b+2}\right],$$

where n *is summed over row vectors* $n=(n_1,\ldots,n_7)\in\mathbb{Z}_{\geq 0}^7$.

Our next result describes the decomposition of $\operatorname{sym}^i V_{(1,0;1)} \otimes \operatorname{sym}^j V_{(1,0;0)} \otimes \operatorname{sym}^k V_{(1,0;0)}$. It is an identity of rational functions in 6 variables. To keep the notation short, we often reflect dependence only on arguments which will vary. Let

$$\begin{aligned} d(t_1,t_2) &= (1-t_1t_3x)(1-t_2x^2)(1-t_1y)(1-t_3xy)(1-t_2t_3xy)(1-t_1x^2y)(1-t_2x^2y^2) \\ &= \left[\sum_{n\in\mathbb{Z}_{\geq 0}^7} t_1^{n\cdot g_1} t_2^{n\cdot g_2} t_3^{n\cdot g_3} x^{n\cdot a} y^{n\cdot b}\right]^{-1}, \end{aligned}$$

where g_1, g_2 and g_3 are the three columns of the matrix g in Corollary 4.2. Define rational functions $\gamma_1,\ldots,\gamma_7$ by

$$\begin{aligned} &\sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2}\sum_{k=0}^{n_1+n_2} u^{n_1}v^{n_2}w^k \\ &\quad = \gamma_1(u,v,w)+\gamma_2(u,v,w)u^{N_1}+\gamma_3(u,v,w)v^{N_2}+\gamma_4(u,v,w)u^{N_1}v^{N_2} \\ &\qquad +\gamma_5(u,v,w)(uw)^{N_1}+\gamma_6(u,v,w)(vw)^{N_2}+\gamma_7(u,v,w)(uw)^{N_1}(vw)^{N_2}, \end{aligned}$$

and let $c_i=\gamma_i(t_1/z, t_1z/t_2, t_2z/t_1)$.

Proposition 4.3 *Let* $\mu_{i,j,k}(n_1,n_2;m)$ *denote the multiplicity of* $V_{(n_1,n_2;m)}$ *in* $\operatorname{sym}^i V_{(1,0;1)} \otimes \operatorname{sym}^j V_{(1,0;0)} \otimes \operatorname{sym}^k V_{(1,0;0)}$. *Then*

$$\sum_{i,j,k,n_1,n_2,m=0}^{\infty} \mu_{i,j,k}(n_1,n_2;m) t_1^{n_1} t_2^{n_2} t_3^m x^i y^j z^k = (1-x^2)^{-1}(1-t_1 z)^{-1}$$
$$\times \frac{c_1 \nu(t_2 z)}{d(z,t_2)} + \frac{c_2 \nu(t_1 t_2)}{d(t_1,t_2)} + \frac{c_3 \nu(t_1 z^2)}{d(z,t_1 z)} + \frac{c_4 \nu(t_1^2 z)}{d(t_1,t_1 z,)} + \frac{c_5 \nu(t_2^2 z)}{d(t_2 z,t_2)}$$
$$+ \frac{c_6 \nu(t_2 z^3)}{d(z,t_2 z^2)} + \frac{c_7 \nu(t_2^2 z^3)}{d(t_2 z,t_2 z^2)},$$

where $c_1, \ldots, c_7$ *and* d *are as above and let* $\nu(u) = 1 - u t_3 x^3 y^2$.

Proof From [23] again, one deduces that

$$[m_1,m_2] \cdot \sum_{\ell=0}^{\infty} [\ell,0] x^\ell = \sum_{\ell=0}^{\infty} \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \sum_{k=0}^{i_1+i_2} [i_1+i_2-k+\ell, m_2-i_2+k] x^{\ell+m_1-i_1+i_2+k}.$$

Combining with Corollary 4.2 gives

$$\sum_{i,j,k,n_1,n_2,m=0}^{\infty} \mu_{i,j,k}(n_1,n_2;m) t_1^{n_1} t_2^{n_2} t_3^m x^i y^j z^k$$
$$= \frac{1}{1-x^2} \frac{1}{1-t_1 z} \sum_{n \in \mathbb{Z}_{\geq 0}^7} x^{n\cdot a} y^{n\cdot b} t_3^{n\cdot g_3} \left(\sum_{i_1=0}^{n\cdot g_1} \sum_{i_2=0}^{n\cdot g_2} \sum_{k=0}^{i_1+i_2} t_1^{i_1+i_2-k} t_2^{n\cdot g_2 - i_2 + k} z^{n\cdot g_1 - i_1 + i_2 + k} \right.$$
$$\left. - x^3 y^2 t_3 \sum_{i_1=0}^{n\cdot g_1+1} \sum_{i_2=0}^{n\cdot g_2+1} \sum_{k=0}^{i_1+i_2} t_1^{i_1+i_2-k} t_2^{n\cdot g_2+1-i_2+k} z^{n\cdot g_1+1-i_1+i_2+k} \right),$$

and and simplifying this rational function gives the result. □

Proposition 4.4 *Let* $V_{(n_1,n_2;n_3;n_4)}$ *denote the irreducible representation of* $Sp_4 \times SL_2 \times SL_2$ *such that* Sp_4 *acts with highest weight* $n_1\varpi_1 + n_2\varpi_2$*, the first* SL_2 *acts with highest weight* n_3*, and the second* SL_2 *acts with highest weight* n_4*. For* $n = (n_1,n_2;n_3;n_4)$*, let* $\mu_{i,j}(n)$ *denote the multiplicity of* V_n *in* $\mathrm{sym}^i V_{(1,0;1;0)} \otimes \mathrm{sym}^j V_{(1,0;0;1)}$*. Then*

$$\sum_{i,j=0}^{\infty} \sum_{n \in \mathbb{Z}_{\geq 0}^4} \mu_{i,j}(n) t_1^{n_1} t_2^{n_2} t_3^{n_3} t_4^{n_4} x^i y^j = \frac{\nu(x,y,t)}{\delta(x,y,t)},$$

where

$$
\begin{aligned}
\nu(x, y, t) = 1 &- t_1t_2t_3t_4^2x^3y^2 - t_1t_2t_3^2t_4x^2y^3 - t_1^2t_3t_4x^3y^3 - t_1^2t_2t_3t_4x^3y^3 - t_1t_2t_3^2t_4x^4y^3 \\
&- t_1t_2t_3t_4^2x^3y^4 - t_1^2t_2t_3^2x^4y^4 - t_1^2t_2t_4^2x^4y^4 + 2t_1^2t_2t_3^2t_4^2x^4y^4 - t_2^2t_3^2t_4^2x^4y^4 \\
&+ t_1^3t_2t_3t_4^2x^5y^4 + t_1t_2^2t_3^3t_4^2x^5y^4 + t_1^3t_2t_3^2t_4x^4y^5 + t_1t_2^2t_3^2t_4^3x^4y^5 + t_1^2t_2t_3^3t_4x^5y^5 \\
&+ t_1^2t_2^2t_3^3t_4x^5y^5 + t_1^2t_2t_3t_4^3x^5y^5 + t_1^2t_2^2t_3t_4^3x^5y^5 + t_1^3t_2t_3^2t_4x^6y^5 + t_1t_2^2t_3^2t_4^3x^6y^5 \\
&+ t_1^3t_2t_3t_4^2x^5y^6 + t_1t_2^2t_3^3t_4^2x^5y^6 - t_1^4t_2t_3^2t_4^2x^6y^6 + 2t_1^2t_2^2t_3^2t_4^2x^6y^6 - t_1^2t_2^2t_3^4t_4^2x^6y^6 \\
&- t_1^2t_2^2t_3^2t_4^4x^6y^6 - t_1^3t_2^2t_3^3t_4^2x^7y^6 - t_1^3t_2^2t_3^2t_4^3x^6y^7 - t_1^2t_2^2t_3^3t_4^3x^7y^7 \\
&- t_1^2t_2^3t_3^3t_4^3x^7y^7 - t_1^3t_2^2t_3^2t_4^3x^8y^7 - t_1^3t_2^2t_3^3t_4^2x^7y^8 + t_1^4t_2^3t_3^4t_4^4x^{10}y^{10},
\end{aligned}
$$

$$
\begin{aligned}
\delta(x, y, t) &= (1 - t_1t_3x)(1 - x^2)(1 - t_2x^2)(1 - t_1t_4y)(1 - y^2)(1 - t_2y^2)(1 - x^2y^2)(1 - t_3t_4xy) \\
&\quad \times (1 - t_2t_3t_4xy)(1 - t_1t_4x^2y)(1 - t_1t_3xy^2)(1 - t_1^2x^2y^2)(1 - t_2t_3^2x^2y^2)(1 - t_2t_4^2x^2y^2)
\end{aligned}
$$

Proof Let p and q be the polynomials such that

$$
\sum_{i,j,k=0}^{\infty} \mu_{i,j,k,n_1,n_2,m}(n_1, n_2; m)t_1^{n_1}t_2^{n_2}t_3^m x^i y^j z^k = \frac{p(x, y, z, t)}{q(x, y, z, t)}.
$$

They may be computed explicitly using Proposition 4.3. Set $t' = (t_1, t_2, t_3)$, and

$$
f(x, y, t', t_4) = \sum_{i,j=0}^{\infty} \sum_{n \in \mathbb{Z}_{\geq 0}^4} \mu_{i,j}(n)t_1^{n_1}t_2^{n_2}t_3^{n_3}t_4^{n_4}x^i y^j.
$$

By regarding $V_{(1,0;0;1)}$ as two copies of $V_{(1,0;0)}$ with the standard torus of the second SL_2 acting with eigenvalues τ and τ^{-1}, say, we see that

$$
\frac{\tau f(x, y, t', \tau) - \tau^{-1} f(x, y, t', \tau^{-1})}{(\tau - \tau^{-1})} = \sum_{r=0}^{n_4} \tau^{n_4 - 2r} = \frac{p(x, y\tau, y\tau^{-1}, t')}{q(x, y\tau, y\tau^{-1}, t')}.
$$

So it suffices to verify that

$$
\begin{aligned}
&p(x, y\tau, y\tau^{-1}, t')(\tau - \tau^{-1})\delta(x, y, t', \tau)\delta(x, y, t', \tau^{-1}) \\
&\quad = q(x, y\tau, y\tau^{-1}, t')[\tau\nu(x, y, t', \tau)\delta(x, y, t', \tau^{-1}) \\
&\qquad - \tau^{-1}\nu(x, y, t', \tau^{-1})\delta(x, y, t', \tau)],
\end{aligned}
$$

which is easily done with a computer algebra system. □

5 Local zeta integrals I

5.1 Definitions and notation

The next step in the analysis of our global integral is to study the corresponding local zeta integrals. We introduce a "local" notation which will be used throughout Sects. 5 and 6, 7 In these section F is a local field which may be archimedean or nonarchimedean. Abusing notation, we denote the F-points of an F-algebraic group H by H as well. We fix an additive character ψ of F, and define a character $\psi_N : N \to \mathbb{C}$ by the same formula used in the global setting. Similarly, if we fix a triple $\chi = (\chi_1, \chi_2, \chi_3)$ of characters of $F^\times$, and $s \in \mathbb{C}^3$, then formula (2) now defines a character of M_P. We write $\mathrm{Ind}_P^G(\chi; s)$ for the corresponding (unnormalized) induced representation (K-finite vectors, relative to some fixed maximal compact K). We shall assume that the characters in χ are unitary, but not that they are normalized, and define $(\chi; s)$ for $s \in \mathbb{C}^2$ by the convention $s_3 = \frac{3s_1+3s_2}{2}$. Thus we have a two parameter family of induced representations and let $\mathrm{Flat}(\chi)$ denote the space of flat sections.

Let $\mathcal{S}(\mathrm{Mat}_{4\times 2})$ be the Bruhat–Schwartz space, which we equip with an action $\omega_{\psi,l}$ of R_Q as in the global setting, and define $I_0 : \mathcal{S}(\mathrm{Mat}_{4\times 2}) \times R_Q \to \mathbb{C}$ by replacing $\mathbb{A}$ by F in (10).

Next, take π to be a ψ_N-generic irreducible admissible representation of C_Q with ψ_N-Whittaker model $\mathcal{W}_{\psi_N}(\pi)$, and with central character $\chi_1^{-3}\chi_2^{-3}\chi_3^2$.

For $W \in \mathcal{W}_{\psi_N}(\pi)$, $f \in \mathrm{Flat}(\chi)$ and $\phi \in \mathcal{S}(\mathrm{Mat}_{4\times 2})$, define the corresponding local zeta integral to be the local analogue of (12), namely:

$$I(W, f, \phi; s) := \int_{ZU_4\backslash C_Q} W(g) \int_{U_Q^w\backslash U_Q} f_{\chi;s}(wug) I_0(\phi, ug)\, du\, dg. \tag{28}$$

In addition to the above notation, for $1 \le i, j \le r, i \ne j$, let $x_{ij}(r) = I_{12} + rE_{i,j} - rE_{13-j,13-i}$, where I_{12} is the 12×12 identity matrix and $E_{i,j}$ is the matrix with a one at the i, j entry and zeros everywhere else, and let

$$\Xi_0 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \Xi_2(a) := \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (a = (a_1, a_2, a_4) \in F^3).$$

5.2 Inital computations

In this section we carry out some initial computations with local zeta integrals which will be used in both the proof of convergence in Sect. 7.1 and in the unramified computations carried out in Sect. 6.

The image of the function x_{23} maps isomorphically onto the one dimensional quotient of $U_4\backslash N$, and the function $g \mapsto f_s^\circ(wg)$ is invariant by the image of x_{23} on the left. Moreover, $W(x_{23}(x_4)g) = \overline{\psi}(x_4)W(g)$, while

$$\left[\omega_\psi\left(x_{23}(x_4)ug\right).\phi\right](\Xi_0+\Xi_2(x_1,x_2,0))=\left[\omega_\psi\left(ut\right).\phi\right](\Xi_0+\Xi_2(x_1,x_2,x_4))$$

Let

$$III(\phi):=\int_{F^3}\phi\begin{pmatrix}r_1&r_2\\1&r_4\\0&1\\0&0\end{pmatrix}\overline{\psi}(r_1+r_4)\,dr,\qquad(\phi\in\mathcal{S}(\mathrm{Mat}_{4\times 2}))$$

$$II(f,\phi,s):=\int_{U_Q^w\backslash U_Q}f_{\chi;s}(wu)III(\omega_{\psi,l}(u).\phi)\,du,\qquad(\phi\in\mathcal{S}(\mathrm{Mat}_{4\times 2}),\ f\in\mathrm{Flat}(\chi))$$

$$I_1(W,f,\phi;s):=\int_{Z\backslash T}W(t)II(R(t).f,\omega_{\psi,l}(t).\phi,s)\,\delta_B^{-1}(t)\,dt,$$

where $\phi\in\mathcal{S}(\mathrm{Mat}_{4\times 2})$, $f\in\mathrm{Flat}(\chi)$, $W\in\mathcal{W}_{\psi_N}(\pi)$, and R is right translation. Then expressing Haar measure on C_Q in terms of Haar measures on T, N and K, and using x_{23} to parametrize $U_4\backslash N$ yields

$$I(W,f,\phi;s)=\int_K I_1(R(k).W,R(k).f,\omega_{\psi,l}(k).\phi;s)\,dk \tag{29}$$

where K is the maximal compact.

The integral $III(\phi)$ is absolutely convergent. Indeed, $III(\phi)=\phi_1(\Upsilon_0)$, where ϕ_1 is the Schwartz function obtained by taking Fourier transform of ϕ in three of the eight variables, and Υ_0 is a matrix with entries 0 and 1. We study the dependence on $u\in U_Q^w\backslash U_Q$ and $t\in Z\backslash T$ using the local analogues of (5). A remark is in order, regarding the Weil index $\gamma_{\psi,\det m_Q^1(g_1,g_2)}$ which appears in the third formula. In order to reconcile the local and global cases, one should think of this as the ratio $\gamma_{\psi,\det m_Q^1(g_1,g_2)}/\gamma_{\psi,1}$. The denominator can be omitted because the global $\gamma_{\psi,1}$ is trivial. In the local setting $\gamma_{\psi,1}$ may not be trivial, but $\gamma_{\psi,a^2}=\gamma_{\psi,1}$ for any a, and $\det m_Q^1(g_1,g_2)$ is always a square, so the ratio is always trivial.

Now, let $U_0\subset U_Q$ be the subgroup corresponding to the variables, x_1, x_2, x_4, y_3, y_5, y_6, z_1, z_2, z_3 and z_5. That is, the subset in which all other variables equal zero. Let $U_7\subset U_Q$ be the subgroup defined by the condition that each variable listed above is 0, and, in addition, $y_7=y_8=0$. Then U_0U_7 maps isomorphically onto the quotient $U_Q^w\backslash U_Q$. We parametrize U_0 and U_7 using the coordinates inherited from U_Q. A direct computation using (5) shows that for $u_0\in U_0$, $u_7\in U_7$, and $t\in T$, $III(\omega_\psi(tu_0u_7).\phi)$ is equal to

$$|t^{\beta_1+\beta_2+\beta_4}||\det t|^{\frac{1}{2}}\psi(x_1t^{\beta_1}+x_4t^{\beta_4}-y_3t^{-\beta_3}+y_6t^{-\beta_6}+z_1+z_5)\phi'\begin{pmatrix}y_1+t^{\beta_1}&y_2\\x_3+t^{-\beta_3}&y_4+t^{\beta_4}\\x_5&x_6+t^{-\beta_6}\\x_7&x_8\end{pmatrix}, \tag{30}$$

where ϕ' is the Schwartz function obtained by taking the Fourier transform of ϕ (relative to ψ) in x_1, x_2 and x_4. If we define $\psi_{U_0,t}(u_0) := \psi(x_1 t^{\beta_1} + x_4 t^{\beta_4} - y_3 t^{-\beta_3} + y_6 t^{-\beta_6} + z_1 + z_5)$, and we define $\delta(t) \in U_7$ and $\pi_7 : U_7 \to \mathrm{Mat}_{4\times 2}$ by

$$\delta(t) = u_Q\left(\begin{pmatrix} 0 & 0 \\ t^{-\beta_3} & 0 \\ 0 & t^{-\beta_6} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ t^{\beta_4} & 0 \\ 0 & t^{\beta_1} \end{pmatrix}, 0\right) \in U_7, \qquad \pi_7(u_7) = \begin{pmatrix} y_1 & y_2 \\ x_3 & y_4 \\ x_5 & x_6 \\ x_7 & x_8 \end{pmatrix},$$

then (30) becomes $|t^{\beta_1+\beta_2+\beta_4}||\det t|^{\frac{1}{2}}\psi_{U_0,t}(u_0)\phi'(\pi_7(\delta(t)^{-1}u_7))$.

The projection π_7 has a two dimensional kernel corresponding to the variables z_4 and z_6. Let U_8 denote this kernel and choose a subset U_8' of U_7 which contains $\delta(t)$ and maps isomorphically onto the quotient. Then we can parametrize $II(R(t).f, \omega_{\psi,l}(t).\phi, s)$ as a triple integral over $U_0 \times U_8 \times U_8'$. The U_8' integral is convergent because ϕ' is Schwartz, after a change of variables it becomes

$$\phi' *_1 f_{\chi;s}(wu_0u_8\delta(t)) := \int_{U_8'} f_{\chi;s}(wu_0u_8\delta(t)u_8')\phi'(\pi_7(u_8'))\,du_8.$$

Thus, conjugating t from right to left, and making a change of variables yields $I_1(W, f, \phi; s) = I_2(W, \phi *_1 f; s)$, where $I_2(W, f, \phi; s)$ is defined as

$$\int_{Z\backslash T} W(t)\delta_B^{-1}(t)|\det t|^{\frac{1}{2}}|t^{\beta_1+\beta_2+\beta_4}|\,\mathrm{Jac}_1(t)(\chi; s)(wtw^{-1})f_{\chi;s}(wu_0u_8\delta(t))\,dt,$$

with $\mathrm{Jac}_1(t)$ being the "Jacobian" of the change of variables $u_0 \to tu_0t^{-1}$, $u_7 \to tu_7t^{-1}$. Notice that $\phi *_1 f$ is simply another smooth section of the same family of induced representations, and that if $f_{\chi;s}$ and ϕ are both unramified, then $\phi' = \phi$ and $\phi' *_1 f_{\chi;s} = f_{\chi;s}$. Thus, we may dispense with the integral over u_8'.

Next, we dispense with the integral over u_8. To do this, we use [5] to replace $f_{\chi;s}$ by a sum of sections of the form

$$\phi_2 *_2 f'_{\chi;s}(g) := \int_{F^2} f_{\chi;s}(gx_{24}(y_1)x_{34}(y_2))\phi_1(y_1, y_2)\,dy, \qquad (s \in \mathbb{C}^2, g \in G).$$

Now, let

$$[II_2.f_{\chi;s}](g) := \int_{U_0} f_{\chi;s}(wu_0g)\psi_{U_0,t}(u_0)\,du_0, \qquad u_9(y_1, y_2) := x_{24}(y_1)x_{34}(y_2).$$

conjugating $u_9(y_1, y_2)$ from right to left shows that

$$[II_2.f_{\chi;s}](\delta(t)u_8u_9(y_1, y_2)) = \psi(-y_1z_6 - y_2(z_4 - t^{\alpha_1}))\cdot[II_2.f_{\chi;s}](\delta(t)u_8).$$

(Recall that z_6 and z_4 are coordinates on U_8). But then

$$\int_{F^2} [II_2.\phi_2 *_2 f_{\chi;s}](\delta(t)u_8)\, dz = \int_{F^2} [II_2.f_{\chi;s}](\delta(t)u_8)\widehat{\phi_2}(z_6, z_4 - t^{\alpha_1})\, dz,$$

which we may write as $[II_2.\widehat{\phi_2} *_3 f_{\chi;s}](\widetilde{\delta}(t))$, where $*_3$ is the action of $\mathcal{S}(U_8)$ by convolution, and $\widetilde{\delta}(t) = \delta(t)x_{29}(t^{\alpha_1})$. Notice that $\widehat{\phi_2} *_3 f_{\chi;s}$ is again another smooth section of the same family of induced representations. Note also that if f is spherical then taking ϕ_2 to be the characteristic function of $\mathfrak{o}^2$ gives $\widehat{\phi_2} *_3 f_{\chi;s} = f_{\chi;s}$.

Thus, we are reduced to the study of the integral

$$I_3(W, f; s) := \int_{Z\backslash T} W(t)\delta_B^{-1/2}(t)\nu_s(t) II_2.f_{\chi;s}(\widetilde{\delta}(t))\, dt, \tag{31}$$

where $\nu_s(t) := \delta_B^{-1/2}(t)|\det t|^{\frac{1}{2}}|t^{\beta_1+\beta_2+\beta_4}|\operatorname{Jac}_1(t)(\chi; s)(wtw^{-1})$. Write $w = w_1w_2w_3$, where $w_1 = w[634]$, $w_2 = w[3236514]$ and $w_3 = w[2356243564]$. Write U for the unipotent radical of our standard Borel of G, and U^- for the unipotent radical of the opposite Borel. For $w \in W$, let $U_w = U \cap w^{-1}U^-w$. Then $w_3U_0w_3^{-1} = U_{w_1w_2} = w_2^{-1}U_{w_1}w_2U_{w_2}$. For $\mathbf{c} := (c_1, \dots, c_6) \in F^6$, define a character $\psi_{\mathbf{c},0}$ of U_0 in terms of the standard coordinates on U_0 by $\psi_{\mathbf{c},0}(u_0) := \psi(c_1x_1 + c_2x_4 + c_3y_6 - c_4y_3 + c_5z_1 + c_6z_5)$. Notice that $\psi_{U_0,t}$ is obtained by taking $\mathbf{c} = (t^{\beta_1}, t^{\beta_4}, t^{-\beta_3}, t^{-\beta_6}, 1, 1)$. In terms of the entries u_{ij} we have $\psi_{\mathbf{c},0}(u) = \psi(c_1u_{15} + c_2u_{26} + c_3u_{27} - c_4u_{38} + c_5u_{19} + c_6u_{2,10})$, Now, w_3 corresponds to the permutation $(2, 4, 11, 9)(3, 8, 10, 5)$. It follows that $u_0' \mapsto \psi_{\mathbf{c},0}(w_3^{-1}u_0'w_3)$ is the character of $w_3U_0w_3^{-1}$ given by $u \mapsto \psi(c_1u_{13}+c_2u_{46}+c_3u_{47}+c_4u_{35}+c_5u_{12}+c_6u_{45})$. In particular, its restriction to $w_2^{-1}U_{w_1}w_2$ is trivial. Let $\psi_{\mathbf{c},2}$ denote the restriction to U_{w_2}. Then

$$\int_{U_0} f_{\chi;s}(wug)\psi_{\mathbf{c},0}(u)\, du = \int_{U_{w_2}} \int_{U_{w_1}} f_{\chi;s}(w_1u_1w_2u_2w_3g)\, du_1\, \psi_{\mathbf{c},2}(u_2)\, du_2, \tag{32}$$

and the u_1 integral is a standard intertwining operator, $M(w_1^{-1}, \chi; s) : \operatorname{Ind}_P^G(\chi; s) \to \underline{\operatorname{Ind}}_{B_G}^G((\chi; s)\delta_{B_G}^{-1/2})^{w_1}$, where $((\chi; s)\delta_B^{-\frac{1}{2}})^{w_1}(t) := ((\chi; s)\delta_B^{-\frac{1}{2}})(w_1tw_1^{-1})$, and $\underline{\operatorname{Ind}}$ denotes normalized induction.

Now let $w_4 = w[32365]$ so that $w_2 = w_4w[14]$. Also, let $w_3' = w[14]w_3$. Observe that w_4 is the long element of the Weyl group of a standard Levi subgroup of GSO_{12} which is isomorphic to $GL_1 \times GL_3 \times GSO_4$. For $c_1, \dots, c_4 \in F$, define a character $\psi_{\mathbf{c},4}$ of U_{w_4} by $\psi_{\mathbf{c},4}(u) = \psi(c_1u_{23} + c_4u_{34} + c_3u_{56} + c_2u_{57})$, and for $f_{\chi;s;w_1} \in \underline{\operatorname{Ind}}_{B_G}^G((\chi; s)\delta_{B_G}^{-1/2})^{w_1}$, let

$$\mathcal{J}_{\psi_{\mathbf{c},4}}.f_{\chi;s;w_1}(g) = \int_{U_{w_4}} f^\circ_{\chi;s,w_1}(w_4ug)\psi_{\mathbf{c},4}(u)\, du,$$

which is a Jacquet integral for this Levi subgroup. Then (32) equals

$$\int_{F^2}[\mathcal{J}_{\psi_{\mathbf{c},4}} \circ M(w_1^{-1}, \chi; s).f_{\chi;s}](x_{21}(r_1)x_{54}(r_2)w_3'g)\psi(c_5r_1 + c_6r_2)\,dr.$$

6 Unramified calculation

We keep the notation from Sect. 5.1, and assume further that F is nonarchimedean, with ring of integers $\mathfrak{o}$ having unique maximal ideal $\mathfrak{p}$. We fix a generator ϖ for $\mathfrak{p}$. The absolute value on F is denoted $|\ |$ and normalized so that $|\varpi| = q := \#\mathfrak{o}/\mathfrak{p}$. The corresponding $\mathfrak{p}$-adic valuation is denoted v. Moreover, we assume that $K = G(\mathfrak{o})$, and that the representation π and characters $\chi_i,\ i = 1, 2, 3$ are unramified, and we let W_π°, f° and ϕ° denote the normalized spherical elements of $\mathcal{W}_{\psi_N}(\pi)$, $\mathrm{Flat}(\chi)$ and $\mathcal{S}(\mathrm{Mat}_{4\times 2})$, respectively.

The (finite Galois form of the) L-group of $GSp_4 \times GSO_4$ is $GSpin_5(\mathbb{C}) \times GSpin_4(\mathbb{C})$. Indeed, one may define $GSpin_{2n+1}$ (resp. $GSpin_{2n}$) as the reductive group with root datum dual to that of GSp_{2n} (resp. GSO_{2n}). However, both $GSpin$ groups appearing here can be understood more explicitly via "coincidences of low rank." Indeed, a simple change of $\mathbb{Z}$-basis reveals that the root datum of GSp_4 is in fact *self* dual. Thus $GSpin_5$ is just GSp_4 in another guise. Note, however, that the isomorphism of GSp_4 with its own dual group does not respect the standard numbering of the simple roots.

Next, we can realize GSO_4 (resp. $GSpin_4$) as a quotient (resp. subgroup) of $GL_2 \times GL_2$. Indeed, we can realize GSO_4 as the similitude group of the four dimensional quadratic space $(\mathrm{Mat}_{2\times 2}, \det)$. Letting $GL_2 \times GL_2$ act by $(g_1, g_2) \cdot X = g_1 X {}^t g_2$ induces a surjection $GL_2 \times GL_2 \to GSO_4$ with kernel $\{(aI_2, a^{-1}I_2) : a \in GL_1\}$, and thence a bijection between representations of GSO_4 and pairs of representations of GL_2 with the same central character. By duality, this induces an isomorphism of $GSpin_4$ with $\{(g_1, g_2) \in GL_2 \times GL_2 : \det g_1 = \det g_2\}$. We remark that the induced map $GSpin_4 \to SO_4$ is *not* the restriction of our chosen map $GL_2 \times GL_2 \to GSO_4$.

We regard $GSp_4 \times GSO_4$ as a subgroup of M_Q containing C_Q in the obvious way, and regard its L group as a subgroup of $GSp_4 \times GL_2 \times GL_2$. We make the identification in such a way that the first GL_2 corresponds to the fifth simple root of $G = GSO_{12}$ and the second GL_2 corresponds to the sixth simple root of G.

Let St_{GSp_4} denote the standard representation of GSp_4. It may also be regarded as the spin representation of $GSpin_5$. For this reason, the associated L function is often called the "Spinor L function." We regard St_{GSp_4} as a representation of of the L group via projection onto the $GSp_4(\mathbb{C})$ factor, and let $St_{GSp_4}^\vee$ denote the dual representation. Let $St_{GL_2^{(1)}}$ (resp. $St_{GL_2^{(2)}}$) denote the representations of the L group obtained by composing the standard representation of GL_2 with projection onto the first (resp. second) $GL_2(\mathbb{C})$ factor.

Theorem 6.1 *Let*

$$N(s,\chi)=L\left(s_1-s_2,\frac{\chi_1}{\chi_2}\right)L\left(s_1-s_2-1,\frac{\chi_1}{\chi_2}\right)L\left(s_1-s_2-2,\frac{\chi_1}{\chi_2}\right)L\left(s_1+s_2-2,\chi_1\chi_2\right)$$
$$\times L(s_1+s_2-3,\chi_1\chi_2)L(s_1+s_2-4,\chi_1\chi_2)L(2s_2,\chi_2^2)L(2s_1-6,\chi_1^2). \tag{33}$$

(Local L functions. The corresponding product of global zeta functions is the normalizing factor of the Eisenstein series). Then $I(W_\pi^\circ, f^\circ, \phi^\circ; s)$ equals

$$\frac{L\left(\frac{s_1-s_2}{2}-1,\pi,St_{GSp_4}^\vee\otimes St_{GL_2^{(1)}}\times\frac{\chi_3}{\chi_1\chi_2^2}\right)L\left(\frac{s_1+s_2}{2}-2,\pi,St_{GSp_4}^\vee\otimes St_{GL_2^{(2)}}\times\frac{\chi_3}{\chi_1\chi_2}\right)}{N(s,\chi)}$$

Proof (*Reduction of the general case to the special case of trivial characters*) For purposes of this proof, write λ_H for the similitude rational character of H where $H=GSO_{12}$, GSp_4, or GSO_4. Our embedding $(GSp_4\times GSO_4)^\circ\hookrightarrow GSO_{12}$ is such that $\lambda_{GSO_{12}}(g,h)=\lambda_{GSp_4}(g)^{-1}=\lambda_{GSO_4}(h)$, and the projection $p: GL_2\times GL_2\to GSO_4$ is such that $\lambda_{GSO_4}(p(g_1,g_2))=\det g_1\det g_2$.

Write $\pi=\Pi\otimes\tau_1\otimes\tau_2$ where Π is an unramified representation of GSp_4 and τ_1 and τ_2 are unramified representations of GL_2 with the same central character (so that $\tau_1\otimes\tau_2$ is a representation of GSO_4). Write $\tau_i=\tau_{i,0}\otimes|\det|^{t_1}$ where $\tau_{i,0}$ is an unramified representation of GL_2 with trivial central character for $i=1,2$, and t_1 is a complex number, and write $\Pi=\Pi_0\otimes|\lambda_{GSp_4}|^{t_2}$, where Π_0 is an unramified representation of GSp_4 with trivial central character and t_2 is a complex number. Then, as representations of $C_Q=(GSp_4\times GSO_4)^\circ$,

$$\pi=\pi_0\otimes|\lambda_{GSO_{12}}|^{t_1-t_2},\qquad \text{where}\quad \pi_0=\Pi_0\otimes\tau_{1,0}\otimes\tau_{2,0}.$$

The operation of twisting Π_0 by $|\lambda_{GSp_4}|^{t_2}$ to obtain Π corresponds, on the L group side, to multiplying the corresponding semisimple conjugacy class in $GSp_4(\mathbb{C})$ by $q^{-t_3}I_4$. Likewise, the operation of twisting $\tau_{i,0}$ by $|\det|^{t_1}$ corresponds to multiplying by q^{-t_1} in $GL_2(\mathbb{C})$ for $i=1,2$. If η is the unramified character $\eta(a)=|a|^r$, then

$$L\left(u,\pi,St_{GSp_4}^\vee\otimes St_{GL_2^{(i)}}\times\eta\right)=L\left(u-t_2+t_1+r,\pi_0,St_{GSp_4}^\vee\otimes St_{GL_2^{(i)}}\right).$$

For $i=1,2,3$ the unramified character χ_i is given by $|\ |^{r_i}$ for some $r_i\in\mathbb{C}$. If $s=(s_1,s_2,\frac{3(s_1+s_2)}{2})\in\mathbb{C}^3$, then let $s'=(s_1+r_1,s_2+r_2,\frac{3(s_1+r_1+s_2+r_2)}{2})$, and let χ_0 be the triple consisting of three copies of the trivial character. Then it follows directly from the definitions that $f^\circ_{\chi;s}=f^\circ_{\chi_0;s'}\cdot|\lambda_{GSO_{12}}|^{r_3-\frac{3r_1+3r_2}{2}}$. The general case now follows from the case $t_1=t_2=r_1=r_2=r_3=0$.

Recall that $I(W,f,\phi;s)$ is only defined when the triple χ and the central character of π are compatible. In the present notation the compatibility condition is that $-3r_1-3r_2+2r_3+2(t_1-t_2)=0$. But then $W_\pi\cdot f^\circ_{\chi;s}=W_{\pi_0}\cdot f^\circ_{\chi_0;s'}$. The general case now reduces to the special case when $\chi=\chi_0$ and $\pi=\pi_0$. □

Remark 6.2 We may now assume that $\pi = \Pi \otimes \tau_1 \otimes \tau_2$, where the central characters of Π, τ_1 and τ_2 are all trivial. Thus, π may be regarded as an unramified representation of $SO_5 \times PGL_2 \times PGL_2$ and corresponds to a semisimple conjugacy class in $Sp_4(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. Define St_{Sp_4}, $St_{SL_2^{(1)}}$ and $St_{SL_2^{(2)}}$ as the restrictions of St_{GSp_4}, $St_{GL_2^{(1)}}$ and $St_{GL_2^{(2)}}$, respectively. Note that all three are self dual representations. In particular, $L(u, \pi, St^{\vee}_{GSp_4} \otimes St_{GL_2^{(i)}}) = L(u, \pi, St_{Sp_4} \otimes St_{SL_2^{(i)}})$ for all $u \in \mathbb{C}$.

Proof (*Proof in the special case of trivial characters*) As χ is trivial we write f_s° instead of $f^\circ_{\chi;s}$. Let $I(s,\pi) := I(W_\pi^\circ, f^\circ, \phi; s)$. As we've seen in Sect. 5.2, it is equal to $I_3(W_\pi^\circ, f^\circ; s)$, defined by (31). Let $\mathbf{I}_1(s; c_1, \dots, c_6)$ equal

$$\int_{U_0} f_{\chi;s}(wu_0)\psi_{\mathbf{c},0}(u_0)\, du_0.$$

Let $a = t^{\beta_1}, b = t^{\beta_4}, c = t^{-\beta_6}, d = t^{-\beta_3}$. Note that $\beta_1 - \beta_3 = \alpha_1$, $\beta_4 - \beta_3 = \alpha_5$, and $\beta_4 - \beta_6 = \alpha_2$. Moreover, the characters $-\beta_3 - \beta_6$ and α_6 are not identical on T_G, but they have the same restriction to the maximal torus T of C_Q. It follows that for t in the support of W_π°, the quantities $|ad|$, $|bd|$, $|bc|$ and $|cd|$ are all ≤ 1.

By plugging in the Iwasawa decomposition for $w_3\widetilde{\delta}(t)$, we find that

$$II_2.f_s^\circ(\widetilde{\delta}(t)) = \begin{cases} \mathbf{I}_1(s; a, b, c, d, 1, 1), & |a|, |b|, |c|, |d| \le 1, \\ |d|^{-s_1-s_2+4}\mathbf{I}_1(s, ad, bd, cd, 1, 1, 0), & |d| > 1, |a|, |b|, |c| \le 1, \\ |c|^{-s_1-s_2+4}\mathbf{I}_1(s, a, bc, 1, cd, 1, 0), & |c| > 1, |a|, |b|, |d| \le 1, \\ |a|^{-s_1+s_2+2}|c|^{-s_1-s_2+4}\mathbf{I}_1(s, 1, bc, 1, acd, 0, 0), & |a|, |c| > 1, |b|, |d| \le 1, \\ |a|^{-s_1+s_2+2}\mathbf{I}_1(s, 1, b, c, ad, 0, 1), & |b|, |c|, |d| \le 1, |a| > 1, \\ |a|^{-s_1+s_2+2}|b|^{-s_1+s_2+2}\mathbf{I}_1(s, 1, 1, bc, abd, 0, 0), & |c|, |d| \le 1, |a|, |b| > 1, \\ |b|^{-s_1+s_2+2}\mathbf{I}_1(s, a, 1, bc, bd, 1, 0), & |d|, |c|, |a| \le 1, |b| > 1. \end{cases} \tag{34}$$

Now let f°_{s,w_1} denote the normalized spherical vector in $\underline{\mathrm{Ind}}_{B_G}^G((\chi_0; s)\delta_B^{-\frac{1}{2}})^{w_1}$, and let

$$Z_1(s) := \frac{\zeta(2s_2-1)}{\zeta(2s_2)}\frac{\zeta(s_1-s_2-1)}{\zeta(s_1-s_2)}\frac{\zeta(s_1+s_2-3)}{\zeta(s_1+s_2-2)}$$

Then, $M(w_1^{-1}, \chi; s).f_s^\circ = Z_1(s) f^\circ_{s,w_1}$, by the Gindikin–Karpelevic formula, and hence

$$\mathbf{I}_1(s; c_1, \ldots, c_6) = Z_1(s) \int_{F^2} \mathcal{J}_{\psi_{\mathbf{c},4}} f^{\circ}_{s,w_1}(x_{21}(r_1)x_{54}(r_2))\psi(c_5r_1 + c_6r_2)\, dr.$$

We remark that $((\chi_0; s)\delta_B^{-\frac{1}{2}})^{w_1}$ maps $\operatorname{diag}(\lambda t_1, \ldots, \lambda t_6, t_6^{-1}, \ldots, t_1^{-1}) \in T_G$ to

$$|t_1|^{s_1-5}|t_2|^{s_1-4}|t_3|^{s_2-2}|t_4|^{-s_2}|t_5|^{s_1-3}|t_6|^{1-s_2}|\lambda|^{s_3-2s_2-\frac{13}{2}}. \tag{35}$$

□

Lemma 6.3 *Assume that each of c_5, c_6 is either zero or a unit. Assume further that if $c_6 = 0$ then at least one of c_2, c_3, c_4 is a unit, and that if $c_5 = 0$ then c_1 is a unit, and set $J_{c_1,c_2,c_3,c_4} = \mathcal{J}_{\psi_{\mathbf{c},4}} f^{\circ}_{s,w_1}(I_{12})$. Then $\mathbf{I}_1(s; c_1, \ldots, c_6)/Z_1(s)$ equals*

$$J_{c_1,c_2,c_3,c_4} - q^{-s_1-s_2+4} J_{\frac{c_1}{\varpi},c_2,c_3,c_4} - q^{-2s_1+6} J_{c_1,\frac{c_2}{\varpi},\frac{c_3}{\varpi},\frac{c_4}{\varpi}} + q^{-3s_1-s_2+10} J_{\frac{c_1}{\varpi},\frac{c_2}{\varpi},\frac{c_3}{\varpi},\frac{c_4}{\varpi}}.$$

Remark 6.4 Observe that all the sextuples $c_1, \ldots, c_6$ appearing in (34) satisfy the conditions of Lemma 6.3.

Proof There exist cocharacters $h_i : GL_1 \to T_G$, $(i = 1, 2)$ such that $\langle h_1, \alpha_i \rangle = \delta_{i,1}$ and $\langle h_2, \alpha_i \rangle = \delta_{i,4}$ (Kronecker δ). It follows that $\mathcal{J}_{\psi_{\mathbf{c},4}} f^{\circ}_{s,w_1}(x_{21}(r_1)x_{54}(r_2))$ depends only on $v(r_1)$ and $v(r_2)$. If c_5 is a unit, then

$$\int_{v(r_1)=-k} \psi(c_5r_1)\, dr_1 = \begin{cases} -1, & k = 1, \\ 0, & k > 1, \end{cases}$$

and similarly with r_2. Since both f°_{s,w_1} and ψ are unramified, it follows that $\mathbf{I}_1(s; c_1, \ldots, c_6)$ equals

$$\mathcal{J}_{\psi_{\mathbf{c},4}} f^{\circ}_{s,w_1}(I_{12}) - \mathcal{J}_{\psi_{\mathbf{c},4}} f^{\circ}_{s,w_1}(\varpi^{-1})) - \mathcal{J}_{\psi_{\mathbf{c},4}} f^{\circ}_{s,w_1}(\varpi^{-1}))$$
$$+\mathcal{J}_{\psi_{\mathbf{c},4}} f^{\circ}_{s,w_1}(x_{21}(\varpi^{-1})x_{54}(\varpi^{-1})).$$

Plugging in the Iwasawa decomposition of $x_{21}(\varpi^{-1})$ and/or $x_{54}(\varpi^{-1})$ gives the result in this case.

Now suppose that c_5 is not a unit. Then it is zero and c_1 is a unit. It follows that

$$\int_{F^2} \mathcal{J}_{\psi_{\mathbf{c},4}} f^{\circ}_{s,w_1}(x_{21}(r_1)x_{54}(r_2))\psi(c_5r_1 + c_6r_2)\, dr$$
$$= \int_F \mathcal{J}_{\psi_{\mathbf{c},4}} f^{\circ}_{s,w_1}(x_{54}(r_2))\psi(c_6r_2)\, dr_2.$$

Indeed the support of J is contained in UT_1K where $K = GSO_{12}(\mathfrak{o})$ is the maximal compact subgroup, and T_1 is the set of torus elements t with $|t^{\alpha_2}| \leq 1$. It follows easily from the Iwasawa decomposition that $x_{21}(r_1)x_{54}(r_2) \in UT_1K$ if and only if $r_1 \in \mathfrak{o}$. If c_6 is a unit then proceeding as before we obtain

$$\int_F \mathcal{J}_{\psi_{\mathbf{c},4}} f^\circ_{s,w_1}(x_{54}(r_2))\psi(c_6r_2)\,dr_2 = J_{c_1,c_2,c_3,c_4} - q^{-2s_1+6} J_{c_1,\frac{c_2}{\varpi},\frac{c_3}{\varpi},\frac{c_3}{\varpi}}.$$

On the other hand, when c_1 is a unit then $J_{\frac{c_1}{\varpi},c_2,c_3,c_4} = J_{\frac{c_1}{\varpi},\frac{c_2}{\varpi},\frac{c_3}{\varpi},\frac{c_4}{\varpi}} = 0$, and the stated result follows in this case as well. Likewise, if c_6 is zero, then integration over r_2 can be omitted and $J_{c_1,\frac{c_2}{\varpi},\frac{c_3}{\varpi},\frac{c_4}{\varpi}} = J_{\frac{c_1}{\varpi},\frac{c_2}{\varpi},\frac{c_3}{\varpi},\frac{c_4}{\varpi}} = 0$, which gives the result in the remaining two cases. □

Now consider the subgroup of the torus consisting of all elements of the form

$$t = \operatorname{diag}\left(t_1t_2, t_2, 1, t_1^{-1}, t_3t_4, t_4, \frac{t_2}{t_4}, \frac{t_2}{t_3t_4}, t_1t_2, t_2, 1, t_1^{-1}\right). \tag{36}$$

This subgroup maps isomorphically onto $Z\backslash T$. For elements of this torus and with coordinates as in (36) we have

$$\mathrm{Jac}_1(t) = |t_1^2t_3t_4^2|^{-1}, \qquad \mathrm{Jac}_2(t) = \left|\frac{t_1^2t_2^3}{t_3t_4^3}\right| \qquad \delta_B^{-\frac{1}{2}} = |t_1^2t_2t_3t_4|^{-1} \qquad |\det t|^{\frac{1}{2}} = |t_2^{-2}t_3^2t_4^4|.$$

$$(\chi_0; s)(wtw^{-1}) = |t_1|^{s_1-s_2}|t_2|^{-s_1-3s_2+s_3}|t_3t_4^2|^{s_2}.$$

Set $x = q^{-(\frac{s_1-3s_2}{2})}$, $y = q^{-s_2+1}$, and let n_i be the $\mathfrak{p} = v(t_i)$ for $1 \le i \le 4$. Then

$$\nu_s(t) = x^{2n_1+n_2}y^{2n_1+n_3+2n_4}. \tag{37}$$

For $l = (l_1, l_2, l_3, l_4) \in \mathbb{Z}^4$, set

$$j_1(l) = 1 - x^{2l_4+2}y^{2l_4+2} - y^{2l_1+2} - x^{2l_1+2l_4+4}y^{4l_1+4l_4+8} + x^{2l_4+2}y^{2l_1+4l_4+6} + x^{2l_1+2l_4+4}y^{4l_1+2l_4+6}$$

$$j_2(l) = 1 - x^{2l_2+2}y^{4l_2+4} \qquad j_3(l) = 1 - x^{2l_3+2}y^{2l_3+2}$$

$$j(l) = \begin{cases} j_1(l)j_2(l)j_3(l), & l_i \ge 0 \forall i, \\ 0, & \text{otherwise.} \end{cases}$$

Then direct computation or the Casselman-Shalika formula shows that

$$\mathcal{J}_{\psi_{\mathbf{c},4}} \cdot f^\circ_{s,w_1}(I_{12}) = \frac{\zeta(s_1+s_2-4)^2}{\zeta(s_1+s_2-3)^2}\frac{\zeta(s_1-s_2-2)^2}{\zeta(s_1-s_2-1)^2}\frac{\zeta(2s_2-2)}{\zeta(2s_2-1)}j(l),$$

where $l_i = v(c_i)$ for each i. Hence, if

$$i_1(l) = j(l) - x^2y^4 j(l-(1,0,0,0)) - x^4y^6 j(l-(0,1,1,1)) + x^6y^{10} j(l-(1,1,1,1)),$$

then

$$\mathbf{I}_1(s; c_1, \ldots, c_6) = \frac{\zeta(s_1-s_2-2)^2\zeta(s_1+s_2-4)^2\zeta(2s_2-2)i_1(n_1,n_2,n_3,n_4)}{\zeta(s_1-s_2)\zeta(s_1-s_2-1)\zeta(s_1+s_2-2)\zeta(s_1+s_2-3)\zeta(2s_2)},$$

for all $c_1, \ldots, c_6$ satisfying the conditions of Lemma 6.3. Consequently,

$$II_2.f_s^\circ(\widetilde{\delta}(t)) = \frac{\zeta(s_1-s_2-2)^2\zeta(s_1+s_2-4)^2\zeta(2s_2-2)i(A,B,C,D)}{\zeta(s_1-s_2)\zeta(s_1-s_2-1)\zeta(s_1+s_2-2)\zeta(s_1+s_2-3)\zeta(2s_2)},$$

where $A = v(t^{\beta_1})$, $B = v(t^{\beta_4})$, $C = v(t^{-\beta_6})$, and $D = v(t^{-\beta_3})$ and i is defined piecewise in terms of i_1 according to the seven cases from (34). It is convenient to use an alternate parametrization. Let $ii(m_1, m_2, m_3, m_4)$ equal

$$i\left(m_1 - \frac{-m_2+m_3+m_4}{2}, \frac{m_2+m_3-m_4}{2}, \frac{m_2-m_3+m_4}{2}, \frac{-m_2+m_3+m_4}{2}\right) x^{2m_1+m_2} y^{2m_1+m_2+m_4} \tag{38}$$

if $m_2 + m_3 + m_4$ is even and zero otherwise. Then

$$\nu_s(t) II_2.f_s^\circ(\widetilde{\delta}(t)) = \frac{\zeta(s_1-s_2-2)^2\zeta(s_1+s_2-4)^2\zeta(2s_2-2)ii(m)}{\zeta(s_1-s_2)\zeta(s_1-s_2-1)\zeta(s_1+s_2-2)\zeta(s_1+s_2-3)\zeta(2s_2)},$$

where now $m_i = v(t^{\alpha_i})$ for $i = 1, 2, 3, 4$. Let $[m_1, m_2; m_3; m_4]$ or $[m]$ denote the trace of the irreducible representation of ${}^L(C_Q/Z) := Sp_4 \times SL_2 \times SL_2$ on which Sp_4 acts with highest weight $m_1\varpi_1 + m_2\varpi_2$, the first SL_2 acts with highest weight m_3, and the second SL_2 acts with highest weight m_4. Then

$$\delta_B^{-\frac{1}{2}}(t) W_\pi(t) = [m_2, m_1; m_3; m_4](\tau_\pi),$$

where τ_π is the semisimple conjugacy class in ${}^L(C_Q/Z)$ attached to π. Note the reversal of order between 1 and 2. The reason for this is that when GSp_4 is identified with its own dual group, the standard numberings for the two dual GSp_4's are opposite to one another. For example the coroot attached to the short simple root α_1 is the long simple coroot, which makes it the long simple root of the dual group.

Now, let $Z_2(x, y) = (1-y^2)(1-x^2y^2)^2(1-x^2y^4)^2$. Then $j(n)$ is divisible by Z_2 for any n. Also, for ϱ the character of a finite dimensional representations of ${}^L(C_Q/Z)$, let $L(u, \varrho) = \sum_{i=0}^\infty u^k \operatorname{Tr}\operatorname{sym}^k(\varrho)$, Then Theorem 6.1 is reduced to the following identity of power series over representation ring of ${}^L(C_Q/Z)$:

$$\frac{1}{Z_2(x,y)} \sum_{m\in\mathbb{Z}^4_{\geq 0}} ii(m_2,m_1,m_3,m_4)[m] = Z_3(x,y)L(xy,[1,0;1;0])L(xy^2,[1,0;0;1]), \tag{39}$$

where $Z_3(x,y) = (1-x^2y^2)(1-x^2y^4)(1-x^4y^6)$.

Define polynomials $P_m(u,v)$ by

$$L(u,[1,0;1;0])L(v,[1,0;0;1]) = \sum_{m\in\mathbb{Z}^4_{\geq 0}} P_m(u,v)[m].$$

Then (39) is equivalent to the family of identities of polynomials,

$$\begin{aligned}&(1-y^2)(1-x^2y^2)^3(1-x^2y^4)^3(1-x^4y^6)P_m(xy,xy^2)\\ &\quad = ii(m_2,m_1,m_3,m_4) \ (\forall m \in \mathbb{Z}^4_{\geq 0}),\end{aligned}$$

or to the identity of power series over a polynomial ring:

$$\begin{aligned}\sum_{m\in\mathbb{Z}^4_{\geq 0}} ii(m_2,m_1,m_3,m_4)t_1^{m_1}t_2^{m_2}t_3^{m_3}t_4^{m_4} &= Z_4(x,y)\sum_{m\in\mathbb{Z}^4_{\geq 0}} P_m(xy,xy^2)t_1^{m_1}t_2^{m_2}t_3^{m_3}t_4^{m_4}\\ &= Z_4(x,y)\frac{\nu(x,y,t)}{\delta(x,y,t)},\end{aligned} \tag{40}$$

where ν and δ are defined as in Proposition 4.4, and $Z_4 = Z_2Z_3$.

The identity (40) can be proved as follows. Let $X = (X_1,X_2,X_3,X_4)$ and $Y = (Y_1,Y_2,Y_3,Y_4)$ be quadruples of indeterminates. Define polynomials,

$$\begin{aligned}\underline{j}_1(x,y,X,Y) := 1 &- x^2y^2X_4^2Y_4^2 - y^2Y_1^2 - x^2y^8X_1^2X_4^2Y_1^4Y_4^4 + x^2y^6X_4^2Y_1^2Y_4^4\\ &+ x^4y^6X_1^2X_4^2Y_1^4Y_4^2\end{aligned}$$

$$\underline{j}_2(x,y,X,Y) = (1-x^2y^4X_2^2Y_2^4); \qquad \underline{j}_3(x,y,X,Y) = (1-x^2y^2X_3^2Y_3^2); \qquad \underline{j} = \underline{j}_1\underline{j}_2\underline{j}_3,$$

so that for $k = (k_1,\ldots,k_4) \in \mathbb{Z}^4_{\geq 0}$, the polynomial $j(k)$ is equal to $\underline{j}(x,y,x^k,y^k)$, where $x^k := (x^{k_1},\ldots,x^{k_4})$ and $y^k := (y^{k_1},\ldots,y^{k_4})$. Likewise, one computes a polynomial $\underline{i}_1(x,y,X,Y)$ such that $i_1(k) = \underline{i}_1(x,y,x^k,y^k)$. It can be expressed as a sum of 12 monomials in X and Y, each with a coefficient which is a polynomial in x and y. Thus $i_1(k) = \sum_{i=1}^{12} c_i(x,y)\prod_{j=1}^{4}(\mu_{i,j}(x,y))^{k_i}$, for some polynomials $c_1,\ldots,c_{12}$ and monomials $\mu_{1,1}\ldots,\mu_{12,4}$ in x and y. Now,

$$\sum_{m\in\mathbb{Z}^4_{\geq 0}} ii(m_2,m_1,m_3,m_4)t_1^{m_1}t_2^{m_2}t_3^{m_3}t_4^{m_4}$$

$$= \sum_{\substack{A,B,C,D\in\mathbb{Z}\\ A+D,B+C,B+D,C+D\geq 0}} i(A,B,C,D)x^{2A+B+C+2D}y^{2A+B+2C+3D}t_1^{B+C}t_2^{A+D}t_3^{B+D}t_4^{C+D}.$$

This is a sum of seven subsums corresponding to the seven cases which appear in (34). The simplest of these is

$$\sum_{A,B,C,D\geq 0} i_1(A,B,C,D)x^{2A+B+C+2D}y^{2A+B+2C+3D}t_1^{B+C}t_2^{A+D}t_3^{B+D}t_4^{C+D}$$
$$=\sum_{i=1}^{12}\frac{c_i(x,y)}{(1-\mu_{i,1}(x,y)t_2x^2y^2)(1-\mu_{i,2}(x,y)t_1t_3xy)(1-\mu_{i,3}(x,y)t_1t_4xy^2)(1-\mu_{i,4}(x,y)t_2t_3t_4x^2y^3)}.$$

In each of the other six sums one can make a substitution to obtain a similar, fourfold sum of A', B', C', D' from 0 to infinity. For example, in the second case listed (34), one has the conditions $A, B, C \geq -D \geq 1$. Substituting $A' = A + D$, $B' = B + D$, $C' = C + D$, and $D' = -D - 1$, yields

$$\sum_{A',B',C',D'=0}^{\infty} i_1(A',B',C',0)t_1^{B'+C'+2D'+2}t_2^{A'}t_3^{B'}t_4^{C'}x^{2A'+B'+C'+4D'+4}y^{2A'+B'+2C'+6D'+6}$$
$$=\sum_{i=1}^{12}\frac{c_i(x,y)x^4y^6t_1^2}{(1-\mu_{i,1}(x,y)t^2x^2y^2)(1-\mu_{i,2}(x,y)t_1t_3xy)(1-\mu_{i,3}(x,y)t_1t_4xy^2)(1-t_1^2x^4y^6)}$$

The other five subsums are treated similarly. Totaling up the resulting rational functions and simplifying gives (40), completing the proof of theorem.

7 Local zeta integrals II

In this section we continue our study of the local zeta integral $I(W, f, \phi; s)$ at the ramified places.

7.1 Convergence

In this section, we prove the convergence of local zeta integrals

Theorem 7.1 *Take* $W \in \mathcal{W}_{\psi_N}(\pi)$, $f \in \mathrm{Flat}(\chi)$ *and* $\phi \in \mathcal{S}(\mathrm{Mat}_{4\times 2})$. *Then the local zeta integral* $I(W, f, \phi; s)$ *converges for* $\mathrm{Re}(s_1-s_2)$ *and* $\mathrm{Re}(s_2)$ *both sufficiently large.*

Proof We need to show that convergence of $I_3(W, f; s)$ defined in (31), for $W \in W_{\psi_N}(\pi)$ and f a smooth section of the family of induced representations $\mathrm{Ind}_P^G(\chi; s)$. To do this, we simply bound $f_{\chi;s}$ by a constant times the spherical section $f^\circ_{\mathrm{Re}(s)}$, where $\mathrm{Re}(s) \in \mathbb{R}^2$ is the real part of s. Then $II_2.f_{\chi;s}(\widetilde{\delta}(t))$ is bounded by a constant multiple of $M(w_2^{-1}w_1^{-1}, \mathrm{Re}(s)).f^\circ_{\mathrm{Re}(s)}(w_3\widetilde{\delta}(t))$, where $M(w_2^{-1}w_1^{-1}, \mathrm{Re}(s))$ is a standard intertwining operator. The unramified character $(\chi_0; \mathrm{Re}(s))\delta_{B_G}^{-1}$ may be identified with an element ς of $X(T_G) \otimes_Z \mathbb{R}$. The integral defining the standard intertwining operator converges provided the canonical pairing $\langle \varsigma, \alpha^\vee\rangle$ is positive for all positive roots α with $w_2^{-1}w_1^{-1}\alpha < 0$. Inspecting this set of roots, one finds it is convergent provided $\mathrm{Re}(s_2) > 1$, $\mathrm{Re}(s_1 - s_2) > 2$, and $\mathrm{Re}(s_1 + s_2) > 5$. Moreover, it converges to a section of the representation induced (via normalized induction) from $\varsigma^{w_1w_2}$.

Next, we need to understand the dependence of $M(w_1w_2, \mathrm{Re}(s)).f^\circ_{\mathrm{Re}(s)}(w_3\widetilde{\delta}(t))$ on T. In order to do this, we write $w_3\widetilde{\delta}(t)$ as $\widetilde{\nu}(t)\widetilde{\tau}(t)\widetilde{\kappa}(t)$ where $\widetilde{\nu}(t) \in U$, $\widetilde{\tau}(t) \in T_G$

and $\widetilde{\kappa}(t)$ varies in a compact set. It is convenient to do so using the basic algebraic substitution

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = \begin{pmatrix} r^{-1} & 1 \\ 0 & r \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & r^{-1} \end{pmatrix}, \tag{41}$$

which corresponds to the Iwasawa decomposition if F is nonarchimedean, but remains valid in the Archimedean case as well.

Recall that $\widetilde{\delta}(t)$ is the product of $x_{37}(-b)$, $x_{36}(-c)$ and $x_{25}(-d)x_{48}(-a)x_{29}(ad)$, which all commute with one another. We can partition T into 16 subsets and use the identity (41) to obtain a uniform expression for $\widetilde{\tau}(t)$ on each subset, and compute $\varsigma^{w_1 w_2}\delta_{B_G}^{1/2}(\widetilde{\tau}(t))$ in each case, obtaining

$$\begin{Bmatrix} 1, & |b| \le 1, \\ |b|^{-2u_1}, & |b| > 1 \end{Bmatrix} \times \begin{Bmatrix} 1, & |c| \le 1, \\ |c|^{-2u_2}, & |c| > 1 \end{Bmatrix}$$
$$\times \begin{Bmatrix} 1, & |a|, |d| \le 1, \\ |d|^{-2u_2}, & |a| \le 1, |d| > 1, \\ |a|^{-2u_1}, & |a| > 1, |ad| \le 1, \\ |a|^{-2u_1-u_2}|d|^{-2u_2}, & |a| > 1, |ad| > 1 \end{Bmatrix},$$

where $u_1 := \mathrm{Re}(\frac{s_1-s_2-2}{2})$, $u_2 := \mathrm{Re}(\frac{s_1+s_2-4}{2})$. Note that most these contributions are already visible in (34). Moreover, as in (37) we have $|\nu_s(t)| = |a|^{2u_1}|b|^{u_1}|c|^{u_2}|d|^{u_1+u_2}$

Next, we consider the quantity $W(t)\delta_B^{-1/2}(t)$ which appears in the integral (31). Using [4] or [26] in the nonarchimedean case, or [35], and [30] as explicated in [31] and [32] in the archimedean case, we have

$$W(t)\delta_B^{-1/2}(t) = \sum_{x \in X_\pi} \Phi_x(ad, bc, bd, cd)x(t), \tag{42}$$

where X_π is a finite set of finite functions depending on the representation π, and Φ_x is a Bruhat-Schwartz function $F^4 \to \mathbb{C}$ for each x.

Thus we obtain a sum of integrals of the form

$$\int_D \Phi(ad, bc, bd, cd)x(t)|a|^{k_1u_1+l_1u_2}|b|^{k_2u_1+l_2u_2}|c|^{k_3u_1+l_3u_2}|d|^{k_4u_1+l_4u_2}\, dt, \tag{43}$$

where D is one of our 16 subsets, $k_1, \dots, k_4$ and $l_1, \dots, l_4$ are explicit integers depending only on D, Φ is a Bruhat-Schwartz function $F^4 \to \mathbb{R}$, and x is a real-valued finite function $Z\backslash T \to \mathbb{R}$.

Now, for each of the seven cases which appear in (34), make a change of variables, as in the unramified case so that $|a|^{k_1u_1+l_1u_2}|b|^{k_2u_1+l_2u_2}|c|^{k_3u_1+l_3u_2}|d|^{k_4u_1+l_4u_2}$ is expressed as powers of the absolute values of the new variables. For example, when $|d| > 1$ and $|a|, |b|, |c| \le 1$, we have $|\nu_s(t)||d|^{-\mathrm{Re}(s_1)-\mathrm{Re}(s_2)+4} = |a|^{2u_1}|b|^{u_1}|c|^{u_2}|d|^{u_1-u_2}$, and substitute $a' = ad, b' = bd, c' = cd$ and $d' = d^{-1}$. After the substitution, each exponent is a nontrivial non-negative linear combination of u_1 and u_2. Also $|d'|$ is bounded, and we have $\Phi(a', b'c'(d')^2, b', c')$, which provides convergence as $|a'|$, $|b'|$ or $|c'| \to \infty$. It follows that the integral converges provided u_1 and u_2 are sufficiently large, relative to the finite function x.

As a second example, we consider the case $|a|, |c| > 1$, $|b|, |d| \le 1$. In this case, we make the change of variables $b' = bc$, $d' = acd$, $a' = a^{-1}$, $c' = c^{-1}$. We obtain the integral

$$\int_{|a'|<1, |c'|<1, |b'c'|\le 1, |a'c'd'|\le 1} \Phi(c'd', b', a'b'(c')^2 d', a'd') x(t) |a'|^{u_1+u_2} |b'|^{u_1} |c'|^{2u_1+2u_2} |d'|^{u_1+u_2} \, da' \, db' \, dc' \, dd',$$

assuming that u_1 and u_2 are positive and sufficiently large (depending on x), the integrals on a' and c' are convergent due to the domain of integration, and the integrals on b' and d' are convergent from the decay of Φ. Indeed, $\Phi(c'd', b', a'd', a'(b')^2c'd') \ll |a'b'c'(d')^2|^{-N}$ for any positive integer N because Φ is Bruhat-Schwartz, and then $|a'b'c'(d')^2|^{-N} \le |b'd'|^{-N}$ on the domain D. The other five cases appearing in (34) are handled similarly.

The nine cases which do not appear in (34) are easier. For example suppose that $|c|$ and $|d|$ are both >1 while $|a|$ and $|b|$ are both ≤ 1. Then the exponents of $|a|$ and $|b|$ are the same as in $\nu_s(t)$, i.e., they are $2u_1$ and u_1 respectively. This gives convergence of the integrals on a and b when $\mathrm{Re}(u_1)$ is sufficiently large (relative to x). The integrals on $|c|$ and $|d|$ converge because of the rapid decay of Φ in cd. The other eight cases are treated similarly, completing the proof of the convergence of $I_1(W, f, \phi; s)$. Now consider $I_1(R(k).W, R(k).f, \omega_\psi(k).\phi; s)$. Each bound used in the analysis of I_1 can be made uniform as k varies in the compact set k. Hence $I_1(R(k).W, R(k).f, \omega_\psi(k).\phi; s)$ varies continuously with k so its integral is again absolutely convergent. □

7.2 Continuation to a slightly larger domain

In this section, we prove that the local zeta integral $I(W, f, \phi; s)$ extends analytically to a domain that includes the point $s_1 = 5$, $s_2 = 1$. This point is of particular interest for global reasons. We keep the notation from the previous section. There are two issues. The first is related to the convergence of the integral $II_2.f_{\chi;s}$. As we have seen, this integral is *not* absolutely convergent at $(5, 1)$. We must show that it extends holomorphically to a domain containing $(5, 1)$. Then we need to prove convergence of the integral over $Z\backslash T$. The domain of absolute convergence for this integral depends on the exponents of the representation π. To make this precise, we use terminology and notation from [2], Sect. 3.1.

Proposition 7.2 *Suppose that* Π *satisfies* $H(\theta_4)$ *and that* τ_1 *and* τ_2 *satisfy* $H(\theta_2)$ *(as in* [2], *section 3.1). Then for any* $\varepsilon > 0$, *the local zeta integral* $I(W, f, \phi; s)$ *extends holomorphically to all* $s \in \mathbb{C}^2$ *satisfying* $\mathrm{Re}(s_1 - s_2) \ge \max(2\theta_4 + 2\theta_2 + 2, 3) + \varepsilon$, $\mathrm{Re}(s_1 + s_2) \ge 5 + \varepsilon$, $\mathrm{Re}(s_2) \ge \frac{1}{2} + \varepsilon$, $\mathrm{Re}(s_1 + 2s_2) \ge 2\theta_2 + 1$.

Proof We first need to extend $II_2.f_{\chi;s}$ beyond its domain of absolute convergence. It suffices to do this for flat K-finite sections, even though the convolution sections encountered in Sect. 5.2 are not, in general, flat of K-finite. Indeed, the integral operator II_2 commutes with the convolution operators considered in 5.2. Moreover, these operators are rapidly convergent, and hence preserve holomorphy.

As we have seen in the unramified computation II_2 can be expressed $II_3 \circ M(w_1^{-1}, \chi; s)$, where II_3 is an operator defined on $\underline{\mathrm{Ind}}_{B_G}^G((\chi; s)\delta_{B_G}^{-1/2})^{w_1}$ by the u_2 integral in (32). Then, $M(w_1^{-1}, \chi; s)$ is absolutely convergent for $\mathrm{Re}(s_2) > \frac{1}{2}, \mathrm{Re}(s_1 - s_2) > 1, \mathrm{Re}(s_1 + s_2 - 3) > 3$. If we insert absolute values into the integral which defines II_3, we obtain a standard intertwining operator attached to w_2^{-1}. We may write is as a composite of rank one intertwining operators attached to $\{\alpha > 0 : w_2^{-1}\alpha < 0\}$. The rank one operator attached to α is absolutely convergent provided that $\langle \alpha^\vee, \varsigma^{w_1}\rangle$ is positive. Running through the eight relevant roots, we find that only one rank one operator diverges at $(5, 1)$. It is attached to the simple root α_3 which satisfies $\langle \alpha^\vee, \varsigma^{w_1}\rangle = 2\,\mathrm{Re}(s_2) - 2$.

Thus, we only need to improve our treatment of the integral over a single one-parameter unipotent subgroup. Thus, we consider

$$\int_F f_{\chi;s}^{w_1}(w[3]x_{34}(r)g)\psi(c_4 r)\,dr, \tag{44}$$

where $c_4 \in F$ and $f_{\chi;s}^{w_1}$ is a section of the family $\underline{\mathrm{Ind}}_{B_G}^G((\chi; s)\delta_{B_G}^{-1/2})^{w_1}$, $s \in \mathbb{C}^2$. Notice that (44) may be regarded as a Jacquet integral for the rank-one Levi attached to the simple root α_3. By [21], this extends to an entire function of s when $f_{\chi;s}^{w_1}$ is flat. If we apply it to the output of $M(w_1^{-1}, \chi; s)$, then it has no poles other than those of $M(w_1^{-1}, \chi; s)$. Now we use again the fact that the asymptotics of a Whittaker function, are controlled by the exponents of the relevant representation. This time we apply it to the induced representation of our rank one Levi. For most values of s, the exponents are $((\chi; s)\delta_{B_G}^{-1/2})^{w_1}$ and $((\chi; s)\delta_{B_G}^{-1/2})^{w_1 w[3]}$ and the Whittaker function is bounded in absolute value by a linear combination of spherical vectors.

On the line $s_2 = 1$, this may fail: if $((\chi; s)\delta_{B_G}^{-1/2})^{w_1} = ((\chi; s)\delta_{B_G}^{-1/2})^{w_1 w[3]}$, then an extra log factor appears in the asymptotics of the Whittaker function (cf. [17], 6.8.11, for example). Bounding $\log|x|$ by $|x|^{-\varepsilon}$ with $\varepsilon > 0$ as $x \to 0$, in this case, we again bound the integral (44) by a sum of spherical sections. In fact, the extra $|x|^\varepsilon$ may be safely ignored, since we obtain convergence for s in an open set and $\varepsilon > 0$ can be taken arbitrarily small. Thus, if $w_2 = w[3]w_2'$, then $II_2.f_{\chi;s}$ extends holomorphically to the domain where the standard intertwining operator attached to w_2' converges on both $f^\circ_{\mathrm{Re}(s),w_1}$, and $f^\circ_{\mathrm{Re}(s),w_1 w[3]}$. Inspecting $\{\alpha > 0 : (w_2')^{-1}\alpha < 0\}$, we see that this means $\mathrm{Re}(2s_2 - 1)$, $\mathrm{Re}(s_1 - s_2 - 3)$ and $\mathrm{Re}(s_1 + s_2 - 5)$ must all be positive. As a side effect, we find that $|II_2.f_{\chi;s}(g)|$ is bounded by a suitable linear combination of $M(w_2^{-1}w_1^{-1}, \mathrm{Re}(s)).f^\circ_{\mathrm{Re}(s)}$ and $M((w_2')^{-1}w_1^{-1}, \mathrm{Re}(s)).f^\circ_{\mathrm{Re}(s)}$.

As before, we obtain a sum of integrals of the form (43) where now the integers $k_1, \ldots, k_4$ and $l_1, \ldots, l_4$ depend on the choice of domain D and on a choice of between w_2 and w_2'.

In order to obtain a precise domain of convergence, we need information about the finite function x. Firstly, since we have taken absolute values and assumed unitary central character, it factors through the map $t \mapsto (|ad|, |bc|, |bd|, |cd|)$. We may assume that x is given in terms of real powers of the coordinates and non-negative integral powers of their logarithms, since such functions span the space of real-valued finite functions. Since a power of $\log y$ may be bounded by an arbitrarily small positive (resp. negative) power of y as $y \to \infty$ (resp. 0), for purposes of determining the domain of convergence, we may assume that there are no logarithms. Thus we may assume $x(t) = |ad|^{\rho_1}|bc|^{\rho_2}|bd|^{\rho_3}|cd|^{\rho_4}$ with $\rho_1, \rho_2, \rho_3, \rho_4 \in \mathbb{R}$. The quadruples $(\rho_1, \rho_2, \rho_3, \rho_4)$ which appear are governed by the exponents of π, by [4] or [26] in the nonarchimedean case, and [35], [30] (see also [32]) in the archimedean case. Hence they are bounded in absolute value by $\max(\theta_2, \theta_4)$, by the definition of $H(\theta_2)$ and $H(\theta_4)$ in [2] and the bound on exponents of tempered representations found in [30], Theorem 15.2.2 in the archimedean case, or [34] in the nonarchimedean case, we see that $|\rho_1|, |\rho_2| \le \theta_4, |\rho_3|, |\rho_4| \le \theta_1$.

What remains is a careful case-by-case analysis along the same lines as the proof of convergence. For each choice of D, after a suitable change of variables we have an integral which is convergent provided u_1 and u_2 are sufficiently large, and "sufficiently large" is given explicitly in terms of $\rho_1, \dots, \rho_4$.

For example, the above integral corresponding to the case $|a|, |c| > 1$ and $|b|, |d| \le 1$ will now feature a Schwartz function integrated against

$$|c'd'|^{\rho_1}|b'|^{\rho_2}|a'b'(c')^2d'|^{\rho_3}|a'd'|^{\rho_4}|a'|^{u_1+u_2}|b'|^{u_1}|c'|^{2u_1+2u_2}|d'|^{u_1+u_2},$$

and so will converge provided $u_1 + u_2 + \rho_3 + \rho_4$, $u_1 + \rho_2 + \rho_3$, $2u_1 + \rho_1 + 2\rho_2$, and $u_1 + u_2 + \rho_1 + \rho_3 + \rho_4$ are all positive. □

7.3 Meromorphic continuation and nonvanishing

Write U_P^- for the unipotent radical of the parabolic opposite P. Notice that $PU_P^- w$ is a Zariski open subset of GSO_{12}. We say that $f \in \mathrm{Flat}(\chi)$ is **simple** if it is supported on PK_1 where K_1 is a compact subset of $U_P^- w$.

Proposition 7.3 *Suppose that f is simple. Then $I(W, f, \phi; s)$ has meromorphic continuation to $\mathbb{C}^2$ for each $\phi \in \mathcal{S}(\mathrm{Mat}_{4\times 2})$ and each $W \in \mathcal{W}_{\psi_N}(\pi)$ Moreover, if s_0 is an element of $\mathbb{C}^2$, then there exist W, f and ϕ such that $I(W, f, \phi; s_0) \neq 0$.*

Proof We begin with some formal manipulations which are valid over any local field. The process requires many of the same subgroups which were defined during the Proof of Theorem 3.1, and we freely use notation from that section.

$$I(W, f, \phi;\ s) = \int_{ZU_4\backslash C_Q}\int_{U_Q^w\backslash U_Q}\int_{\mathrm{Mat}_{1\times 2}} W(g)f(wug, s)[\omega_\psi(ug).\phi]\begin{pmatrix} r \\ I_2 \\ 0\end{pmatrix}\overline{\psi}\left(r\cdot\begin{bmatrix}1\\0\end{bmatrix}\right) dr\, du\, dg.$$

Define $U_1(a, b, c)$ as in (23). Then

$$[\omega_\psi(ug).\phi]\begin{pmatrix} r \\ I_2 \\ 0\end{pmatrix} = [\omega_\psi(U_1(r_1, r_2, c)ug).\phi](\Xi_0),$$

(for any c) where $\Xi_0 := \begin{pmatrix} 0 \\ I_2 \\ 0 \end{pmatrix}$ and $r = (r_1\ r_2)$. Also $W(U_1(r_1, r_2, c)g) = \overline{\psi}(r_1)W(g) = \overline{\psi}\left(r \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) W(g)$. Hence

$$I(W, f, \phi; s) = \int_{ZU_5\backslash C_Q} \int_{U_Q^w\backslash U_Q} W(g) f(wug, s)[\omega_\psi(ug).\phi](\Xi_0) du\, dg,$$

where U_5 is the product of U_6, U_2 and the center $Z(U_1)$ of U_1. □

Recall that C_Q^w is a standard parabolic subgroup of C_Q. Let $U(C_Q^w)^-$ denote the unipotent radical of the opposite parabolic. Then $C_Q^w \cdot U(C_Q^w)^-$ is a subset of full measure in C_Q and we can factor the Haar measure on C_Q as the product of (suitably normalized) left Haar measure on C_Q^w and Haar measure on $U(C_Q^w)^-$. Hence

$$I(W, f, \phi; s) = \int_{ZU_5\backslash C_Q^w} \int_{U(C_Q^w)^-} \int_{U_Q^w\backslash U_Q} W(gu_1) f(wugu_1, s)[\omega_\psi(ugu_1).\phi](\Xi_0) du\, du_1\, d_\ell g$$

Conjugating g across u, making a change of variables, and making use of the equivariance of f yields

$$I(W, f, \phi; s) = \int_{U(C_Q^w)^-} \int_{U_Q^w\backslash U_Q} J(R(u_1).W, \omega_\psi(uu_1).\phi; s) f(wuu_1, s)\, du\, du_1.$$

where

$$J(W, \phi, s) = \int_{ZU_5\backslash C_Q^w} W(g)[\omega_\psi(g).\phi](\Xi_0)\ \mathrm{Jac}_1(g)(\chi; s)(wgw^{-1})\, d_\ell g.$$

with $\mathrm{Jac}_1(g)$ being the Jacobian of the change of variables in u. Now conjugation by w maps $U_Q^w\backslash U_Q \times U(C_Q^w)$ isomorphically onto the unipotent radical U_P^- of the parabolic opposite P. Hence, if Φ is any smooth compactly supported function on $U_Q^w\backslash U_Q \times U(C_Q^w)^-$, then there is a flat section f such that $f(wuu_1, s_0) = \Phi(u, u_1)$.

We claim that the integral $J(W, \phi; s)$ converges provided $\mathrm{Re}(s_1 - s_2)$ and $\mathrm{Re}(s_2)$ are both sufficiently large, and that $J(R(u_1).W, \omega_\psi(uu_1).\phi; s)$ extends meromorphically to $\mathbb{C}^2$ and is a continuous function of uu_1 away from the poles. Granted this claim, is clear that if the integral of $J(R(u_1).W, \omega_\psi(uu_1).\phi; s_0)$ against the arbitrary test function $f(wuu_1, s_0)$ is always zero, then $J(W, \phi; s_0)$ is zero for all W and ϕ.

Now, $C_Q^w = (P_1 \times P_2)^\circ$ is the intersection of C_Q with the product of the Klingen parabolic P_1 of GSp_4 and the Siegel parabolic P_2 of GSO_4. Let $C' = (P_1 \times M_2)^\circ$ denote the subgroup of elements whose GSO_4 component lies in the Levi, and let $U_5' = C' \cap U_5 = U_6 Z(U_1)$. Then C' surjects onto $ZU_5\backslash C_Q^w$, which is thus canonically identified with $ZU_5'\backslash C'$. Expressing the measure on C_Q^w in terms of Haar measures on U_1, U_2 and $(M_1 \times M_2)^\circ$, and then identifying $Z(U_1)\backslash U_1$ with $\mathrm{Mat}_{2\times 1}(F)$ via the map $\bar{u}_1(\begin{bmatrix} r_1 & r_2 \end{bmatrix}) = U_1(r_1, r_2, 0)$, yields the following expression for $J(W, \phi; s)$:

$$\int_{\mathrm{Mat}_{1\times 2}} \int_{(M_1\times M_2)^\circ} W(\bar{u}_1(r)m)[\omega_\psi(\bar{u}_1(r)m).\phi](\Xi_0)\ \mathrm{Jac}_1(m)(\chi; s)(wmw^{-1})\, \delta^{-1}_{C_Q^w}(m)dm\, dr, \tag{45}$$

where $\delta_{C_Q^w}$, is the modular quasicharacter.

Now, elements of C' map under j into the Siegel Levi of Sp_{16}. So that

$$[\omega_\psi \circ j(c').\phi](\xi) = |\det c'|^{\frac{1}{2}} \phi(\xi \cdot c'),$$

where $\cdot$ is the rational right action of C' on $\mathrm{Mat}_{4\times 2}$ by

$$\xi \cdot m_Q^1(g_1, g_2) = g_1^{-1}\xi g_2.$$

[with m_Q^1 as in (4)]. The stabilizer of the matrix Ξ_0 is precisely the group M_5 introduced in the Proof of Theorem 3.1.

In (45), conjugate m across $\bar{u}_1(r)$, make a change of variables in r, and let $\mathrm{Jac}_2(m)$ denote the Jacobian. Define

$$\mu_s(m) = (\chi; s)(wmw^{-1})\delta^{-1}_{C_Q^w}(m)|\det m|^{\frac{1}{2}}\, \mathrm{Jac}_1(m)\, \mathrm{Jac}_2(m).$$

Then replace m by $m_5 m_5'(g)$ where $m_5 \in M_5$ and $m_5'(g) = m(1, I_2, g)$, [with m as in (21)]. Observe that

$$\Xi_0 \cdot m_5 m_5'(g)\bar{u}_1(r) = \begin{pmatrix} r \cdot g \\ g \\ 0 \end{pmatrix}.$$

Hence if $x(g, r) = m_5'(g)\bar{u}_1(rg^{-1})$, then

$$J(W, \phi; s) = \int_{\mathrm{Mat}_{1\times 2}} \int_{GL_2} J'(R(x(g, r)).W, s)\phi \begin{pmatrix} r \\ g \\ 0 \end{pmatrix} \mu_s(m_5'(g))|\det g|^{-1}\, dg\, dr, \tag{46}$$

$$\text{where} \qquad J'(W, s) := \int_{ZU_6\backslash M_5} W(m_5)\mu_s(m_5)\, dm_5. \tag{47}$$

Direct computation shows that $\mu_s(m_5'(g)) = |\det g|^{s_2}\chi_2(\det g)$.

Write $M_5 = U_6 T_5 K_5$, where $T_5 = T \cap M_5$ and K_5 is the maximal compact subgroup of the GL_2 factor. Then

$$J'(W, s) := \int_{K_5} \int_{Z\backslash T_5} W(tk)\mu_s(t)\delta^{-1}_{B_5}(t)$$

where δ_{B_5} is the modular quasicharacter of the standard Borel subgroup B_5 of M_5. Set $t_6'(a) = \mathrm{diag}(a, 1, 1, a^{-1}, 1, 1, 1, 1, a, 1, 1, a^{-1})$, and write $t \in T_5$ as $t_6 t_6'(a)$ for $t_6 \in T_6$ and $a \in F^\times$. Then

$$J'(W, s) = \int_{K_5} \int_{F^\times} J''(R(t_6'(a)k).W, s)\mu_s(t_6'(a))\, dt,$$

$$\text{where } J''(W, s) = \int_{Z\backslash T_6} W(t_6)\mu_s(t_6)\delta_{B_5}^{-1}(t_6)\, dt_6.$$

Observe that $J'(W, s)$ may be written formally as

$$\int_{M_6\backslash M_5} J''(R(g_1).W, s)\mu_s(g_1) dg_1.$$

Also, direct computation shows that $\mu_s(t_6'(a)) = |a|^{s_1 - s_2 - 4}\chi_1(a)/\chi_2(a)$.

For ϕ_1 a smooth function of compact support $F^2 \to \mathbb{C}$ let

$$[\phi_1 *_1 W](g) = \int_{F^2} W(gU_1(r_1, r_2, 0))\phi(r_1, r_2)\, dr.$$

Observe that

$$[\phi_1 *_1 W](M_5(t, g_3)) = W(M_5(t, g_3))\widehat{\phi}_1\left(g_3^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \det g_3\right).$$

Thus, by replacing W by $\phi_1 *_1 W$ (which is justified by [5]) we may introduce what amounts to an arbitrary test function on $M_6\backslash M_5$. Hence

$$J'(W, s_0) = 0 \forall W \iff J''(W, s_0) = 0 \forall s_0.$$

Similarly, if

$$[\phi_2 *_2 W](g) := \int_{U_6} W(gu_6)\phi_2(u_6)\, du_6, \qquad \phi_2 \in C_c^\infty(U_6),$$

then $J''(\phi_2 *_2 W, s_0) = 0 \forall \phi_2 \in C_c^\infty(U_6)$ if and only if W vanishes identically on T_6. In particular, if $J''(W, s_0)$ vanishes identically on $\mathcal{W}_{\psi_N}(\pi)$, then $\mathcal{W}_{\psi_N}(\pi)$ is trivial–a contradiction.

This completes the formal arguments for Proposition 7.3. It remains to check the convergence and continuity statements made above. These will be proved based on facts about asymptotics of Whittaker functions and the Mellin transform

$$Z_{\xi,n}.\Phi(u) = \int_{F^\times} \Phi(x)\xi(x)(\log|x|)^n |x|^u\, d^\times x, \tag{48}$$

where $\Phi \in \mathcal{S}(F)$, $\xi : F^\times \to \mathbb{C}$, a character, $n \geq 0 \in \mathbb{Z}$, $u \in \mathbb{C}$. We recall some properties.

Proposition 7.4 *Fix a character ξ and a non-negative integer n.*

(1) *There is a real number c depending on ξ such that the integral defining $Z_{\xi,n}.\Phi$ converges absolutely and uniformly on $\{u \in \mathbb{C} : \mathrm{Re}(u) \geq c + \varepsilon\}$ for all $\varepsilon > 0$ and all $\Phi \in \mathcal{S}(F)$.*
(2) *There is a discrete subset S_ξ of $\mathbb{C}$ such that $Z_{\xi,n}.\Phi$ extends meromorphically to all of $\mathbb{C}$ with poles only at points in S_ξ. Moreover, there is an integer $o_{\xi,n}$ such that no pole of $Z_{\xi,n}.\Phi$ has order exceeding $o_{\xi,n}$, for any Φ.*
(3) *If F is archimedean, then $Z_{\xi,n}.\Phi = Q_\Phi(q^u)$ for some rational function Q.*
(4) *We have*

$$Z_{\xi,n+1}.\Phi(u) = \frac{d}{du} Z_{\xi,n}.\Phi(u). \tag{49}$$

Proof If $n = 0$ then the first three parts are proved in [33]. Convergence for $n > 0$ is straightforward, since $\log|x|$ grows slower than any positive power of $|x|$ at infinity, and slower than any negative power as $|x| \to 0$. Equation (49) is clear in the domain of convergence, and follows elsewhere by continuation. The first three parts for general n then follow. □

Next, we need a version of the expansion (42). Specifically, if we replace $W(t)$ by $W(tk)$ then each Φ_x in (42) will be in $\mathcal{S}(F^4 \times K)$ (see [31], especially the remark on p. 20).

Let us now consider the convergence issues raised by our formal computations more carefully. Recall that $I(W, f, \phi; s)$ was initially expressed as an integral over $U_Q^w \backslash U_Q \times ZU_5 \backslash C_Q$. In the course of our arguments, we have expressed it as an iterated integral over

$$(U_Q^w \backslash U_Q \times U(C_Q^w)^-) \times (F^2 \times GL_2) \times (F^\times \times K_5) \times Z\backslash T_6.$$

In order to perform the integration on $Z\backslash T_6$ we may identify with with $\{\bar{t}_6(a) = \mathrm{diag}(1, 1, a^{-1}, a^{-1}, 1, a^{-1}, 1, a^{-1}, 1, 1, a^{-1}, a^{-1}), a \in F^\times\}$. Then $\mu_s(\bar{t}_6(a)) = |a|^{\frac{s_1-s_2}{2}-2} \frac{\chi_1^2\chi_2}{\chi_3}(a)$.

Now, the integral $J''(W, s)$ is the Mellin transform taken along the one dimensional torus we use to parametrize $Z\backslash T_6$. Its convergence and analytic continuation follow directly from Proposition 7.4 and (42). Now consider $J''(R(t_6'(a)k).W, s)$ with $a \in F^\times, k \in K_5$. We claim that it is smooth and of rapid decay in a. In the domain of absolute convergence, this is easily seen. For s outside of the domain of convergence, we use (49) to pass to the Mellin transform of a suitable derivative at a point *inside* the domain of convergence. To get $J'(W, s)$ we integrate k over the compact set k and take another Mellin transform in the variable a. This of course converges absolutely, and by similar reasoning, we see that $J'(R(x(g, r).W), s)$ is smooth. Now, set g equal to $\begin{pmatrix} t_1 & b \\ 0 & t_2 \end{pmatrix} k$ and consider the integral

$$\int_{\mathrm{Mat}_{1\times 2}} \int_{F^\times} \int_{F^\times} \int_{F} \int_{K_{GL_2}} J'(R(x(g,r)).W,s)\phi\begin{pmatrix} r \\ g \\ 0 \end{pmatrix} \chi_2(t_1t_2)|t_1|^{s_4}|t_2|^{s_5}\, dk\, db\, dr\, d^\times t_1\, d^\times t_2,$$

where s_4 and s_5 are two more complex variables, and K_{GL_2} is the standard maximal compact subgroup of GL_2. The integrals on k, r and b converge absolutely and uniformly because K_{GL_2} is compact and ϕ is Schwartz-Bruhat. The integrals on t_1 and t_2 take two more Mellin transforms, yielding a meromorphic function of four complex variables. The restriction to a suitable two-dimensional subspace of $\mathbb{C}^4$ is $J(W,\phi,s)$. Moreover, $J(R(u_1).W, \omega_\psi(uu_1).\phi, s)$ remains continuous in $u_1 \in U(C_Q^w)^-$ and $u \in U_Q^w \backslash U_Q$, which completes the proof.

8 Global identity

We now return to the global situation. Thus F is again a number field with adele ring $\mathbb{A}$, while and ψ_N, and $\mathrm{Flat}(\chi)$, are defined as in Sects. 3 and 2, respectively. In addition, let $\pi = \otimes_v \pi_v$ be an irreducible, globally ψ_N-generic cuspidal automorphic representation of $GSp_4(\mathbb{A}) \times GSO_4(\mathbb{A})$, with normalized central character ω_π, and φ be a cusp form from the space of π, etc.

For r a representation of ${}^LG$ define $L(u,\pi,r\times\eta)$ to be the twisted L function. Thus at an unramified place v the local factor is

$$L_v(u,\pi_v,r\times\eta_v) = \det(I - q^{-u}\eta_v(\varpi_v)r(\tau_{\pi_v}))^{-1},$$

where ϖ_v is a uniformizer, q_v is the cardinality of the residue class field, τ_{π_v} is the semisimple conjugacy class attached to π_v, and η_v is the local component of η at v.

Theorem 8.1 *Suppose that* $f_\chi = \prod_v f_{\chi_v} \in \mathrm{Flat}(\chi)$, $\phi = \prod_v \phi_v \in \mathcal{S}(\mathrm{Mat}_{4\times 2}(\mathbb{A}))$ *and* $W_\varphi = \prod_v W_v$ *(the Whittaker function of* π *as in Theorem 3.1 are factorizable. Let* $I(f_{\chi_v;s}, W_v, \phi_v)$ *be the local zeta integral, defined as in (28), and let* S *be a finite set of places* v *and all data is unramified for all* $v \notin S$. *Then for* $\mathrm{Re}(s_1 - s_2)$ *and* $\mathrm{Re}(s_2)$ *both sufficiently large, the global integral* $I(f_{\chi;s}, \varphi, \phi)$, *defined as in (7), is equal to*

$$\frac{L^S\left(\frac{s_1-s_2}{2}-1, \pi, St^\vee_{GSp_4}\otimes St_{GL_2^{(1)}} \times \frac{\chi_3}{\chi_1\chi_2^2}\right) L^S\left(\frac{s_1+s_2}{2}-2, \pi, St^\vee_{GSp_4}\otimes St_{GL_2^{(2)}} \times \frac{\chi_3}{\chi_1\chi_2}\right)}{N^S(s,\chi)}$$

times

$$\prod_{v\in S} I(f_{\chi_v}, W_v, \phi_v),$$

where $N^S(s,\chi)$ *is the product of partial zeta functions corresponding to* (33)

Remark 8.2 Let η_1 and η_2 be any two characters of $F^\times\backslash\mathbb{A}^\times$. Fix π and let ω_π be its central character. Then the system

$$\frac{\chi_1^3\chi_2^3}{\chi_3^2} = \omega_\pi, \qquad \frac{\chi_3}{\chi_1\chi_2^2} = \eta_1, \qquad \frac{\chi_3}{\chi_1\chi_2} = \eta_2$$

has a unique solution. If $\eta_1 = \eta_2$ is trivial, then it is given by $\chi_1 = \chi_3 = \omega_\pi$ and $\chi_2 \equiv 1$.

Proof It follows from Theorem 6.1 the bound obtained in [25] that for any cuspidal representation $\pi = \otimes_v \pi_v$ of $GSp_4(\mathbb{A}) \times GSO_4(\mathbb{A})$ the infinite product $\prod_{v\in S} I(f_{\chi_v}, W_v, \phi_v)$ is convergent for $\mathrm{Re}(s_1 - s_2)$ and $\mathrm{Re}(s_2)$ sufficiently large. It then follows from Theorem 3.1, and the basic results on integration over restricted direct products presented in [33] that

$$I(f_{\chi;s}, \varphi, \phi) = \prod_v I(f_{\chi_v;s}, W_v, \phi_v),$$

which, in conjunction with Theorem 6.1 again gives the result. □

Corollary 8.3 *Let π_v be the local constituent at v of a cuspidal representation π. Then the local zeta integral $I_v(W_v, f_v, \phi_v; s)$ has meromorphic continuation to $\mathbb{C}^2$ for any W_v, f_v and ϕ_v.*

Proof This follows from a globalization argument. We create a section of the global induced representation which is f at one place and simple at every other place. Meromorphic continuation of the global zeta integral follows from that of the Eisenstein series. Having shown meromorphic continuation at every other place in Proposition 7.3, we deduce it at the last place. □

9 Application

In this section we give an application relating periods, poles of L functions, and functorial lifting. The connection between L functions and functorial lifting in this case was obtained in [2].

Let Π be a globally generic cuspidal automorphic representation of GSp_4, and let τ_1 and τ_2 be two cuspidal automorphic representations of GL_2. Assume that Π, τ_1 and τ_2 have the same central character. Then $\tau_1 \otimes \tau_2$ may be regarded as a representation of GSO_4 via the realization of GSO_4 as a quotient of $GL_2 \times GL_2$ discussed in Sect. 6, and when $\Pi \otimes \tau_1 \otimes \tau_2$ is restricted to the group C_Q (which we identify C_Q with subgroup of $GSp_4 \times GSO_4$ as in Sect. 2 its central character is trivial.

Now take s_1 and s_2 to be two complex numbers. Let $\chi_1 = \chi_2 = \chi_3$ be trivial. Consider the space $\mathrm{Flat}(\chi)$ of flat sections as in Sect. 2. Its elements are functions $\mathbb{C}^3 \times G(\mathbb{A}) \to \mathbb{C}$, but we regard each as a function $\mathbb{C}^2 \times G(\mathbb{A}) \to \mathbb{C}$ by pulling it back through the function $(s_1, s_2) \mapsto (s_1, s_2, \frac{3s_1+3s_2}{2})$. Then Theorem 8.1 relates the global integral (7) with the product of L functions

$$L^S\left(\frac{s_1-s_2-2}{2}, \widetilde{\Pi}\times\tau_1\right) L^S\left(\frac{s_1+s_2-4}{2}, \widetilde{\Pi}\times\tau_2\right).$$

For $f \in \mathrm{Flat}(\chi)$, let

$$\underline{r}(f, g) = \mathrm{Res}_{s_1-s_2=4}\,\mathrm{Res}_{s_1+s_2=6}\, E(f_{\chi;s}, g)$$

be the iterated residue of the Eisenstein series along the plane $s_1 + s_2 = 6$ and then the plane $s_1 - s_2 = 4$. (It follows from Theorem 8.1 and Proposition 7.3 that this residue is nonzero. It can also be checked directly.) As f varies we obtain a residual automorphic representation which we denote $\mathcal{R}$. Given $\underline{r} \in \mathcal{R}$ and $\phi \in \mathcal{S}(\mathrm{Mat}_{4\times 2}(\mathbb{A}))$, we define the Fourier coefficient $\underline{r}^{\theta(\phi)}$ exactly as in 6. Varying $\underline{r}$ and ϕ we obtain a space of smooth, K-finite functions of moderate growth $Z(\mathbb{A})C_Q(F)\backslash C_Q(\mathbb{A}) \to \mathbb{C}$. We denote this space $FC(\mathcal{R})$. Write V_Π for the space of the representation Π and V_τ for that of τ. Then, define the period $\mathcal{P} : V_\Pi \times V_\tau \times FC(\mathcal{R}) \to \mathbb{C}$, by the formula

$$\mathcal{P}(\varphi_\Pi, \varphi_\tau, \underline{r}^{\theta(\phi)}) = \int_{Z\backslash C_Q} \underline{r}^{\theta(\phi)}(g)\varphi_\Pi(g_1)\varphi_\tau(g_2)\, dg.$$

Theorem 9.1 *First suppose that $\tau_1 \neq \tau_2$. Then the following are equivalent:*

(1) $L^S(s, \widetilde{\Pi}\times\tau_1)$ *and* $L^S(s, \widetilde{\Pi}\times\tau_2)$ *have poles at* $s = 1$.
(2) Π *is the weak lift of* $\tau_1 \times \tau_2$
(3) *the period* $\mathcal{P}$ *does not vanish identically on* $V_\Pi \times V_\tau \times FC(\mathcal{R})$.

Similarly, if $\tau_1 = \tau_2$, then the following are equivalent:

(1) $L^S(s, \Pi\times\tau_1)$ *has a pole at* $s = 1$.
(2) $\widetilde{\Pi}$ *is the weak lift of* $\tau_1 \times \tau'$ *for some cuspidal representation* τ' *of* $GL_2(\mathbb{A})$,
(3) *the period* $\mathcal{P}$ *does not vanish identically on* $V_\Pi \times V_\tau \times FC(\mathcal{R})$.

Proof The relationship between poles and the similitude theta correspondence was established in [2]. What is new here is the period condition, which follows from our earlier results. Indeed, for $f \in \mathrm{Flat}(\chi)$, $\phi \in \mathcal{S}(\mathrm{Mat}_{4\times 2}(\mathbb{A}))$ $\varphi_\Pi \in V_\Pi$ and $\varphi_\tau \in V_\tau$, the period $\mathcal{P}(\varphi_\Pi, \varphi_\tau, R(f)^{\theta(\phi)})$ vanishes if and only if

$$\mathrm{Res}_{s_1-s_2=4}\,\mathrm{Res}_{s_1+s_2=6}\, I(f_{\chi;s}, \varphi_\Pi, \varphi_\tau, \phi) \neq 0.$$

By [2], the local components of Π all satisfy $H(15/34)$ and the local components of τ all satisfy $H(1/9)$. Hence each ramified local zeta integral is holomorphic at (5, 1) by Proposition 7.2. Moreover, by Proposition 7.3, each ramified local zeta integral is nonzero at (5, 1) for a suitable choice of data. The result follows. □

Remark 9.2 Inspecting the various intertwining operators which appear in the constant term of our Eisenstein series along the Borel, one finds that some have poles along $s_1 - s_2 = 4$ and $s_1 + s_2 = 6$ of orders as high as three, as well as simple poles along $s_1 = 5$ and $s_2 = 4$. However, it follows from Theorem 8.1 and Proposition 7.2 that the global integral can have at most a simple pole along $s_1 - s_2 = 4$ and a simple pole along $s_1 + s_2 = 6$. It follows that any automorphic forms obtained by considering higher order singularities of the Eisenstein series either do not support our Fourier–Jacobi coefficient or have the property that their Fourier–Jacobi coefficients, regarded as smooth functions of moderate growth on $C_Q(F)\backslash C_Q(\mathbb{A})$, are orthogonal to cuspforms.

10 A similar integral on GSO_{18}

In this section we consider the global integral (7) in the case $n = 3$. Our unfolding does not produce an integral involving the Whittaker functions attached to our cusp forms, but it does reveal another intriguing connection with the theta correspondence.

As before, the space of double cosets $P\backslash GSO_{18}/R_Q$ is represented by elements of the Weyl group, and

$$I(f_{\chi;s}, \varphi, \phi) = \sum_{w \in P\backslash GSO_{18}/R_Q} I_w(f_{\chi;s}, \varphi, \phi), \qquad \text{where}$$

$$I_w(f_{\chi;s}, \varphi, \phi) = \int_{C_Q^w(F)\backslash C_Q(\mathbb{A})} \varphi(g) \int_{U_Q^w(\mathbb{A})\backslash U_Q(\mathbb{A})} f_{\chi;s}(wu_2 g) \int_{[U_Q^w]} \theta(\phi, u_1 u_2 g)\, du_1\, du_2\, dg,$$

which is zero if $\psi_l\big|_{Z(U_Q)\cap U_Q^w}$ is nontrivial, or if some parabolic subgroup of C_Q stabilizes the flag $0 \subset \overline{U}_Q^w \subset (\overline{U}_Q^w)^\perp$ in $U_Q/Z(U_Q)$, where $\overline{U}_Q^w$ is the image of U_Q^w and $(\overline{U}_Q^w)^\perp$ is its perp space relative to the symplectic form defined by composing $l : Z(U_Q) \to \mathbb{G}_a$ with the commutator map $U_Q/Z(U_Q) \to Z(U_Q)$.

Lemma 10.1 *Let w_ℓ denote the longest element of the Weyl group of GSO_{18}, let w_1 be the shortest element of $(W \cap P) \cdot w_\ell \cdot (W \cap Q)$ and let $w_2 = w_1 \cdot w[32]$. Then Pw_2R_Q is a Zariski open subset of GSO_{18}.*

Proposition 10.2 *$I_w(f_{\chi;s}, \varphi, \phi)$ is zero unless w is in the open double coset.*

Proposition 10.3 *Let $U_0 \subset C_Q$ be given by*

$$\left\{ u_0(x, x') := \left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ & 1 & 0 & 0 & x_6 & * \\ & & 1 & 0 & 0 & * \\ & & & 1 & 0 & * \\ & & & & 1 & * \\ & & & & & 1 \end{pmatrix}, \right.\right.$$

$$\left.\left. \begin{pmatrix} 1 & x_1' & x_4' & x_7 & x_8 & * \\ & 1 & x_6' & x_9 & * & * \\ & & 1 & 0 & -x_9 & * \\ & & & 1 & -x_6 & * \\ & & & & 1 & -x_1 \\ & & & & & 1 \end{pmatrix} \right) : \begin{array}{l} x \in \mathbb{G}_a^9, \\ x' \in \mathbb{G}_a^3 \end{array} \right\},$$

where entries marked $*$ *are determined by symmetry, and for* $x \in \mathbb{A}^9$ *and* $x' \in \mathbb{A}^3$*, let*

$$\psi_{U_0}(u_0(x,x')) = \psi(x_1 + x_6 - x_1' - x_6' + x_9).$$

Let $SL_2^{\alpha_3}$ *be the copy of* SL_2 *generated by* $U_{\pm\alpha_3}$*, and let* R_0 *be the product of* U_0 *and* $SL_2^{\alpha_3}$*. Let* ψ_{R_0} *be the character of* R_0 *which restricts to* ψ_{U_0} *and to the trivial character of* $SL_2^{\alpha_3}$*. Let*

$$\varphi^{(R_0,\psi_{R_0})}(c) = \int_{[R_0]} \varphi(rc)\psi_{R_0}(r)\,dr = \int_{[U_0]}\int_{[SL_2^{\alpha_3}]} \varphi(uhc)\psi_{U_0}(u)\,dh\,du, \qquad (c \in C_Q(\mathbb{A})).$$

Let $V_4 = \{u(x,x') : x_i = x_i', i = 1,4,6\} \subset U_0$*, and let*

$$\xi_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{6\times 3}(F).$$

Then

$$I_{w_2}(f_{\chi;s},\varphi,\phi) = \int_{Z(\mathbb{A})V_4(\mathbb{A})\backslash C_Q(\mathbb{A})} \varphi^{(R_0,\psi_{R_0})}(c) \int_{U^{w_2}(\mathbb{A})\backslash U(\mathbb{A})} f_{\chi;s}(w_2uc)\left[\omega_\psi(uc).\phi\right](\xi_0)\,du\,dc.$$

Remark 10.4 It was shown in [15] that the period we obtain in the GSp_6 characterizes the image of the theta lift from SO_6 to Sp_6.

Proof First, $U_Q^{w_2}$ is the set of all $u_Q(0,Y,0)$ such that rows 2, 5 and 6 of Y are zero. It follows that

$$\theta^{U_Q^{w_2}}(\phi, u_1u_2g) := \int_{[U_Q^{w_2}]} \theta(\phi, u_1u_2g)\,du_1 = \sum_\xi [\omega_\psi(u_2g).\phi](\xi),$$

where the sum is over $\xi \in \mathrm{Mat}_{6\times 3}(F)$ such that rows 3, 4 and 1 are zero. Next, the identification of C_Q with a subgroup of $GSp_6 \times GSO_6$ identifies $C_Q^{w_2}$ with the subset of elements of the form

$$\left(\begin{pmatrix} t & x_1 & x_2 & x_3 & x_4 & x_5 \\ & a & 0 & 0 & b & * \\ & & a' & b' & 0 & * \\ & & c' & d' & 0 & * \\ & c & & & d & * \\ & & & & & t\lambda \end{pmatrix}, \begin{pmatrix} g & W \\ & {}_tg^{-1}\lambda \end{pmatrix} \right),$$

$$t \in GL_1, \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in GL_2, \; g \in GL_3. \tag{50}$$

Now we can expand φ along the abelian unipotent subgroup which consists of elements of the form $u_1(W) := \left(I_6, \begin{pmatrix} I_3 & W \\ & I_3 \end{pmatrix}\right)$, $W \in {}^2\wedge_3$. The constant term is of course zero. The group $C_Q^{w_2}$ acts transitively on the nontrivial characters. As a representative for the open orbit we select the character $\psi_{2,1}(u_1(W)) := \psi(W_{2,1})$. The stabilizer of this representative can be described in terms of the coordinates from (50) as the set of elements of $C_Q^{w_2}$ such that $g \in GL_3$ is of the form $\begin{pmatrix} t_1 & u \\ & g_1 \end{pmatrix}$ with $g_1 \in GL_2$ and $\det g_1 = \lambda$. Denote this group by $C_1^{w_2}$. Now we write the integral as a double integral, with the inner integral being

$$\int\limits_{[{}^2\wedge_3]} \int\limits_{U_Q^{w_2}(\mathbb{A})\backslash U_Q(\mathbb{A})} f_{\chi;s}(w_2u_2u_1(W)g)\theta^{U_Q^{w_2}}(\phi, u_2u_1(W)g)\, du_2\psi_{2,1}^{-1}(W)\, dW. \quad (51)$$

Now,

$$u_Q(\xi, 0, 0)u_1(W) = u_1(W)u_Q(\xi, 0, 0)u_Q(0, \xi W, -\xi W\, {}_t\xi), \qquad (W \in {}^2\wedge_3,\ \xi \in \mathrm{Mat}_{6\times 3}).$$

It follows that (51) is equal to

$$\int\limits_{[{}^2\wedge_3]} \int\limits_{U_Q^{w_2}(\mathbb{A})\backslash U_Q(\mathbb{A})} f_{\chi;s}(w_2u_2g) \sum_{\xi} [\omega_\psi(u_2g).\phi](\xi)\psi_l(\xi W\, {}_t\xi)\psi_{2,1}^{-1}(W)\, du_2\, dW,$$

with ξ summed over 6×3 matrices such that rows 3, 4 and 6 are zero. Clearly the integral on W picks off the terms such that $\psi_l(\xi W\, {}_t\xi)\psi_{2,1}^{-1}(W)$ is trivial. Now, direct calculation shows that

$$\xi = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_4 & \xi_5 & \xi_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \xi_7 & \xi_8 & \xi_9 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} y_1 & y_2 & 0 \\ y_3 & 0 & -y_2 \\ 0 & -y_3 & -y_1 \end{pmatrix}, \implies \psi_l(\xi W\, {}_t\xi)$$
$$= \psi\left(\det \begin{pmatrix} y_3 & -y_1 & y_2 \\ \xi_4 & \xi_5 & \xi_6 \\ \xi_7 & \xi_8 & \xi_9 \end{pmatrix}\right).$$

So, in the coordinates above, the condition for $\psi_l(\xi W\, {}_t\xi)\psi_{2,1}^{-1}(W)$ to be trivial is $\xi_4 = \xi_7 = 0$ and $\det \begin{pmatrix} \xi_5 & \xi_6 \\ \xi_8 & \xi_9 \end{pmatrix} = 1$. Observe that if ξ_1 is also zero, then the function $g \mapsto [\omega_\psi(g).\phi](\xi)$ is invariant on the left by $\left\{ \left(I_6, \begin{pmatrix} u & \\ & {}_tu^{-1} \end{pmatrix}\right) \in C_Q^{w_2} : u = \begin{pmatrix} 1 & x & y \\ & 1 & \\ & & 1 \end{pmatrix} \in GL_3 \right\}$. Thus, the contribution from such ξ is trivial by cuspidality. The group $C_1^{w_2}$ permutes the remaining terms transitively, and the stabilizer of ξ_0 is

$$C_2^{w_2} := \left\{ \left(\begin{pmatrix} t & x_1 & x_2 & x_3 & x_4 & x_5 \\ & a & 0 & 0 & b & * \\ & & a' & b' & 0 & * \\ & & c' & d' & 0 & * \\ & c & & & d & * \\ & & & & & t\lambda \end{pmatrix}, \begin{pmatrix} g & W \\ & {}_t g^{-1}\lambda \end{pmatrix} \right) \in C_1^{w_2} : \right.$$
$$\left. g = \begin{pmatrix} t & x_1 & x_4 \\ & a & b \\ & c & d \end{pmatrix} \right\}.$$

Expanding first on x_1 and x_4, and then on the unipotent radical of the diagonally embedded GL_2, and using Lemma 3.7 two more times gives the result. □

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