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Normal families and fixed-points of meromorphic functions

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Abstract In this paper, we obtain some normality criteria of families of meromorphic functions, which improve and generalize the related results of Gu, Pang-Yang-Zalcman, and Zhang-Pang-Zalcman, respectively. Some examples are given to show the sharpness of our results.

Keywords Meromorphic function · Fixed-point · Normal family

Mathematics Subject Classification 30D45

1 Introduction and main results

Let D be a domain in the complex plane $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined on D. $\mathcal{F}$ is said to be normal on D, in the sense of Montel, if for any sequence $\{f_n\} \subset \mathcal{F}$ there exists a subsequence $\{f_{n_k}\}$, such that $\{f_{n_k}\}$ converges spherically locally uniformly on D, to a meromorphic function or ∞ (see [3],[8],[13]).

The following well-known normality criterion was conjectured by Hayman[3], and proved by Gu [2].

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Theorem A *Let k be a positive integer. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. If for each $f \in \mathcal{F}$, $f \neq 0$ and $f^{(k)} \neq 1$, then $\mathcal{F}$ is normal in D.*

This result has undergone various extensions and improvements. In [5] (cf. [6], [11]), Pang-Yang-Zalcman obtained.

Theorem B *Let k be a positive integer. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, all of whose zeros have multiplicity at least $k+2$ and whose poles are multiple. Let $h(z)(\not\equiv 0)$ be a holomorphic functions on D. If for each $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$, then $\mathcal{F}$ is normal in D.*

When $k = 1$, an example [19, Example 1] (cf. [6]) shows that the condition on the multiplicity of zeros of functions in $\mathcal{F}$ cannot be weakened. Zhang-Pang-Zalcman[14] proved that when $k \geq 2$ the multiplicity of zeros of functions in $\mathcal{F}$ can be reduced from $k+2$ to $k+1$ in Theorem B.

Theorem E *Let $k \geq 2$ be a positive integer. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, all of whose zeros have multiplicity at least $k+1$ and whose poles are multiple. Let $h(z)(\not\equiv 0)$ be a holomorphic functions on D. If for each $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$, then $\mathcal{F}$ is normal in D.*

Also in [14], they indicated that one cannot further reduce the assumption on the multiplicity of the zeros from $k+1$ to k, by considering the following example.

Example 1 (see [14]) Let $\Delta = \{z : |z| < 1\}$, $h(z) = z$, and let

$$\mathcal{F} = \left\{ f_n(z) = nz^k \right\}.$$

Clearly, all zeros of f_n are of multiplicity k, and $f_n^{(k)}(z) = nk! \neq z$ on Δ. However, $\mathcal{F}$ fails to be equicontinuous at 0, and then $\mathcal{F}$ is not normal in Δ.

In this paper, we consider the case $h(z) = z$, then $f^{(k)}(z) \neq h(z)$ means that $f^{(k)}$ has no fixed-points. We reduce the multiplicity of zeros of functions in $\mathcal{F}$ to k, but restricting the values $f^{(k)}$ can take at the zeros of f, as follows.

Theorem 1 *Let $k \geq 4$ be a positive integer, $A > 1$ be a constant. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. If, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least k and satisfies the following conditions:*

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z|$.
(b) $f^{(k)}(z) \neq z$.
(c) *All poles of f are multiple.*

Then $\mathcal{F}$ is normal in D.

For the case $k = 2$ or 3, the multiplicity of poles of $f \in \mathcal{F}$ need be at least three.

Theorem 2 *Let $k = 2$ or 3, $A > 1$ be a constant. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. If, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least k and satisfies the following conditions:*

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z|$.
(b) $f^{(k)}(z) \neq z$.
(c) *All poles of* f *have multiplicity at least* 3.

Then $\mathcal{F}$ *is normal in* D.

Example 1 shows that condition (a) in Theorems 1 and 2 cannot be removed. For the case $k = 1$, the above theorems are no longer true even if the multiplicities of poles of $f \in \mathcal{F}$ are large enough, as is shown by the next example.

Example 2 Let j be a positive integer, $\Delta = \{z : |z| < 1\}$, and let

$$\mathcal{F} = \left\{ f_n(z) = \frac{z^{j+2} - 1/n^{j+2}}{2z^j} \right\}.$$

Clearly,

$$f_n'(z) = z + \frac{j}{2n^{j+2}z^{j+1}} \neq z.$$

For each n, f_n has one pole $z = 0$ with multiplicity j, and $j + 2$ simple zeros $z_m = \frac{1}{n}e^{i\frac{2m\pi}{j+2}}$ $(m = 0, 1, \ldots, j + 1)$ in Δ. We have

$$f_n'(z_m) = z_m + \frac{j}{2n^{j+2}z_m^{j+1}} = \frac{j+2}{2n}e^{i\frac{2m\pi}{j+2}},$$

and then

$$|f_n'(z_m)| \leq \frac{j+2}{2}|z_m|,$$

that is, $f_n(z) = 0 \Rightarrow |f_n'(z)| \leq \frac{j+2}{2}|z|$. But, since $f_n(1/n) = 0$ and $f_n(0) = \infty$, $\mathcal{F}$ fails to be equicontinuous at $z = 0$, and then $\mathcal{F}$ is not normal in Δ.

The following example shows that condition (c) in Theorem 2 is necessary, and the number 3 is best possible.

Example 3 Let $\Delta = \{z : |z| < 1\}$, and let

$$\mathcal{F} = \left\{ f_n(z) = \frac{(z - 1/n)^3(z + 1/n)^3}{24z^2} \right\}.$$

Clearly,

$$f_n^{(3)}(z) = z + \frac{1}{n^6z^5} \neq z.$$

For each n, f_n has two zeros $z_1 = 1/n$ and $z_2 = -1/n$ of multiplicity 3. We have

$$f_n^{(3)}(\frac{1}{n}) = \frac{2}{n}, \quad f_n^{(3)}(-\frac{1}{n}) = -\frac{2}{n},$$

and $|f_n^{(3)}(z_i)| \le 2|z_i| (i = 1, 2)$, then $f_n(z) = 0 \Rightarrow |f_n^{(3)}(z)| \le 2|z|$. However $\mathcal{F}$ is not normal at 0 since $f_n(1/n) = 0$ and $f_n(0) = \infty$.

The next example shows that condition (c) cannot be omitted in Theorem 1.

Example 4 Let k be a positive integer, $\Delta = \{z : |z| < 1\}$ and

$$\mathcal{F} = \left\{ f_n(z) = \frac{1}{(k+1)!} \frac{(z - 1/n)^{k+2}}{z - (k+2)/n} \right\}.$$

Clearly, the zero of f_n is of multiplicity $k + 2$, so that $f(z) = 0 \Rightarrow |f^{(k)}(z)| \le |z|$; the pole of f_n is simple. On the other hand, since

$$f_n(z) = \frac{1}{(k+1)!} \left(z^{k+1} + P_{k-1}(z) + \frac{a}{z - (k+2)/n} \right),$$

where $P_{k-1}(z)$ is a polynomial of degree $k - 1$ and a is a nonzero constant, we have $f_n^{(k)}(z) \ne z$. But $\mathcal{F}$ is not normal at 0 since $f_n(1/n) = 0$ and $f_n((k+2)/n) = \infty$.

In this paper, we write $\Delta = \{z : |z| < 1\}$ and $\Delta' = \{z : 0 < |z| < 1\}$. For $z_0 \in \mathbb{C}$ and $r > 0$, we write $\Delta(z_0, r) = \{z : |z - z_0| < r\}$, and $\Delta'(z_0, r) = \{z : 0 < |z - z_0| < r\}$.

2 Preliminary results

To prove our results, we need the following lemmas.

Lemma 1 [4, Lemma 2] *Let k be a positive integer and let $\mathcal{F}$ be a family of meromorphic functions in a domain D, all of whose zeros have multiplicity at least k, and suppose that there exists $A \ge 1$ such that $|f^{(k)}(z)| \le A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each α, $0 \le \alpha \le k$, there exist a sequence of complex numbers $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^{\alpha}} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \le g^{\#}(0) = kA + 1$. Moreover, $g(\zeta)$ has order at most 2.

Here, as usual, $g^{\#}(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

Lemma 2 [11, Lemma 5] *Let f be a transcendental meromorphic function, $k(\ge 2)$, ℓ be positive integers. If f has only zeros of order at least 3, then $f^{(k)} - z^{\ell}$ has infinitely many zeros.*

The next is a generalization of Hayman inequality, which is due to Yang [12].

Lemma 3 *Let f be a transcendental meromorphic function, φ be a small meromorphic function of f, and $k \in \mathbb{N}$. Then*

$$T(r, f) \leq 3N\left(r, \frac{1}{f}\right) + 4N\left(r, \frac{1}{f^{(k)} - \varphi}\right) + S(r, f).$$

Lemma 4 [1, Corollary 2] *Let f be meromorphic in $\mathbb{C}$ and of finite order ρ and E be the set of its critical values. If f has at most $2\rho + card E'$ asymptotic values, where E' is the derived set of E.*

Lemma 5 [7, Lemma 2.2] *Let f be meromorphic in $\mathbb{C}$ and suppose that the set of all finite critical and asymptotic values of f is bounded. Then there exists $R > 0$ such that if $|z| > R$ and $|f(z)| > R$, then*

$$|f'(z)| \geq \frac{|f(z)| \log |f(z)|}{16\pi |z|}.$$

Lemma 6 *Let f be a transcendental meromorphic function of finite order ρ, and let $k(\geq 2)$ be a positive integer. If f has only zeros of multiplicity at least k, and there exists $A > 1$ such that $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z|$, then $f^{(k)}$ has infinitely many fix-points.*

Proof Suppose that $f^{(k)}$ has finitely many fix-points. Lemma 3 implies that f has infinitely many zeros, say $z_n (n = 1, 2, \ldots)$. Clearly, $z_n \to \infty$. Now set

$$g(z) = \frac{z^2}{2} - f^{(k-1)}(z).$$

Then g is also of finite order ρ, and $g'(z) = z - f^{(k)}(z)$ has only finitely many zeros. By Lemma 4 or Denjoy-Carleman-Ahlfors' theorem, g has at most 2ρ asymptotic values, and then satisfies the hypotheses of Lemma 5 for some $R > 0$. It follows that

$$\frac{|z_n g'(z_n)|}{|g(z_n)|} \geq \frac{\log |g(z_n)|}{16\pi}$$

for large n. Since $g(z_n) = z_n^2/2$ and $|g'(z_n)| = |z_n - f^{(k)}(z_n)| \leq (A + 1)|z_n|$, we have

$$2(A + 1) \geq \frac{1}{16\pi}[2 \log |z_n| - \log 2] \to \infty$$

as $n \to \infty$, a contradiction. Lemma 6 is proved. □

Lemma 7 [10, Lemma 5] *Let f be meromorphic in $\mathbb{C}$ and of finite order, and let $k \geq 2$ be a positive integer and K be a positive number. Suppose that f has only zeros of multiplicity at least k, $|f^{(k)}(z)| < K$ whenever $f(z) = 0$, and $f^{(k)}(z) \neq 1$. Then one of the following two cases must occur:*

(1)

$$f(z) = \alpha(z-\beta)^k, \tag{1}$$

where $\alpha, \beta \in \mathbb{C}$, *and* $\alpha \cdot k! \neq 1$.

(2) *If* $k = 2$, *then*

$$f(z) = \frac{(z-c_1)^2(z-c_2)^2}{2(z-c)^2}, \tag{2}$$

or

$$f(z) = \frac{(z-c_1)^3}{2(z-c)}. \tag{3}$$

If $k \geq 3$, *then*

$$f(z) = \frac{1}{k!}\frac{(z-c_1)^{k+1}}{(z-c)}. \tag{4}$$

Here c_1, c_2, c *are distinct complex numbers.*

Lemma 8 [9, Lemma 8] *Let* f *be a non-polynomial rational function and* k *be a positive integer. If* $f^{(k)}(z) \neq 1$, *then*

$$f(z) = \frac{1}{k!}z^k + a_{k-1}z^{k-1} + \cdots + a_0 + \frac{a}{(z-b)^m},$$

where $a_{k-1}, \ldots, a_0, a(\neq 0), b$ *are constants and* m *is a positive integer.*

Lemma 9 *Let* $k(\geq 2)$ *be a positive integer, and* f *be a rational function, all of whose zeros are of multiplicity at least* k. *If* $f^{(k)}(z) \neq z$, *then one of the following three cases must occur:*

(1)

$$f(z) = \frac{(z+c)^{k+1}}{(k+1)!}; \tag{5}$$

(2)

$$f(z) = \frac{(z-c_1)^{k+2}}{(k+1)!(z-b)}; \tag{6}$$

(3)

$$f(z) = \frac{(z-c_1)^2(z-c_2)^3}{6(z-b)^2} \ (for\ k=2), \tag{7}$$

$$f(z) = \frac{(z-c_1)^3(z-c_2)^3}{24(z-b)^2} \ (for\ k=3), \tag{8}$$

where c is nonzero constant, and c_1, c_2 and b are distinct constants.

Proof Suppose first that f is a polynomial. Then $f^{(k)}(z) = z + c$, where $c(\neq 0)$ is a constant, so that

$$f^{(k-1)}(z) = \frac{z^2}{2} + cz + d$$

where d is a constant. If f vanishes at z_0, then $f^{(k-1)}(z_0) = z_0^2/2 + cz_0 + d = 0$ since f has only zeros of multiplicity at least k. It follows that f has at most two zeros. So f has either only one zero of multiplicity $k+1$ or two distinct zeros of multiplicity exactly k. If f has two distinct zeros of multiplicity exactly k, then $\deg f = 2k$ and $\deg f^{(k)} = k$, which contradicts the fact that $f^{(k)}(z) = z + c$ and $k \geq 2$. Thus, f has only one zero of multiplicity $k+1$, and hence f has the form (5).

Suppose then that f is a nonpolynomial rational function. Set

$$g(z) = f(z) - \frac{1}{(k+1)!}z^{k+1} + \frac{1}{k!}z^k.$$

Then $g^{(k)}(z) \neq 1$, so by Lemma 8

$$g(z) = \frac{1}{k!}z^k + a_{k-1}z^{k-1} + \cdots + a_0 + \frac{a}{(z-b)^m},$$

where $a_{k-1}, \ldots, a_0, a(\neq 0), b$ are constants and m is a positive integer. Thus

$$f(z) = p(z) + \frac{a}{(z-b)^m} = \frac{p(z)(z-b)^m + a}{(z-b)^m}, \tag{9}$$

where

$$p(z) = \frac{1}{(k+1)!}z^{k+1} + a_{k-1}z^{k-1} + \cdots + a_0.$$

Let $c_1, c_2, \cdots, c_q$ be q distinct zeros of $p(z)(z-b)^m + a$, with multiplicity $n_1, n_2, \cdots, n_q$. Clearly, $n_i \geq k$, $c_i \neq b$, and c_i is a zero of $(p(z)(z-b)^m + a)'$ with multiplicity $n_i - 1 \geq k - 1 (1 \leq i \leq q)$. Since

$$\left(p(z)(z-b)^m + a\right)' = (z-b)^{m-1}\left(p'(z)(z-b) + mp(z)\right), \tag{10}$$

then c_i must be a zero of $p'(z)(z-b) + mp(z)$ with multiplicity $n_i - 1 (\geq k-1)$. Note that $\deg[p'(z)(z-b) + mp(z)] = k + 1$. Now we divide into three cases.

Case 1. $k = 2$.

Then $\deg[p'(z)(z-b) + mp(z)] = 3$, and hence

(a) $p'(z)(z-b)+mp(z)$ has three simple zeros c_1, c_2, and c_3; or
(b) $p'(z)(z-b)+mp(z)$ has one simple zero c_1 and one zero c_2 with multiplicity 2; or
(c) $p'(z)(z-b)+mp(z)$ has only one zero c_1 with multiplicity 3.

For case (a), we deduce that $m=3$, and

$$p'(z)(z-b)+3p(z)=(z-c_1)(z-c_2)(z-c_3),$$

$$p(z)(z-b)^3+a=\frac{1}{6}(z-c_1)^2(z-c_2)^2(z-c_3)^2.$$

These, together with (10) give

$$(z-b)^2=\frac{1}{3}[(z-c_1)(z-c_2)+(z-c_1)(z-c_3)+(z-c_2)(z-c_3)].$$

Equating coefficients, we have $b=(c_1+c_2+c_3)/3$ and $b^2=(c_1c_2+c_1c_3+c_2c_3)/3$, so that

$$c_1^2+c_2^2+c_2^2=c_1c_2+c_1c_3+c_2c_3,$$

that is,

$$(c_1-c_2)^2+(c_1-c_3)^2+(c_2-c_3)^2=0,$$

and hence $c_1=c_2=c_3$, a contradiction. Thus case (1) is ruled out.

For case (b), we deduce that $m=2$ and

$$p(z)(z-b)^2+a=\frac{1}{6}(z-c_1)^2(z-c_2)^3.$$

Then, by (9), f has the form (7).

For case (c), we can deduce that $m=1$ and

$$p(z)(z-b)+a=\frac{1}{6}(z-c_1)^4,$$

This, together with (9), gives that f has the form (6).

Case 2. $k=3$.

Since $\deg[p'(z)(z-b)+mp(z)]=4$, $p'(z)(z-b)+mp(z)$ has two zeros c_1, c_2 with multiplicity 2 *or* one zero c_1 with multiplicity 4. It follows that $m=2$ and

$$p(z)(z-b)^2+a=\frac{1}{24}(z-c_1)^3(z-c_2)^3$$

or $m = 1$ and

$$p(z)(z-b)+a=\frac{1}{24}(z-c_1)^5.$$

Then, by (9), f has the form (6) or (8).

Case 3. $k \geq 4$.

Noting that $\deg[p'(z)(z-b)+mp(z)] = k+1$, we conclude that $p'(z)(z-b)+mp(z)$ has only one zero c_1 with multiplicity $k+1$. In fact, if $p'(z)(z-b)+mp(z)$ has at least two zeros c_1, c_2 with multiplicity $n_1, n_2, \geq k-1$, then $2(k-1) \leq k+1$, and thus $k \leq 3$, a contradiction. Thus $m = 1$ and $p(z)(z-c)+b = \frac{1}{k!}(z-c_1)^{k+2}$, and hence f has the form (6). This completes the proof of Lemma 9. □

Lemma 10 *Let $k \geq 3$ be a positive integer, $A > 1$ be a constant. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. Suppose that, for every $f \in \mathcal{F}$, f has only zeros of multiplicity at least k, and satisfies the following conditions:*

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z|$.
(b) $f^{(k)}(z) \neq z$.
(c) *all poles of f are multiple.*

Then $\mathcal{F}$ is normal in $D\backslash\{0\}$.

Proof Suppose that $\mathcal{F}$ is not normal at a point $z_0 \in D\backslash\{0\}$. Giving a small $r > 0$ such that $\Delta(z_0, r) \subset D\backslash\{0\}$ and $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z_0| + 1$ for $f \in \mathcal{F}$ and $z \in \Delta(z_0, r)$. Then by Lemma 1, for $\alpha = k$, there exist a sequence of functions $f_n \in \mathcal{F}$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$g_n(\zeta) = \rho_n^{-k} f_n(z_n + \rho_n\zeta) \to g(\zeta)$$

converges sphericaly uniformly on compact subsets of $\mathbb{C}$, where g is a non-constant meromorphic functionon on $\mathbb{C}$, all zeros of g have multiplicity at least k, and

$$g^{\#}(\zeta) \leq g^{\#}(0) = k(A|z_0| + 1) + 1$$

for all $\zeta \in \mathbb{C}$. Moreover, , g is of finite order. By Hurwitz's theorem, all poles of g are multiple.

We claim: (1) $g = 0 \Rightarrow |g^{(k)}| \leq A|z_0|$; (2) $g^{(k)}(\zeta) \neq z_0$.

Let ζ_0 be a zero of $g(\zeta)$. Then there exist $\zeta_n, \zeta_n \to \zeta_0$, such that $g_n(\zeta_n) = \rho_n^{-k} f_n(z_n + \rho_n\zeta_n) = 0$ for n sufficiently large. Thus $f_n(z_n + \rho_n\zeta_n) = 0$, so that $|f_n^{(k)}(z_n + \rho_n\zeta_n)| \leq A|z_n + \rho_n\zeta_n|$ for sufficiently large n. Since

$$g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n\zeta_n) \to g^{(k)}(\zeta_0),$$

we have $|g^{(k)}(\zeta_0)| \leq A|z_0|$. We have proved (i).

Suppose that there exists ζ_0 such that $g^{(k)}(\zeta_0) = z_0$. Since

$$0 \neq f_n^{(k)}(z_n + \rho_n\zeta) - (z_n + \rho_n\zeta) = g_n^{(k)}(\zeta) - (z_n + \rho_n\zeta) \to g^{(k)}(\zeta) - z_0,$$

Hurwitz's theorem implies that $g^{(k)}(\zeta) \equiv z_0$. Note that g has only zeros of multiplicity at least k, we have

$$g(\zeta) = \frac{z_0}{k!}(z-\alpha)^k, \quad \alpha \in \mathbb{C}.$$

A simple calculation shows that

$$g^{\#}(0) \le \begin{cases} k/2 & \text{if } |\alpha| \ge 1; \\ |z_0| & \text{if } |\alpha| < 1. \end{cases}$$

But this contradicts $g^{\#}(0) = k(A|z_0| + 1) + 1$, and thus (2) is proved.

By Lemma 7, g has the form (1) or (4) in Lemma 7. Similarly as above, we exclude the case that g has the form (1), so that g has the form (4). But g has only multiple poles, a contradiction. This completes the proof of Lemma 10. □

Lemma 11 *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, $A > 1$ be a constant. Suppose that, for every $f \in \mathcal{F}$, f has only zeros of multiplicity at least k, and satisfies the following conditions:*

(a) $f(z) = 0 \Rightarrow |f''(z)| \le A|z|$.
(b) $f''(z) \ne z$.
(c) *all poles of f are of multiplicity at least 3.*

Then $\mathcal{F}$ is normal in $D \backslash \{0\}$.

This lemma can be proved almost the same as Lemma 10. We omit the details here.

3 Proof of Theorems 1 and 2

Proof of Theorem 1 Since normality is a local property, by Lemma 10, we only need to prove that $\mathcal{F}$ is normal at $z = 0$. Without loss of generality, we may assume $D = \Delta$. Suppose, on the contrary, $\mathcal{F}$ is not normal at the origin. Our goal is to obtain a contradiction in the sequel.

Consider the family

$$\mathcal{G} = \left\{ g(z) = \frac{f(z)}{z} : f \in \mathcal{F} \right\}.$$

We claim that $f(0) \ne 0$ for every $f \in \mathcal{F}$. Otherwise, if $f(0) = 0$, by the assumption of Theorem 1, $|f^{(k)}(0)| \le 0$, and then $f^{(k)}(0) = 0$. But $f^{(k)}(z) \ne z$, a contradiction. Thus, for each $g \in \mathcal{G}$, $g(0) = \infty$. Furthermore, all zeros of $g(z)$ have multiplicity at least k. On the other hand, by simple calculation, we have

$$g^{(k)}(z) = \frac{f^{(k)}(z)}{z} - \frac{kg^{(k-1)}(z)}{z}. \tag{11}$$

Since $f(z) = 0 \Rightarrow |f^{(k)}(z)| \le A|z|$, we deduce that $g(z) = 0 \Rightarrow |g^{(k)}(z)| \le A$.

We first prove that $\mathcal{G}$ is normal at 0. Suppose not; by Lemma 1, there exist functions $g_n \in \mathcal{G}$, points $z_n \to 0$ and positive numbers $\rho_n \to 0$ such that

$$G_n(\zeta) = \frac{g_n(z_n + \rho_n\zeta)}{\rho_n^k} \to G(\zeta), \tag{12}$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, where G is a non-constant meromorphic functionon on $\mathbb{C}$ and of finite order, all zeros of G have multiplicity at least k, and $G^{\#}(\zeta) \le G^{\#}(0) = kA + 1$ for all $\zeta \in \mathbb{C}$.

We distinguish two cases.

Case 1. $z_n/\rho_n \to \infty$. Since $G_n(-z_n/\rho_n) = g_n(0)/\rho_n^k$, the pole of G_n corresponding to that of g_n at 0 drifts to infity. Then, by Hurwitz's theorem, G has only mutiple poles. By (11) and (12), we have

$$\begin{aligned} G_n^{(k)}(\zeta) &= g_n^{(k)}(z_n + \rho_n\zeta) \\ &= \frac{f_n^{(k)}(z_n + \rho_n\zeta)}{z_n + \rho_n\zeta} - k\frac{g_n^{(k-1)}(z_n + \rho_n\zeta)}{\rho_n}\frac{\rho_n}{z_n + \rho_n\zeta}. \end{aligned}$$

Noting that

$$\frac{\rho_n}{z_n + \rho_n\zeta} \to 0$$

uniformly on compact subsets of $\mathbb{C}$, and $g_n^{(k-1)}(z_n + \rho_n\zeta)/\rho_n$ is locally bounded on $\mathbb{C}\backslash G^{-1}(\infty)$ since $g_n(z_n + \rho_n\zeta)/\rho_n^k \to G(\zeta)$. Thus

$$\frac{f_n^{(k)}(z_n + \rho_n\zeta)}{z_n + \rho_n\zeta} \to G^{(k)}(\zeta), \tag{13}$$

uniformly on compact subsets of $\mathbb{C}\backslash G^{-1}(\infty)$.

Claim: (I) $G(\zeta) = 0 \Rightarrow |G^{(k)}(\zeta)| \le A$; (II) $G^{(k)}(\zeta) \ne 1$.

Indeed, if $G(\zeta_0) = 0$, Hurwitz's theorem and (12) imply that there exist $\zeta_n, \zeta_n \to \zeta_0$, such that $g_n(z_n + \rho_n\zeta_n) = 0$, and then $f_n(z_n + \rho_n\zeta_n) = 0$ for n sufficiently large. By assumption, $|f_n^{(k)}(z_n + \rho_n\zeta_n)| \le A|z_n + \rho_n\zeta_n|$. It follows from (13) that $|G^{(k)}(\zeta_0)| \le A$. Claim (I) is proved.

Since $f_n^{(k)}(z) \ne z$, Hurwitz's theorem and (13) yield that either $G^{(k)}(\zeta) \ne 1$ or $G^{(k)}(\zeta) \equiv 1$ for any $\zeta \in \mathbb{C}\backslash G^{-1}(\infty)$. Clearly, these also hold for all $\zeta \in \mathbb{C}$. If $G^{(k)}(\zeta) \equiv 1$, noting that all zeros of G have multiplicity at least k, we have $G(\zeta) = (\zeta - \alpha)^k/k!(\alpha \in \mathbb{C})$. As in the proof of Lemma 10,

$$G^{\#}(0) \le \begin{cases} k/2 & \text{if } |\alpha| \ge 1; \\ 1 & \text{if } |\alpha| < 1. \end{cases}$$

which contradicts $G^{\#}(0) = kA + 1$. Then Claim (II) is proved. Then by Lemma 7, G has the form (1) or (4) in Lemma 7. The form (1) can be ruled out similarly as above.

Thus

$$G(\zeta) = \frac{1}{k!}\frac{(\zeta - c_1)^{k+1}}{(\zeta - c)},$$

where c_1, c are distinct complex numbers. But, this contradicts that G has only mutiple poles.

Case 2. $z_n/\rho_n \not\to \infty$. Taking subsequence, we can assume that $z_n/\rho_n \to \alpha$, a finite complex number. Then

$$\frac{g_n(\rho_n\zeta)}{\rho_n^k} = G_n(\zeta - z_n/\rho_n) \overset{\chi}{\to} G(\zeta - \alpha) = \widetilde{G}(\zeta)$$

on $\mathbb{C}$. Clearly, all zeros of $\widetilde{G}$ have multiplicity at least k, and all poles of $\widetilde{G}$ are multiple, except possibly the pole at 0.

Set

$$H_n(\zeta) = \frac{f_n(\rho_n\zeta)}{\rho_n^{k+1}}. \tag{14}$$

Then

$$H_n(\zeta) = \frac{f_n(\rho_n\zeta)}{\rho_n^{k+1}} = \zeta\frac{g_n(\rho_n\zeta)}{\rho_n^k} \to \zeta\widetilde{G}(\zeta) = H(\zeta) \tag{15}$$

spherically uniformly on compact subsets of $\mathbb{C}$, and

$$H_n^{(k)}(\zeta) = \frac{f_n^{(k)}(\rho_n\zeta)}{\rho_n} \to H^{(k)}(\zeta) \tag{16}$$

locally uniformly on $\mathbb{C} \setminus H^{-1}(\infty)$. Obviously, all zeros of H have multiplicity at least k, and all poles of H are multiple. Since $\widetilde{G}(0) = \infty$, $H(0) \neq 0$.

Claim: (III) $H(\zeta) = 0 \Rightarrow |H^{(k)}(\zeta)| \leq A|\zeta|$; (IV) $H^{(k)}(\zeta) \neq \zeta$.

If $H(\zeta_0) = 0$, by Hurwitz's theorem and (15), there exist $\zeta_n \to \zeta_0$ such that $f_n(\rho_n\zeta_n) = 0$ for for n sufficiently large. By the assumption, $|f_n^{(k)}(\rho_n\zeta_n)| \leq A|\rho_n\zeta_n|$. Then, it follows from (16) that $|H^{(k)}(\zeta_0)| \leq A|\zeta_0|$. Claim (III) is proved.

Suppose that there exists ζ_0 such that $H^{(k)}(\zeta_0) = \zeta_0$. By (16),

$$0 \neq \frac{f_n^{(k)}(\rho_n\zeta) - \rho_n\zeta}{\rho_n} = H_n^{(k)}(\zeta) - \zeta \to H^{(k)}(\zeta) - \zeta,$$

uniformly on compact subsets of $\mathbb{C}\setminus H^{-1}(\infty)$. Hurwitz's theorem implies that $H^{(k)}(\zeta) \equiv \zeta$ on $\mathbb{C}\setminus H^{-1}(\infty)$, and then on $\mathbb{C}$. It follows that H is a polynomial of degree $k+1$. Since all zeros of H have multiplicity at least k, and noting that $k \geq 4$, we know that H has a single zero ζ_1 with multiplicity $k+1$, so that $H^{(k)}(\zeta_1) = 0$, and hence $\zeta_1 = 0$ since $H^{(k)}(\zeta) \equiv \zeta$. But $H(0) \neq 0$, we arrive at a contradiction. This proves claim (IV).

Then, by Lemma 6, H must be a rational function, and thus Lemma 9 implies that H has the form (5) or (6) in Lemma 9. The form (6) can be excluded since all poles of H are multiple. Thus we have

$$H(\zeta) = \frac{(\zeta + c)^{k+1}}{(k+1)!} \tag{17}$$

where $c(\neq 0)$ is a constant.

Next we will show that (17) is impossible. Indeed, combining (15) and (17) gives

$$\frac{f_n(\rho_n \zeta)}{\rho_n^{k+1}} \to \frac{(\zeta + c)^{k+1}}{(k+1)!}. \tag{18}$$

Note that all zeros of f_n have multiplicity at least k and $k \geq 4$, there exist points $\zeta_{n,0} \to -c$ such that $z_{n,0} = \rho_n \zeta_{n,0}$ is a zero of f_n with multiplicity $k+1$.

We now consider two subcases.

Case 2.1 There exists $0 < \delta \leq 1$ such that the functions $f_n(z)$ (for large n) are all holomorphic on $\Delta(0, \delta)$.

Since $\{f_n\}$ is normal on $\Delta'(0, \delta)$, but not normal at 0, it follows from the maximum modulus principle that $f_n \to \infty$ locally uniformly on $\Delta'(0, \delta)$.

Suppose that there exists $0 < \sigma < \delta$ such that each f_n has only one zero $z_{n,0}$ in $\Delta(0, \sigma)$. Set

$$K_n(z) = \frac{f_n(z)}{(z - z_{n,0})^{k+1}}. \tag{19}$$

Then $\{K_n\}$ is a sequence of nonvanishing holomorphic functions on $\Delta(0, \sigma)$, and $K_n(z) \to \infty$ locally uniformly on $\Delta'(0, \sigma)$. It follows that $\{1/K_n\}$ is holomorphic on $\Delta(0, \sigma)$, and $1/K_n(z) \to 0$ locally uniformly on $\Delta'(0, \sigma)$, and hence on $\Delta(0, \sigma)$ by the maximum modulus principle. So $K_n(z) \to \infty$ locally uniformly on $\Delta(0, \sigma)$. In particular, $K_n(2z_{n,0}) \to \infty$. But, by (18) and (19),

$$K_n(2z_{n,0}) = \frac{f_n(2z_{n,0})}{z_{n,0}^{k+1}} = \frac{f_n(2\rho_n \zeta_{n,0})}{\rho_n^{k+1} \zeta_{n,0}^{k+1}} \to \frac{1}{(k+1)!},$$

a contradiction.

Hence, taking a subsequence if necessary, for any $0 < \sigma < \delta$, f_n has at least two distinct zeros in $\Delta(0, \sigma)$ for sufficiently large n. We assume that $z_{n,1}$ is a zero of f_n on $\Delta(0, \sigma) \backslash \{z_{n,0}\}$. Clearly, $z_{n,1} \to 0$. Let $\zeta_{n,1} = z_{n,1}/\rho_n$, it follows froms (18) that $\zeta_{n,1} \to \infty$. Hence $z_{n,0}/z_{n,1} = \zeta_{n,0}/\zeta_{n,1} \to 0$. Set

$$L_n(z) = \frac{f_n(z_{n,1} z)}{z_{n,1}^{k+1}}.$$

Then, for sufficiently large n, $\{L_n\}$ is well-defined and holomorphic on each bounded set of $\mathbb{C}$, and all of whose zeros have multiplicity at least k. By the assumption, we have $L_n(z) = 0 \Rightarrow |L_n^{(k)}(z)| \le A|z|$, and $L_n^{(k)}(z) \ne z$. By Lemma 10, $\{L_n\}$ is normal on the punctured complex plane $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. We claim that $\{L_n\}$ is also normal at 0. Otherwise, the maximum modulus principle implies that $L_n \to \infty$ locally uniformly on $\mathbb{C}^*$. But, this is impossible since $L_n(1) = 0$. Hence $\{L_n\}$ is normal on the whole plane $\mathbb{C}$.

Taking a subsequence and renumbering, we assume that

$$L_n(z) \to L(z),$$

and then

$$L_n^{(k)}(z) \to L^{(k)}(z) \tag{20}$$

locally uniformly on $\mathbb{C}$, where L is entire, all zeros of L have multiplicity at least k. Clearly, $L(1) = 0$. On the other hand, $L_n(z_{n,0}/z_{n,1}) = 0$ and $z_{n,0}/z_{n,1} \to 0$, we get that $L(0) = 0$. Since $L_n(z) = 0 \Rightarrow |L_n^{(k)}(z)| \le A|z|$, an argument similar to that in Claim III yields that $L(z) = 0 \Rightarrow |L^{(k)}(z)| \le |z|$. So it follows from $L(0) = 0$ that $L^{(k)}(0) = 0$. Since $L_n^{(k)}(z) \ne z$, Hurwitz's theorem and (20) imply that $L^{(k)}(z) \equiv z$. Note that all zeros of L have multiplicity at least k and $L(0) = 0$, we deduce that $L(z) = z^{k+1}/(k+1)!$. But, this in impossible since $L(1) = 0$.

Case 2.2 By taking a subsequence, if necessary, for any $\delta > 0$, f_n has at least one pole on $\Delta(0, \delta)$ for all n.

Then there exist points $z_{n,\infty} \to 0$ such that $f_n(z_{n,\infty}) = \infty$. We may assume that $z_{n,\infty}$ is the pole of f_n of smallest modulus. Let $\zeta_{n,\infty} = z_{n,\infty}/\rho_n$. It follows from (18) that $\zeta_{n,\infty} \to \infty$, and then $z_{n,0}/z_{n,\infty} = \zeta_{n,0}/\zeta_{n,\infty} \to 0$. Now set

$$M_n(z) = \frac{f_n(z_{n,\infty}z)}{z_{n,\infty}^{k+1}}.$$

Then, for sufficiently large n, $\{M_n\}$ is well-defined for each $z \in \mathbb{C}$, all of whose zeros have multiplicity at least k and whose poles are are multiple. Moreover, $\{M_n\}$ is holomorphic on Δ for sufficiently large n. By the assumption, we have $M_n(z) = 0 \Rightarrow |M_n^{(k)}(z)| \le A|z|$, and $M_n^{(k)}(z) \ne z$. Lemma 10 implies that $\{M_n\}$ is normal on $\mathbb{C}^*$. We claim that $\{M_n\}$ is also normal at 0. Otherwise, $\{M_n\}$ is normal on Δ', but not normal at 0. Since $\{M_n\}$ is holomorphic on Δ, he maximum modulus principle implies that $M_n \to \infty$. But $M_n(z_{n,0}/z_{n,\infty}) = 0$ and $z_{n,0}/z_{n,\infty} \to 0$. This contradiction proves our claim. Hence, $\{M_n\}$ is normal on $\mathbb{C}$.

Then, taking a subsequence and renumbering,

$$M_n(z) \to M(z)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where M is meromorphic, all of whose zeros have multiplicity at least k. Clearly, $M(1) = \infty$. On the other hand,

$M_n(z_{n,0}/z_{n,\infty}) = 0$ and $z_{n,0}/z_{n,\infty} \to 0$, we obtain $M(0) = 0$. Arguing as in Case 2.1 (for $L(z)$), we have $M(z) = z^{k+1}/(k+1)!$. But, $M(1) = \infty$, a contradiction. *Then we have shown that* (17) *is impossible*.

We thus have proved that $\mathcal{G}$ is normal at 0.

We now turn to show that is normal at $z = 0$. Since $\mathcal{G}$ is normal at 0, then the family $\mathcal{G}$ is equicontinuous at 0 with respect to the spherical distance. On the other hand, $g(0) = \infty$ for each $g \in \mathcal{G}$, so there exists $\delta > 0$ such that $|g(z)| \geq 1$ for all $g \in \mathcal{G}$ and each $z \in \Delta(0, \delta)$. It follows that $f(z) \neq 0$ for all $f \in \mathcal{F}$ and $z \in \Delta(0, \delta)$. Since $\mathcal{F}$ is normal on Δ' but not normal at $z = 0$, the family $1/\mathcal{F} = \{1/f : f \in \mathcal{F}\}$ is holomorphic in D_δ and normal on $\Delta'(0, \delta)$, but not normal at $z = 0$. Thus there exists a sequence $\{1/f_n\} \subset 1/\mathcal{F}$ which converges locally uniformly in $\Delta'(0, \delta)$, but not on $\Delta(0, \delta)$. The maximum modulus principle implies that $1/f_n \to \infty$ in $\Delta'(0, \delta)$. Thus $f_n \to 0$ converges locally uniformly in $\Delta'(0, \delta)$, and hence so does $\{g_n\} \subset \mathcal{G}$, where $g_n(z) = f_n(z)/z$. But $|g_n(z)| \geq 1$ for $z \in \Delta(0, \delta)$, a contradiction. This completes the proof of Theorem 1. □

Proof of Theorem 2 Using the same argument as in the proof of Theorem 1, we can prove Theorem 2. We here omit the details. □

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