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Continued fractions for some transcendental numbers

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Abstract We consider series of the form

$$\frac{p}{q}+\sum_{j=2}^{\infty}\frac{1}{x_j},$$

where $x_1 = q$ and the integer sequence (x_n) satisfies a certain non-autonomous recurrence of second order, which entails that $x_n | x_{n+1}$ for $n \geq 1$. It is shown that the terms of the sequence, and multiples of the ratios of successive terms, appear interlaced in the continued fraction expansion of the sum of the series, which is a transcendental number.

Keywords Continued fraction · Non-autonomous recurrence · Transcendental number

Mathematics Subject Classification Primary 11J70; Secondary 11B37

1 Introduction

In recent work [5], we considered the integer sequence

$$1, 1, 2, 12, 936, 68408496, 342022190843338960032, \ldots \tag{1.1}$$

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(sequence A112373 in Sloane's Online Encyclopedia of Integer Sequences), which is generated from the initial values $x_0 = x_1 = 1$ by the nonlinear recurrence relation

$$x_{n+2}\, x_n = x_{n+1}^2 (x_{n+1} + 1), \tag{1.2}$$

and proved some observations of Hanna, namely that the sum

$$\sum_{j=1}^{\infty} \frac{1}{x_j} \tag{1.3}$$

has the continued fraction expansion

$$[x_0; y_0, x_1, y_1, x_2, \ldots, y_{j-1}, x_j, \ldots], \tag{1.4}$$

where $y_j = x_{j+1}/x_j \in \mathbb{N}$ and we use the notation

$$[a_0; a_1, a_2, a_3, \ldots, a_n, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots \cfrac{1}{a_n + \ldots}}}}$$

for continued fractions. Furthermore, we generalized this result by obtaining the explicit continued fraction expansion for the sum of reciprocals (1.3) in the case of a sequence (x_n) generated by a nonlinear recurrence of the form

$$x_{n+1}\, x_{n-1} = x_n^2\, F(x_n), \tag{1.5}$$

with $F(x) \in \mathbb{Z}_{\geq 0}[x]$ and $F(0) = 1$; so (1.2) corresponds to the particular case $F(x) = x + 1$.

All of the recurrences (1.5) exhibit the Laurent phenomenon [4], and starting from $x_0 = x_1 = 1$ they generate a sequence of positive integers satisfying $x_n | x_{n+1}$. The latter fact means that the sum (1.3) is an Engel series (see Theorem 2.3 in Duverney's book [3], for instance).

The purpose of this note is to present a further generalization of the results in [5], by considering a sum

$$S = \frac{p}{q} + \sum_{j=2}^{\infty} \frac{1}{x_j}, \tag{1.6}$$

with the terms x_n satisfying the recurrence

$$x_{n+1}\, x_{n-1} = x_n^2\, (z_n x_n + 1), \tag{1.7}$$

for $n \geq 2$, where (z_n) is a sequence of positive integers, $x_1 = q$, and x_2 is specified suitably. Observe that, in contrast to (1.5), the recurrence (1.7) can be viewed as a non-autonomous dynamical system for x_n, because the coefficient z_n can vary independently (unless it is taken to be $G(x_n)$, for some function G). The same argument as used in [5], based on Roth's theorem, shows the transcendence of any number S defined by a sum of the form (1.6) with such a sequence (x_n).

2 The main result

We start with a rational number written in lowest terms as p/q, and suppose that the continued fraction of this number is given as

$$\frac{p}{q} = [a_0; a_1, a_2, a_3, \dots, a_{2k}] \tag{2.1}$$

for some $k \geq 0$. Note that, in accordance with a comment on p. 230 of [7], there is no loss of generality in assuming that the index of the final coefficient is even. For the convergents we denote numerators and denominators by p_n and q_n, respectively, and use the correspondence between matrix products and continued fractions, which says that

$$\mathbf{M}_n := \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

yielding the determinantal identity

$$\det \mathbf{M}_n = p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}. \tag{2.3}$$

Now for a given sequence (z_n) of positive integers, we define a new sequence (x_n) by

$$x_1 = q, \qquad x_{n+1} = x_n y_{n-1}(x_n z_n + 1) \quad \text{for} \quad n \geq 1, \tag{2.4}$$

where

$$y_0 = q_{2k-1} + 1, \qquad y_n = \frac{x_{n+1}}{x_n} \quad \text{for} \quad n \geq 1. \tag{2.5}$$

It is clear from (2.4) and (2.5) that (x_n) is an increasing sequence of positive integers such that $x_n | x_{n+1}$ for all $n \geq 1$; (y_n) also consists of positive integers, and is an increasing sequence as well. The recurrence (1.7) for $n \geq 2$ follows immediately from (2.4) and (2.5).

Theorem 2.1 *The partial sums of (1.6) are given by*

$$S_n := \frac{p}{q} + \sum_{j=2}^{n} \frac{1}{x_j} = [a_0; a_1, \dots, a_{2(k+n-1)}]$$

for all $n \geq 1$, where the coefficients appearing after a_{2k} are

$$a_{2k+2j-1} = y_{j-1} z_j, \qquad a_{2k+2j} = x_j \quad for \quad j \geq 1.$$

Proof For $n = 1$, S_1 is just (2.1), and we note that $q_{2k-1} = y_0 - 1$ and $q_{2k} = q = x_1$. Proceeding by induction, we suppose that $q_{2k+2n-3} = y_{n-1} - 1$ and $q_{2k+2n-2} = x_n$, and calculate the product

$$\begin{aligned}\mathbf{M}_{2k+2n} &= \mathbf{M}_{2k+2n-2}\begin{pmatrix} a_{2k+2n-1} & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} a_{2k+2n} & 1 \\ 1 & 0\end{pmatrix} \\ &= \mathbf{M}_{2k+2n-2}\begin{pmatrix} y_{n-1}z_n & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} x_n & 1 \\ 1 & 0\end{pmatrix} \\ &= \begin{pmatrix} p_{2k+2n-2} & p_{2k+2n-3} \\ q_{2k+2n-2} & q_{2k+2n-3}\end{pmatrix}\begin{pmatrix} x_n y_{n-1}z_n & y_{n-1}z_n \\ x_n & 1\end{pmatrix}.\end{aligned}$$

By making use of (2.4) and (2.5), this gives $p_{2k+2n} = (x_n y_{n-1} z_n + 1) p_{2k+2n-2} + x_n p_{2k+2n-3}$,

$$\begin{aligned} q_{2k+2n-1} &= y_{n-1}z_n\, q_{2k+2n-2} + q_{2k+2n-3} = x_n y_{n-1} z_n + y_{n-1} - 1 \\ &= \frac{x_{n+1}}{x_n} - 1 = y_n - 1,\end{aligned}$$

and

$$\begin{aligned} q_{2k+2n} &= (x_n y_{n-1} z_n + 1) q_{2k+2n-2} + x_n q_{2k+2n-3} \\ &= (x_n y_{n-1} z_n + 1) x_n + x_n (y_{n-1} - 1) = x_{n+1},\end{aligned}$$

which are the required denominators for the $(2k + 2n - 1)$th and $(2k + 2n)$th convergents. Thus we have

$$S_{n+1} = S_n + \frac{1}{x_{n+1}} = \frac{p_{2k+2n-2}}{q_{2k+2n-2}} + \frac{1}{q_{2k+2n}} = \frac{1}{q_{2k+2n}}\left(\frac{x_{n+1}}{x_n} p_{2k+2n-2} + 1\right).$$

From (2.3) and (2.4), the bracketed expression above can be rewritten as

$$\begin{aligned} &\Big(y_{n-1}(x_n z_n + 1) - q_{2n+2k-3}\Big) p_{2k+2n-2} + q_{2n+2k-2}\, p_{2k+2n-3} \\ &\quad = \Big(y_{n-1}(x_n z_n + 1) - y_{n-1} + 1\Big) p_{2k+2n-2} + x_n p_{2k+2n-3},\end{aligned}$$

giving

$$S_{n+1} = \frac{1}{q_{2k+2n}}\Big((x_n y_{n-1} z_n + 1) p_{2k+2n-2} + x_n p_{2k+2n-3}\Big) = \frac{p_{2k+2n}}{q_{2k+2n}},$$

which is the required result. □

Upon taking the limit $n \to \infty$ we obtain the infinite continued fraction expansion for the sum S, which is clearly irrational. To show that S is transcendental, we need the following growth estimate for x_n:

Lemma 2.2 *The terms of a sequence defined by (2.4) satisfy*

$$x_{n+1} > x_n^{5/2}$$

for all $n \geq 3$.

Proof Since (x_n) is an increasing sequence, the recurrence relation (1.7) gives

$$x_{n+1} > \frac{x_n^3}{x_{n-1}} > x_n^2$$

for $n \geq 2$. Hence $x_{n-1} < x_n^{1/2}$ for $n \geq 3$, and putting this back into the first inequality above yields $x_{n+1} > x_n^3/x_n^{1/2} = x_n^{5/2}$, as required. □

The preceding growth estimate for x_n means that S can be well approximated by rational numbers.

Theorem 2.3 *The sum*

$$S = \frac{p}{q} + \sum_{j=2}^{\infty} \frac{1}{x_j} = [a_0; a_1, \ldots, a_{2k}, y_0 z_1, x_1, y_1 z_2, \ldots, y_{j-1} z_j, x_j, \ldots]$$

is a transcendental number.

Proof This is the same as the proof of Theorem 4 in [5], which we briefly outline here. Let $P_n = p_{2k+2n-2}$ and $Q_n = q_{2k+2n-2}$. Approximating the irrational number S by the partial sum $S_n = P_n/Q_n$, then using Lemma 2.2 and a comparison with a geometric sum, gives the upper bound

$$\left| S - \frac{P_n}{Q_n} \right| = \sum_{j=n+1}^{\infty} \frac{1}{x_j} < \frac{1}{x_n^{5/2-\epsilon}} = \frac{1}{Q_n^{5/2-\epsilon}}$$

for any $\epsilon > 0$, whenever n is sufficiently large. Roth's theorem [6] (see also chapter VI in [1]) says that, for an arbitrary fixed $\kappa > 2$, an irrational algebraic number α has only finitely many rational approximations P/Q for which $\left| \alpha - \frac{P}{Q} \right| < \frac{1}{Q^\kappa}$; so S is transcendental. □

For other examples of transcendental numbers whose continued fraction expansion is explicitly known, see [2] and references therein.

3 Examples

The autonomous recurrences (1.5) considered in [5], where the polynomial F has positive integer coefficients and $F(0) = 1$, give an infinite family of examples. In that case, one has $p = 1$ and $x_1 = q = 1$, so that $k = 0$, $y_0 = 1$ and $z_n =$

$(F(x_n) - 1)/x_n$. More generally, one could take $z_n = G(x_n)$ for any non-vanishing arithmetical function G.

In general, it is sufficient to take the initial term in (1.6) lying in the range $0 < p/q \leq 1$, since going outside this range only alters the value of a_0. As a particular example, we take

$$\frac{p}{q} = \frac{2}{7} = [0; 3, 2], \qquad z_n = n \quad \text{for} \quad n \geq 1,$$

so that $k = 1$, and $q_1 = 3$ which gives $y_0 = 2$. Hence $x_1 = 7$, $x_2 = 112$, and the sequence (x_n) continues with

$$403200, 1755760043520000, 53695136666462381094317154204367872000000, \ldots.$$

The sum S is the transcendental number

$$\frac{2}{7} + \frac{1}{112} + \frac{1}{403200} + \frac{1}{1755760043520000} + \cdots \approx 0.2946453373015879,$$

with continued fraction expansion

$$[0; 3, 2, 2, 7, 32, 112, 10800, 403200, 17418254400, 1755760043520000, \ldots].$$

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