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Characterizations of a Clifford hypersurface in a unit sphere via Simons' integral inequalities

Sung-Hong Min[1] · Keomkyo Seo[2]

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Abstract Let M be an $n(\geq 3)$-dimensional closed hypersurface in a unit sphere with constant m-th order mean curvature and with two distinct principal curvatures. We obtain a sharp curvature integral for M in terms of Ricci curvature, which gives a characterization of a Clifford hypersurface. Moreover we give a generalization of Simons' integral inequality for closed hypersurface with vanishing m-th order mean curvature by making use of the Laplacian of the function of principal curvatures.

Keywords Clifford hypersurface · Simons' integral inequality · Ricci curvature · Higher-order mean curvature

Mathematics Subject Classification 53C40 · 53C42

1 Introduction

Let M be an n-dimensional closed minimal hypersurface in an $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$. In his celebrated paper [14], Simons proved that either M is totally

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✉ Keomkyo Seo
kseo@sookmyung.ac.kr
http://sites.google.com/site/keomkyo/

Sung-Hong Min
shmin@math.ntu.edu.tw

1 Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan

2 Department of Mathematics, Sookmyung Women's University, Hyochangwongil 52, Yongsan-ku, Seoul 140-742, Korea

geodesic, or $|A|^2 \equiv n$, or $|A|^2(x) > n$ at some point $x \in M$. Here $|A|^2$ denotes the squared norm of the second fundamental form on M. Motivated by this result, Chern et al. [9] and Lawson [8] proved independently that if $|A|^2 \equiv n$, then M is isometric to a Clifford minimal hypersurface. This characterization was generalized into the case of hypersurfaces with constant mean curvature by Alencar and do Carmo [1].

On the other hand, Otsuki [12] classified minimal hypersurfaces in a unit sphere with two distinct principal curvatures. He proved that if the multiplicities of two distinct principal curvatures are at least 2, then the minimal hypersurface is locally congruent to a Clifford minimal hypersurface. Furthermore, he was able to prove that if one of the two distinct principal curvatures is simple, then there are infinitely many minimal hypersurfaces other than Clifford minimal hypersurfaces. In this direction, Perdomo [13] and Wang [15] independently obtained a curvature integral inequality for closed minimal hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures. More precisely, they proved

Theorem [13,15] *Let M be an $n(\geq 3)$-dimensional closed minimal hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures, one of them being simple. Then*

$$\int_M |A|^2 \leq n\mathrm{Vol}(M),$$

where Vol(M) *denotes the volume of M. Moreover, equality holds if and only if M is isometric to a Clifford minimal hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-1}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{1}{n}}\right)$.*

It turned out that the similar curvature integral inequality holds when the m-th order mean curvature H_m vanishes, which was obtained by Wei [17]. More generally, we consider n-dimensional closed hypersurfaces with constant m-th order mean curvature in a unit sphere with two distinct principal curvatures. Applying the similar argument as in [12] shows that if the multiplicities of two distinct principal curvatures are at least 2, then a closed hypersurface with constant m-th order mean curvature is congruent to a Clifford hypersurface (see also [20]). Thus it suffices to consider the case where one of the two distinct principal curvatures is simple.

In this paper, we first obtain a sharp curvature integral inequality in case of constant m-th order mean curvature. In fact, we prove the following (Theorem 4.1):

Theorem *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures λ and μ, μ being simple* (*i.e., multiplicity* 1). *For the unit principal direction vector e_n corresponding to μ, we have*

$$\int_M \mathrm{Ric}(e_n, e_n) \geq 0,$$

where Ric *denotes the Ricci curvature. Moreover, equality holds if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$.*

We remark that if $H_m \equiv 0$ for $1 \leq m < n$, then

$$\mathrm{Ric}(e_n, e_n) = (n-1)(1+\lambda\mu) = (n-1)\left(1 - \frac{m(n-m)}{n(m^2-2m+n)}|A|^2\right).$$

Therefore it can be regarded as an extension of [13,15,17]. It immediately follows from the above theorem that if $\mathrm{Ric}(e_n, e_n) \leq 0$ on such a hypersurface M, then M is isometric to a Clifford hypersurface.

Furthermore, we analyse the Laplacian of the function of principal curvatures to give a characterization theorem under a pointwise Ricci curvature assumption. More precisely, we obtain the following theorem (see Corollary 4.3 and Theorem 4.4):

Theorem *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures λ and μ, μ being simple. Denote by e_n the unit principal direction vector corresponding to μ. Either if $\mathrm{Ric}(e_n, e_n) \geq 0$ or $\mathrm{Ric}(e_n, e_n) \leq 0$ on M, then M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$.*

As a consequence of the above theorem, when $H_m \equiv 0$ for $1 \leq m < n$, the Ricci curvature assumption on M is equivalent to the condition

$$|A|^2 \leq \frac{n(m^2-2m+n)}{m(n-m)} \quad \text{or} \quad |A|^2 \geq \frac{n(m^2-2m+n)}{m(n-m)}.$$

Thus the above theorem generalizes the previous results in [3,5,6,12,16,18,20].

The Ricci curvature of a Clifford minimal hypersurface $\mathbb{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right) \times \mathbb{S}^k\left(\sqrt{\frac{k}{n}}\right)$ varies between $\frac{n(k-1)}{k}$ and $\frac{n(n-k-1)}{n-k}$. Thus it is natural to ask whether it is isometric to a Clifford minimal hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-1}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{1}{n}}\right)$, provided M is a closed minimal hypersurface in $\mathbb{S}^{n+1}$ and the Ricci curvature of M satisfies $0 \leq \mathrm{Ric}(M) \leq \frac{n(n-2)}{n-1}$. Li [10] gave an affirmative answer for the minimal hypersurfaces in $\mathbb{S}^4$. Hasanis and Vlachos [7] extended this result to minimal hypersurfaces in a unit sphere of arbitrary dimension. Zhang [21] also generalized Li's result to compact hypersurfaces with constant mean curvature in a unit sphere. Moreover, Cheng and Ishikawa [4] proved that there exists a constant $\epsilon(n) > 0$ such that if M is a closed minimal hypersurface with Ricci curvature $\mathrm{Ric}(M) \geq n/2$ and $n \leq |A|^2 \leq n + \epsilon(n)$, then M is isometric to a Clifford minimal hypersurface. As to the case of hypersurface with constant m-th order mean curvature and with two distinct principal curvatures, our above theorem can be thought of as an extension of these results as well.

Finally we also generalize Simons' integral inequality to the case of closed hypersurfaces with $H_m \equiv 0$ as follows (Theorem 4.7):

Theorem *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ and with two distinct principal curvatures, one of them being simple. Then we have*

$$\begin{cases} \int_M |A|^p \left(|A|^2 - \frac{n(m^2-2m+n)}{m(n-m)}\right) \leq 0 \text{ if } p < \frac{n-2}{n}m, \\ \int_M |A|^p \left(|A|^2 - \frac{n(m^2-2m+n)}{m(n-m)}\right) \geq 0 \text{ if } p > \frac{n-2}{n}m. \end{cases}$$

Moreover, equalities hold if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

When $m = 1$ and $p = 2$, the above theorem is exactly the same as Simons' result. When $m = 2$ and $p = 2$, Li [11] obtained some pointwise estimates on $|A|^2$, which gives the above theorem. For $p = 2$ and $3 < m < n$, Wei [19] obtained the above theorem for closed and rotational hypersurfaces in a unit sphere with $H_m \equiv 0$. Thus our theorem unifies the previous results for closed hypersurfaces with $H_m \equiv 0$ and with two distinct principal curvatures.

2 Preliminaries

Let M be an $n(\geq 3)$-dimensional hypersurface in the unit sphere $\mathbb{S}^{n+1}$. Denote the Riemannian connection of M by ∇. We choose orthonormal frame fields $e_1, \ldots, e_n, e_{n+1}$ of the unit sphere such that $e_1, \ldots, e_n$ are tangent to M. Let $\omega^1, \ldots, \omega^n, \omega^{n+1}$ be the dual coframe. Then $\omega^{n+1} = 0$ on M. Let h and A be the second fundamental form and the shape operator of M, respectively. Note that h is a symmetric 2-form and A is a self-adjoint endomorphism of the tangent space at each $p \in M$ such that

$$\langle A(X), Y \rangle = h(X, Y)$$

for all $X, Y \in T_pM$, where T_pM denotes the tangent space of M at $p \in M$. The covariant derivative ∇h of h is a differential 3-form, $\nabla h = \sum_{i,j,k=1}^n h_{ijk}\omega^i \otimes \omega^j \otimes \omega^k$, where h_{ijk} is the coefficient function of ∇h such that

$$\begin{aligned} h_{ijk} \equiv h_{ij;k} &= \left(\nabla_{e_k} h\right)(e_i, e_j) \\ &= \nabla_{e_k} h(e_i, e_j) - h(\nabla_{e_k} e_i, e_j) - h(e_i, \nabla_{e_k} e_j). \end{aligned}$$

From the Codazzi equation, it follows

$$h_{ijk} = h_{ikj}.$$

The m-th order mean curvature H_m of an n-dimensional hypersurface $M \subset \mathbb{S}^{n+1}$ is defined by the elementary symmetric polynomial of degree m in the principal curvatures $\lambda_1, \lambda_2, \ldots, \lambda_n$ on M as follows:

$$\binom{n}{m} H_m = \sum_{1 \le i_1 < \cdots < i_m \le n} \lambda_{i_1} \ldots \lambda_{i_m}.$$

In the following, we give an important example of closed hypersurfaces in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature and with two distinct principal curvatures which is called a *Clifford hypersurface*. If an n-dimensional Clifford hypersurface in $\mathbb{S}^{n+1}$ has two distinct principal curvatures λ and μ of multiplicities $n-k$ and k, respectively, then it is given by

$$\mathbb{S}^{n-k}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^k\left(\frac{1}{\sqrt{1+\mu^2}}\right)$$

with $\lambda\mu + 1 = 0$, that is,

$$\mathbb{S}^{n-k}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^k\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right),$$

where λ satisfies the following identity:

$$\begin{aligned}\binom{n}{m} H_m = \binom{n-k}{m}\lambda^m + \binom{n-k}{m-1}\binom{k}{1}\lambda^{m-1}\mu + \cdots \\ + \binom{n-k}{1}\binom{k}{m-1}\lambda\mu^{m-1} + \binom{k}{m}\mu^m\end{aligned}$$

with $\lambda\mu + 1 = 0$.

In particular, if one of the principal curvatures is simple, say $k = 1$, then

$$\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right),$$

where λ satisfies the following identity:

$$(n-m)\lambda^m - m\lambda^{m-2} = nH_m.$$

Moreover, if $k = 1$ and $H_m = 0$, then $\lambda = \pm\sqrt{\frac{m}{n-m}}$ and $\mu = \mp\sqrt{\frac{n-m}{m}}$. Thus a Clifford hypersurface is given by

$$\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right).$$

As mentioned in the introduction, we have the following fact by using the similar argument as in [12] if the multiplicities of the principal curvatures are at least 2.

Lemma 2.1 [20] *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures of multiplicities $n-k$ and k for $2 \leq k \leq n-2$. Then M is isometric to a Clifford hypersurface $\mathbb{S}^{n-k}(\sqrt{1-c^2}) \times \mathbb{S}^k(c)$ for some $0 < c < 1$.*

From now on, let M be a closed hypersurface in a unit sphere with constant m-th order mean curvature H_m and with two distinct principal curvatures with multiplicities $n-1$, 1. Since M has two distinct principal curvatures and one of them is simple, we may assume that $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$. We choose the orthonormal frame tangent to M such that $h_{ij} = \lambda_i \delta_{ij}$, that is,

$$\begin{aligned} Ae_i &= \lambda e_i \quad \text{for} \quad i = 1, \ldots, n-1, \\ Ae_n &= \mu e_n. \end{aligned}$$

Since M has two distinct principal curvatures λ and μ,

$$\binom{n}{m} H_m = \binom{n-1}{m} \lambda^m + \binom{n-1}{m-1} \lambda^{m-1} \mu.$$

Therefore

$$H_m = \frac{m}{n} \lambda^{m-1} \left(\frac{n-m}{m} \lambda + \mu \right). \tag{1}$$

We claim that $\lambda^m - H_m$ never vanishes on M. To see this, we consider two cases: $m = 1$ and $m \geq 2$. Suppose $m = 1$. If $H_1 = 0$, then $\lambda \neq 0$ by our assumption. Thus $\lambda - H_1 \neq 0$. If $H_1 \neq 0$, then $\lambda - H_1 = \frac{\lambda - \mu}{n}$ by the identity (1). Since $\lambda \neq \mu$, it never vanishes. Suppose $m \geq 2$. If $H_m \neq 0$, then the identity (1) implies $\lambda \neq 0$. Therefore $\lambda^m - H_m = \frac{m}{n} \lambda^{m-1}(\lambda - \mu) \neq 0$. If $H_m = 0$ and $\lambda \neq 0$, then $\lambda^m - H_m$ never vanishes. If $H_m = 0$ and $\lambda = 0$ at some point, then it follows from the equation $\lambda^{m-1}(\frac{n-m}{n}\lambda + \mu) = 0$ that $\lambda \equiv 0$ by the continuity of λ and μ. Thus M has constant sectional curvature 1 by the Gauss equation, which implies that M is totally geodesic. However this is a contradiction because $\lambda \neq \mu$. Hence we conclude that $\lambda^m - H_m$ never vanishes on M. We also note that λ never vanishes on M for $1 \leq m \leq n$.

Now define a function $w := |\lambda^m - H_m|^{-\frac{1}{n}}$. Wu [20] obtained the following useful second order ordinary differential equation on M:

$$\frac{d^2 w}{dv^2} = -w \left(\frac{nH_m - (n-m)\lambda^m}{m\lambda^{m-2}} + 1 \right), \tag{2}$$

where v is the arclength parameter of the integral curve with respect to μ. In particular, if $H_1 \equiv 0$, then this equation was originally obtained by Otsuki [12].

3 Laplacian of the function of principal curvatures

Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures λ and μ, μ being simple:

$\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$. Let $w = |\lambda^m - H_m|^{-\frac{1}{n}}$. For more computations of w, we need the following lemma.

Lemma 3.1 [12] *Let M be an n-dimensional closed hypersurface in a unit sphere. If the multiplicities of principal curvatures are all constant, then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. Moreover, if the multiplicity of a principal curvature is greater than* 1*, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

From Lemma 3.1, it follows, for each $1 \leq i \leq n-1$,

$$\nabla_{e_i}\lambda = e_i\lambda = 0. \tag{3}$$

Since $w = |\lambda^m - H_m|^{-\frac{1}{n}} = (s(\lambda^m - H_m))^{-\frac{1}{n}}$ for a fixed constant $s = \pm 1$, we can compute the directional derivative of w with respect to e_i, $i = 1, \ldots, n$ as follows:

$$\begin{aligned}\nabla_{e_i} w &= \nabla_{e_i}(|\lambda^m - H_m|^{-\frac{1}{n}})\\ &= -\frac{m}{n} s w^{n+1}\lambda^{m-1}\nabla_{e_i}\lambda.\end{aligned} \tag{4}$$

For $i = 1, \ldots, n-1$, the Eq. (3) implies

$$\nabla_{e_i} w = 0.$$

For a function $f = f(w)$ on M, we compute the Laplacian of f on M.

$$\begin{aligned}\Delta f = \operatorname{div}\nabla f &= \sum_{i=1}^{n}\left\langle \nabla_{e_i}\nabla f, e_i\right\rangle = \sum_{i,j=1}^{n}\left\langle \nabla_{e_i}\left((e_j f)e_j\right), e_i\right\rangle\\ &= \sum_{i,j=1}^{n}\left\langle \nabla_{e_i}\left((e_j w) f'(w) e_j\right), e_i\right\rangle\\ &= \sum_{i=1}^{n}\left\langle \nabla_{e_i}\left((e_n w) f'(w) e_n\right), e_i\right\rangle\\ &= \sum_{i=1}^{n}\left\langle e_i((e_n w) f'(w)) e_n, e_i\right\rangle + f'(w)(e_n w)\sum_{i=1}^{n-1}\left\langle \nabla_{e_i} e_n, e_i\right\rangle.\end{aligned}$$

Taking the covariant derivative of h, we have

$$\begin{aligned} h_{iin} &= \nabla_{e_n} h(e_i, e_i) - h(\nabla_{e_n} e_i, e_i) - h(e_i, \nabla_{e_n} e_i) \\ &= e_n\lambda - 2\langle A(e_i), \nabla_{e_n} e_i\rangle \\ &= e_n\lambda - 2\lambda\langle e_i, \nabla_{e_n} e_i\rangle \\ &= e_n\lambda - \lambda\nabla_{e_n}\langle e_i, e_i\rangle \\ &= e_n\lambda. \end{aligned}$$

Moreover

$$\begin{aligned} h_{ini} &= \nabla_{e_i} h(e_i, e_n) - h(\nabla_{e_i} e_i, e_n) - h(e_i, \nabla_{e_i} e_n) \\ &= -\langle \nabla_{e_i} e_i, A(e_n)\rangle - \langle A(e_i), \nabla_{e_i} e_n\rangle \\ &= -\mu\langle \nabla_{e_i} e_i, e_n\rangle - \lambda\langle e_i, \nabla_{e_i} e_n\rangle \\ &= \mu\langle e_i, \nabla_{e_i} e_n\rangle - \lambda\langle e_i, \nabla_{e_i} e_n\rangle \\ &= (\mu - \lambda)\langle e_i, \nabla_{e_i} e_n\rangle. \end{aligned}$$

Since $h_{iin} = h_{ini}$,

$$\langle e_i, \nabla_{e_i} e_n\rangle = \frac{e_n\lambda}{\mu - \lambda}.$$

Note that

$$\mu - \lambda = \frac{n(H_m - \lambda^m)}{m\lambda^{m-1}} = -\frac{nsw^{-n}}{m\lambda^{m-1}}$$

by the equation $w = (s(\lambda^m - H_m))^{-\frac{1}{n}}$. Combining this with (4), we get

$$\frac{e_n\lambda}{\mu - \lambda} = \frac{e_n w}{s^2 w} = \frac{e_n w}{w}.$$

Therefore

$$\Delta f = f''(w)(e_n w)^2 + f'(w) e_n e_n w + (n-1) f'(w)\frac{(e_n w)^2}{w}. \tag{5}$$

Since v is the arc length parameter of the integral curve with respect to μ,

$$\frac{\partial}{\partial v} = e_n.$$

Thus by the Eq. (2), w satisfies

$$e_n e_n w = -w\left(\frac{nH_m - (n-m)\lambda^m}{m\lambda^{m-2}} + 1\right). \tag{6}$$

Combining the Eqs. (1), (5) and (6), we get

$$\Delta f = -f'(w)w\left(\frac{nH_m - (n-m)\,\lambda^m}{m\lambda^{m-2}} + 1\right) + \left[f''(w) + (n-1)\frac{f'(w)}{w}\right](e_n w)^2$$
$$= -f'(w)w\,(\lambda\mu + 1) + \left[f''(w) + (n-1)\frac{f'(w)}{w}\right](e_n w)^2.$$

From Gauss equation, it follows

$$\Delta f = -f'(w)w\; R^i{}_{nin} + \left[f''(w) + (n-1)\frac{f'(w)}{w}\right](e_n w)^2,$$

where $R^i{}_{jkl} e_i = R(e_k, e_l)e_j$ is the Riemann curvature tensor of M. Thus we obtain

$$\Delta f = -\frac{1}{n-1} f'(w)w\,\mathrm{Ric}(e_n, e_n) + \left[f''(w) + (n-1)\frac{f'(w)}{w}\right](e_n w)^2, \tag{7}$$

where $\mathrm{Ric}(e_n, e_n)$ denotes the Ricci curvature in the direction of e_n.

4 Curvature integral inequalities

Theorem 4.1 *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures λ and μ, μ being simple. Denote by e_n the unit principal direction vector corresponding to μ. Then*

$$\int_M \mathrm{Ric}(e_n, e_n) \geq 0,$$

where Ric *denotes the Ricci curvature. Moreover, equality holds if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$.*

Proof Define a function $f(w) = \log w$, where $w = |\lambda^m - H_m|^{-\frac{1}{n}}$. From the Eq. (7), we see

$$\Delta f = -\frac{\mathrm{Ric}(e_n, e_n)}{n-1} + \frac{n-2}{w^2}(e_n w)^2.$$

Integrating Δf over M gives

$$0 = \int_M -\frac{\mathrm{Ric}(e_n, e_n)}{n-1} + \frac{n-2}{w^2}(e_n w)^2.$$

Therefore

$$\int_M \mathrm{Ric}(e_n, e_n) = (n-1)(n-2)\int_M \frac{(e_n w)^2}{w^2} \geq 0.$$

Moreover equality holds if and only if $e_n w \equiv 0$ on M in the above inequality, which is equivalent that $e_n \lambda \equiv 0$. Thus both λ and μ are constant, which implies that M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$ by Cartan [2], where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$. □

In particular, if $H_m \equiv 0$ for $1 \leq m < n$, then one sees

$$\mathrm{Ric}(e_n, e_n) = (n-1)\left(1 - \frac{m(n-m)}{n(m^2-2m+n)}|A|^2\right). \tag{8}$$

Therefore from Theorem 4.1 and this observation we have the following, which can be regarded as an extension of [13,15,17].

Corollary 4.2 *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leq m < n)$ and with two distinct principal curvatures, one of them being simple. Then*

$$\int_M |A|^2 \leq \frac{n(m^2-2m+n)}{m(n-m)}\mathrm{Vol}(M),$$

where equality holds if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

As another application of Theorem 4.1, we have

Corollary 4.3 *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature and with two distinct principal curvatures, one of them being simple. Denote by e_n the unit principal direction vector corresponding to μ. If $\mathrm{Ric}(e_n, e_n) \leq 0$ on M, then M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$.*

Furthermore, by making use of the Laplacian of a function of principal curvatures, we obtain a characterization theorem under a pointwise nonnegative Ricci curvature assumption:

Theorem 4.4 *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures λ and μ, μ being simple. Denote by e_n the unit principal direction vector corresponding to μ. If $\mathrm{Ric}(e_n, e_n) \geq 0$ on M, then M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$.*

Proof Define a function $f(w) = w^k$ for a number $k < 2 - n$. Then from the identity (7) we obtain

$$\Delta f = -\frac{1}{n-1} f'(w)w \operatorname{Ric}(e_n, e_n) + \left[f''(w) + (n-1)\frac{f'(w)}{w} \right] (e_n w)^2$$
$$= -\frac{1}{n-1} k w^k \operatorname{Ric}(e_n, e_n) + k(k+n-2)w^{k-2}(e_n w)^2.$$

Integrating Δf over M, we have

$$0 = \int_M \Delta f = \int_M \left[-\frac{1}{n-1} k w^k \operatorname{Ric}(e_n, e_n) + k(k+n-2)w^{k-2}(e_n w)^2 \right].$$

Therefore

$$\frac{1}{n-1} \int_M w^k \operatorname{Ric}(e_n, e_n) = (k+n-2) \int_M w^{k-2}(e_n w)^2. \tag{9}$$

From the equality (9) and the assumption that $\operatorname{Ric}(e_n, e_n) \geq 0$, it follows that $e_n w \equiv 0$ on M. Therefore one can conclude that M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$ as in the proof of Theorem 4.1. □

We remark that the proof of Theorem 4.4 still works for the case where $\operatorname{Ric}(e_n, e_n) \leq 0$, which gives another proof of Corollary 4.3. (In this case, the constant k in the proof of Theorem 4.4 should be chosen to be bigger than $2 - n$.) When $H_m \equiv 0$, using Corollary 4.3, Theorem 4.4, and the Eq. (8), we give a simple proof of the previous results in [3,5,6,12,16,18,20].

Corollary 4.5 [3,5,6,12,16,18,20] *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leq m < n)$ and with two distinct principal curvatures, one of them being simple. Either if $|A|^2 \geq \frac{n(m^2-2m+n)}{m(n-m)}$ or $|A|^2 \leq \frac{n(m^2-2m+n)}{m(n-m)}$ on M, then M is isometric to the Riemannian product $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.*

Combining Lemma 2.1, Corollary 4.3, and Theorem 4.4, we obtain the following.

Corollary 4.6 *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature and with two distinct principal curvatures. If the Ricci curvature on M is nonnegative, then M is isometric to a Clifford hypersurface.*

It would be interesting to ask if the above results are still true without assuming that M has two distinct principal curvatures.

It is well-known that a minimal hypersurface M in $\mathbb{S}^{n+1}$ supports the following integral inequality which is due to Simons [14]:

$$\int_M |A|^2 \left(|A|^2 - n\right) \geq 0.$$

As mentioned in the introduction, when $H_m \equiv 0$, we obtain a generalized Simons' integral inequality for closed hypersurfaces with two distinct principal curvatures as follows:

Theorem 4.7 *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leq m < n)$ and with two distinct principal curvatures, one of them being simple. Then we have*

$$\begin{cases} \int_M |A|^p \left(|A|^2 - \frac{n(m^2-2m+n)}{m(n-m)}\right) \leq 0 & \textit{if} \quad p < \frac{n-2}{n}m, \\ \int_M |A|^p \left(|A|^2 - \frac{n(m^2-2m+n)}{m(n-m)}\right) \geq 0 & \textit{if} \quad p > \frac{n-2}{n}m. \end{cases}$$

Moreover, equalities hold if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

Proof Since $H_m \equiv 0$, the Eq. (1) gives

$$\mu = -\frac{n-m}{m}\lambda.$$

Thus as in the proof of Theorem 4.4, we have

$$\int_M w^k \operatorname{Ric}(e_n, e_n) \geq 0 \quad \text{if} \quad k > 2-n, \tag{10}$$

$$\int_M w^k \operatorname{Ric}(e_n, e_n) \leq 0 \quad \text{if} \quad k < 2-n, \tag{11}$$

where $w = |\lambda|^{-\frac{m}{n}}$. Let $p = -\frac{km}{n}$. Then

$$\begin{cases} \int_M |\lambda|^p \geq \frac{n-m}{m} \int_M |\lambda|^{p+2} & \text{if} \quad p < \frac{n-2}{n}m, \\ \int_M |\lambda|^p \leq \frac{n-m}{m} \int_M |\lambda|^{p+2} & \text{if} \quad p > \frac{n-2}{n}m. \end{cases}$$

Therefore using

$$|A|^2 = (n-1)\lambda^2 + \mu^2 = \frac{n(m^2-2m+n)}{m^2}\lambda^2,$$

we finally obtain

$$\begin{cases} \int_M |A|^p \left(|A|^2 - \frac{n(m^2-2m+n)}{m(n-m)}\right) \leq 0 & \text{if} \quad p < \frac{n-2}{n}m, \\ \int_M |A|^p \left(|A|^2 - \frac{n(m^2-2m+n)}{m(n-m)}\right) \geq 0 & \text{if} \quad p > \frac{n-2}{n}m. \end{cases}$$

Furthermore, equalities hold in the inequalities (10) and (11) if and only if $e_n w \equiv 0$. Thus equalities in Theorem 4.7 hold if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$. □

Using Theorem 4.7, one can obtain the lower bound of the total absolute Gauss-Kronecker curvature $\int_M |GK|$ in terms of the L^{n-2} norm of the second fundamental form on M as follows:

Corollary 4.8 *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ and with two distinct principal curvatures, one of them being simple. Then*

$$\int_M |GK| \geq \left(\frac{m^2}{n(m^2 - 2m + n)}\right)^{\frac{n-2}{2}} \int_M |A|^{n-2},$$

where GK is the Gauss-Kronecker curvature of M. Moreover, equalities hold if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

(*In other words, $\int_M |GK| \geq \int_M |\lambda|^{n-2}$ for the principal curvature λ with multiplicity $n-1$.*)

Proof Since the Gauss-Kronecker curvature of M is given by

$$GK = \lambda^{n-1}\mu = -\frac{n-m}{m}\lambda^n,$$

taking $p = n - 2 > \frac{n-2}{n}m$ in Theorem 4.7 shows

$$\int_M |GK| = \frac{n-m}{m}\int_M |\lambda|^n \geq \int_M |\lambda|^{n-2}.$$

As before, equalities hold if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$. □

It would be interesting if one can obtain the same curvature integral inequalities as in Theorem 4.7 and Corollary 4.8 without assuming that M has two distinct principal curvatures.

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References

1. Alencar, H., do Carmo, M.: Hypersurfaces with constant mean curvature in spheres. Proc. Am. Math. Soc. **120**(4), 1223–1229 (1994)
2. Cartan, E.: Familles de surfaces isoparametriques dans les espaces a courvure constante. Ann. Mat. Pura Appl. **17**(1), 177–191 (1938)

3. Cheng, Q.M.: Hypersurfaces in a unit sphere $S^{n+1}(1)$ with constant scalar curvature. J. Lond. Math. Soc. (2) **64**(3), 755–768 (2001)
4. Cheng, Q.M., Ishikawa, S.: A characterization of the Clifford torus. Proc. Am. Math. Soc. **127**(3), 819–828 (1999)
5. Hasanis, T., Savas-Halilaj, A., Vlachos, T.: Complete minimal hypersurfaces in a sphere. Monatsh. Math. **145**(4), 301–305 (2005)
6. Hasanis, T., Vlachos, T.: A pinching theorem for minimal hypersurfaces in a sphere. Arch. Math. (Basel) **75**(6), 469–471 (2000)
7. Hasanis, T., Vlachos, T.: Ricci curvature and minimal submanifolds. Pac. J. Math. **197**(1), 13–24 (2001)
8. Lawson, H.B., Jr.: Local rigidity theorems for minimal hypersurfaces. Ann. Math. (2) **89**, 187–197 (1969)
9. Chern, S.S., do Carmo, M., Kobayashi, S.: Minimal submanifolds of a sphere with second fundamental form of constant length. In: Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago, Chicago, Ill., vol. 1970, pp. 59–75. Springer, New York (1968)
10. Li, H.: A characterization of Clifford minimal hypersurfaces in S^4. Proc. Am. Math. Soc. **123**(10), 3183–3187 (1995)
11. Li, H.: Hypersurfaces with constant scalar curvature in space forms. Math. Ann. **305**(4), 665–672 (1996)
12. Otsuki, T.: Minimal hypersurfaces in a Riemannian manifold of constant curvature. Am. J. Math. **92**, 145–173 (1970)
13. Perdomo, O.: Ridigity of minimal hypersurfaces of spheres with two principal curvatures. Arch. Math. (Basel) **82**(2), 180–184 (2004)
14. Simons, J.: Minimal varieties in Riemannian manifolds. Ann. Math. (2) **88**, 62–105 (1968)
15. Wang, Q.: Rigidity of Clifford minimal hypersurfaces. Monatsh. Math. **140**(2), 163–167 (2003)
16. Wei, G.: Complete hypersurfaces with constant mean curvature in a unit sphere. Monatsh. Math. **149**(3), 251–258 (2006)
17. Wei, G.: Rigidity theorem for hypersurfaces in a unit sphere. Monatsh. Math. **149**(4), 343–350 (2006)
18. Wei, G.: Complete hypersurfaces with $H_k = 0$ in a unit sphere. Differ. Geom. Appl. **25**(5), 500–505 (2007)
19. Wei, G.: J. Simons' type integral formula for hypersurfaces in a unit sphere. J. Math. Anal. Appl. **340**(2), 1371–1379 (2008)
20. Wu, B.Y.: On hypersurfaces with two distinct principal curvatures in a unit sphere. Differ. Geom. Appl. **27**(5), 623–634 (2009)
21. Zhang, Y.T.: Pinching theorems of hypersurfaces in a unit sphere. Differ. Geom. Appl. **29**(6), 730–736 (2011)