Monatsh Math (2016) 179:281–292
DOI 10.1007/s00605-015-0832-6

Monotonic functions related to the q-gamma function

Ahmed Salem[1]

Received: 4 April 2013 / Accepted: 12 November 2015 / Published online: 26 November 2015

Abstract In this paper, the monotonicity property for two functions involving the logarithmic of the q-gamma function is proven for all $q > 0$. As a consequence, sharp inequalities for the q-gamma function are established. Our results are shown to be as a generalization of results which were obtained by Anderson and Qiu (Proc Am Math Soc 125:3355–3362, 1997).

Keywords Inequalities · q-Gamma function · q-Polygamma functions · Monotonic function

Mathematics Subject Classification 33D05 · 26D07 · 26A48

1 Introduction

Euler's gamma function is defined for positive real numbers x by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0$$

which is one of the most important special functions and has many extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. Anderson and Qiu [1] used the increasing monotonicity of the function

Communicated by A. Constantin.

✉ Ahmed Salem
ahmedsalem74@hotmail.com

1 Department of Basic Science, Faculty of Information Systems and Computer Science, October 6 University, Sixth of October City, Egypt

$$f(x)=\frac{\log\Gamma(x+1)}{x\log x},\quad x>1 \tag{1.1}$$

to establish a sharp inequality

$$x^{(1-\gamma)x-1}<\Gamma(x)<x^{x-1},\quad x>1 \tag{1.2}$$

where $\gamma=0.577215\ldots$ is the Euler–Mascheroni constant, which has attracted the attention of many researches, because of its simple form, and of its usefulness in practical applications in pure mathematics or other branches of science such as probabilities, engineering, or statistical physics. They conjectured that f is concave on the interval $[1,\infty)$. The concavity of f on $[1,\infty)$ was established by Elbert and Laforgia [2]. A short and simple proof of the increasing of the function f which extended the increasing on $(0,\infty)$, has been presented by Alzer [3]. It is worth mentioning that in 1989, Anderson et al. [4] conjectured that the function

$$G(x)=\frac{\log\Gamma(\frac{x}{2}+1)}{x\log x},\quad x\geq 2 \tag{1.3}$$

is strictly increasing on $[2,\infty)$. This conjecture was proved by Anderson and Qiu [1].

Many of the classical facts about the ordinary gamma function have been extended to the q-gamma function (see [5–8] and the references given therein). The aim of this paper is to extend the inequality (1.2) to the q-gamma function for all positive real numbers x and q by means of the study of the monotonicity property of the function

$$F_q(x)=\frac{\log\Gamma_q(x+1)-\frac{x(x-1)}{2}H(q-1)\log q}{x\log[x]_q-x(x-1)H(q-1)\log q},\quad x>0,\ q>0 \tag{1.4}$$

where $[x]_q=(1-q^x)/(1-q)$, $H(\cdot)$ denotes the Heaviside step function and $\Gamma_q(x)$ is the q-gamma function defined as

$$\Gamma_q(x)=(1-q)^{1-x}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+x}},\quad 0<q<1, \tag{1.5}$$

and

$$\Gamma_q(x)=(q-1)^{1-x}q^{\frac{x(x-1)}{2}}\prod_{n=0}^{\infty}\frac{1-q^{-(n+1)}}{1-q^{-(n+x)}},\quad q>1. \tag{1.6}$$

From the previous definitions, for a positive x and $q\geq 1$, we get

$$\Gamma_q(x)=q^{\frac{(x-1)(x-2)}{2}}\Gamma_{q^{-1}}(x). \tag{1.7}$$

Also, we extend the function $G(x)$ to $F_q(x)$, defined in (1.4), which contains the q-gamma function, for all $q\in(0,\infty)$ and $x\in(0,1)\cup[2,\infty)$. This means that the function $G(x)$ is also increasing on the interval $(0, 1)$. Furthermore, we use these results to establish new inequalities for the q-gamma function.

An important fact for gamma function in applied mathematics as well as in probability is the Stirling's formula that gives a pretty accurate idea about the size of gamma function. With the Euler–Maclaurin formula, Moak [7] obtained the following q-analogue of Stirling's formula (see also [9])

$$\log\Gamma_q(x) \sim \left(x-\frac{1}{2}\right)\log[x]_q + \frac{\mathrm{Li}_2(1-q^x)}{\log q} + \frac{1}{2}H(q-1)\log q + C_{\hat{q}} + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(\frac{\log\hat{q}}{\hat{q}^x-1}\right)^{2k-1}\hat{q}^x P_{2k-3}(\hat{q}^x), \quad x\to\infty \tag{1.8}$$

where B_k is the Bernoulli numbers,

$$\hat{q} = \begin{cases} q & \text{if } 0 < q \le 1 \\ q^{-1} & \text{if } q \ge 1, \end{cases}$$

$\mathrm{Li}_2(z)$ is the dilogarithm function defined for complex argument z as [10]

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt, \quad z \notin (1, \infty), \tag{1.9}$$

P_k is a polynomial of degree k satisfying

$$P_k(z) = (z - z^2)P'_{k-1}(z) + (kz+1)P_{k-1}(z), \quad P_0 = P_{-1} = 1, \quad k = 1, 2, \cdots \tag{1.10}$$

and

$$C_q = \frac{1}{2}\log(2\pi) + \frac{1}{2}\log\left(\frac{q-1}{\log q}\right) - \frac{1}{24}\log q + \log\left(\sum_{m=-\infty}^{\infty}\left(r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right)\right) \tag{1.11}$$

where $r = \exp(4\pi^2/\log q)$. It is easy to see that

$$\lim_{q\to1} C_q = C_1 = \frac{1}{2}\log(2\pi), \quad \lim_{q\to1}\frac{\mathrm{Li}_2(1-q^x)}{\log q} = -x \quad \text{and} \quad P_k(1) = (k+1)! \tag{1.12}$$

and so (1.8) when letting $q \to 1$, tends to the ordinary Stirling's formula [10]

$$\log\Gamma(x) \sim \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}\frac{1}{x^{2k-1}}, \quad x \to \infty. \tag{1.13}$$

2 Useful lemmas

In order to prove our main results we need to study the monotonicity properties of some functions which are connected with the q-digamma function $\psi_q(x)$ and its derivative which is defined as the logarithmic derivative of the q-gamma function

$$\psi_q(x) = \frac{d}{dx}(\log\Gamma_q(x)) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} \tag{2.1}$$

The q-digamma function $\psi_q(x)$ appeared in the work of Krattenthaler and Srivastava [11] when they studied the summations for basic hypergeometric series. Some of its properties are presented and proven in their work. Also, in their work, they proved that $\psi_q(x)$ tends to the digamma function $\psi(x)$ when letting $q \to 1$. For more details on the q-digamma function (see [12] and the references therein). From (1.5), we get for $0 < q < 1$ and for all real variable $x > 0$

$$\psi_q(x) = -\log(1-q) + \log q \sum_{k=1}^{\infty}\frac{q^{xk}}{1-q^k}, \tag{2.2}$$

and from (1.6) we obtain for $q > 1$ and $x > 0$

$$\psi_q(x) = -\log(q-1) + \log q\left[x - \frac{1}{2} - \sum_{k=1}^{\infty}\frac{q^{-xk}}{1-q^{-k}}\right]. \tag{2.3}$$

It is worth mentioning that many papers recently have introduced inequalities related to the q-gamma, q-digamma and q-polygamma functions, see [9,13–21] and the references therein.

Lemma 2.1 *Let x and q be real numbers such that $0 < q < 1$. Then the function $\log\Gamma_q(x+1) \ge 0$ for all $x \ge 1$ and $\log\Gamma_q(x+1) \le 0$ for all $0 \le x \le 1$.*

Proof Replacing x by $x+1$ in (2.2) followed by integrating from 0 to x, the logarithmic of the q-gamma function can be represented as

$$\log\Gamma_q(x+1) = -x\log(1-q) + \sum_{k=1}^{\infty}\frac{q^{(x+1)k}-q^k}{k(1-q^k)}$$

which can be also rewritten as

$$\log \Gamma_q(x+1) = \sum_{k=1}^{\infty} \frac{q^k}{k(1-q^k)} \alpha(y), \quad y = q^k,$$

where $\alpha(y) = x(1-y) + y^x - 1$ which has the derivative $\alpha'(y) = -x(1-y^{x-1})$. It is clear that $\alpha'(y) \le 0$ if $x \ge 1$ and $\alpha'(y) \ge 0$ if $x \le 1$ which reveals that $\alpha(y)$ is decreasing on $(0, 1)$ if $x \ge 1$ and increasing on $(0, 1)$ if $x \le 1$. Since $\alpha(1) = 0$ for all $x \ge 0$, then $\alpha(y) \ge 0$ if $x \ge 1$ and $\alpha(y) \le 0$ if $x \le 1$ which give the desired results.

Lemma 2.2 *Let q be a positive real number such that $0 < q < 1$. Then the function*

$$f_q(x) = \psi_q''(x+1) - \frac{\log q}{1-q^x} \psi_q'(x+1) \tag{2.4}$$

is strictly positive for all $x \in \mathbb{R}^+$.

Proof The relation (2.2) and the Cauchy product rule gives

$$\frac{\log q}{1-q^x} \psi'(x+1) = \log^3 q \sum_{k=1}^{\infty} q^{xk} \sum_{r=1}^{k} \frac{rq^r}{1-q^r}$$

which yields that

$$f_q(x) = -\log^3 q \sum_{k=1}^{\infty} q^{xk} \ell(k)$$

where

$$\ell(k) = \sum_{r=1}^{k} \frac{rq^r}{1-q^r} - \frac{k^2 q^k}{1-q^k}.$$

Forward shift operator gives

$$\ell(k+1) - \ell(k) = \frac{(k+1)q^{k+1}}{1-q^{k+1}} - \frac{(k+1)^2 q^{k+1}}{1-q^{k+1}} + \frac{k^2 q^k}{1-q^k}$$

which can be simplified as

$$\ell(k+1) - \ell(k) = \frac{kq^k(k(1-q) - q(1-q^k))}{(1-q^k)(1-q^{k+1})}.$$

Since $1 - q^k = (1-q)(1+q+q^2+\cdots+q^{k-1}) \le k(1-q)$ for all $k \in \mathbb{N}$, then we get $\ell(k+1) \ge \ell(k)$ for all $k \in \mathbb{N}$ which gives that $\ell(k) \ge \ell(1) = 0$ for all $k \in \mathbb{N}$ and so the function $f_q(x) \ge 0$ for all $x > 0$.

Lemma 2.3 *Let q be a positive real number such that $0 < q < 1$. Then the function*

$$h_q(x) = x\psi_q(x+1) - \log\Gamma_q(x+1) \tag{2.5}$$

is non-negative and increasing on $[0, \infty)$.

Proof Differentiation gives

$$h'_q(x) = x\psi'_q(x+1) = x\sum_{k=1}^{\infty}\frac{kq^{(x+1)k}\log^2 q}{1-q^k} \geq 0 \quad x \geq 0.$$

Hence, the monotonicity of h_q follows. Obviously, $h_q(0) = 0$.

Lemma 2.4 *Let q be a positive real number such that $0 < q < 1$. Then the function*

$$g_q(x) = x^2\psi'_q(x+1) - 2h_q(x) - \frac{x\log q}{1-q^x}h_q(x) \tag{2.6}$$

is strictly positive for all $x \in (0, \infty)$, where $h_q(x)$ is defined as in Lemma 2.3.

Proof Differentiation gives

$$g'_q(x) = x^2\psi''_q(x+1) - \frac{x^2\log q}{1-q^x}\psi'_q(x+1) - \frac{(1-q^x+xq^x\log q)\log q}{(1-q^x)^2}h_q(x).$$

Let $\lambda(y) = y\log y + 1 - y$ where $y = q^x$. A short calculation shows that

$$\lambda(y) = y\sum_{n=2}^{\infty}\frac{\log(1/y)}{n!} \geq 0, \quad 0 < y < 1. \tag{2.7}$$

Since $h_q(x) \geq 0$ according to Lemma 2.3, then we get $g'_q(x) \geq x^2 f_q(x)$ where $f_q(x)$ defined as in Lemma 2.2. This concludes that $g'_q(x) > 0$ for all $x > 0$ and so that the function $g_q(x)$ is increasing on $(0, \infty)$ for all $0 < q < 1$. It is clear that from (2.6) and Lemma 2.3 that $\lim_{x\to 0} g_q(x) = 0$ which concludes that $g_q(x) > 0$ for all $x > 0$ and $0 < q < 1$.

Lemma 2.5 *Let q be a positive real number such that $0 < q < 1$. Then the function*

$$H_q(x) = \log[x]_q + \frac{xq^x\log q}{1-q^x}\frac{\log\Gamma_q(x+1)}{h_q(x)} \tag{2.8}$$

is strictly positive on $(0, \infty)$, where $h_q(x)$ is defined as in Lemma 2.3.

Proof Differentiation gives

$$
\begin{aligned}
H_q'(x) &= -\frac{q^x \log q}{1-q^x} + \frac{xq^x \log q}{1-q^x} \frac{\psi_q(x+1)h_q(x) - h_q'(x)\log\Gamma_q(x+1)}{h_q^2(x)} \\
&\quad + \frac{q^x \log q(1-q^x + x\log q)}{(1-q^x)^2} \frac{\log\Gamma_q(x+1)}{h_q(x)} \\
&= -\frac{q^x \log q}{(1-q^x)h_q^2(x)} \Big(h_q^2(x) - x\psi_q(x+1)h_q(x) + x^2\psi_q'(x+1)\log\Gamma_q(x+1) \\
&\quad - \frac{h_q(x)\log\Gamma_q(x+1)(1-q^x + x\log q}{1-q^x} \Big) \\
&= -\frac{q^x \log q \log\Gamma_q(x+1)}{(1-q^x)h_q^2(x)} g(x)
\end{aligned}
$$

where $g(x)$ defined as in Lemma 2.4. According to the results obtained in Lemmas 2.1 and 2.4, we see that $H_q'(x) \geq 0$ if $x \geq 1$ and $H_q'(x) \leq 0$ if $x \leq 1$ which yields that $H_q(x)$ is increasing on $[1, \infty)$ and decreasing on $(0, 1]$. It is obvious from (2.8) that $H_q(1) = 0$ which gives that $H_q(x) > 0$ for all $x > 0$.

Lemma 2.6 *Let x and q be positive real numbers. Then the function*

$$
S_q(x) = \frac{x\log[x]_q - x(x-1)H(q-1)\log q}{x\log[2x]_q - x(2x-1)H(q-1)\log q} \tag{2.9}
$$

is strictly increasing on $(0, 1/2) \cup (1/2, \infty)$ *and* $S_q(x) \geq 0$ *if* $x \in (0, 1/2) \cup [1, \infty)$ *and* $S_q(x) \leq 0$ *if* $x \in (1/2, 1]$.

Proof When $0 < q < 1$, differentiation gives

$$
S_q'(x) = -\frac{q^x \log q}{(1+q^x)\log^2[x]_q} \beta(x) \tag{2.10}
$$

where

$$
\beta(x) = \log[x]_q + \frac{1+q^x}{1-q^x} \log(1+q^x)
$$

which has the derivative

$$
\beta'(x) = \frac{2q^x \log q}{(1-q^x)^2} \log(1+q^x) < 0, \quad x > 0.
$$

Since $\lim_{x\to\infty} \beta(x) = -\log(1-q) > 0$ and $\beta'(x) < 0$, then $\beta(x) > 0$ for all $x > 0$ which yields that $S_q'(x) > 0$ for all $x \in (0, 1) \cup (1, \infty)$ and so the function $S_q(x)$ is increasing on $(0, 1/2) \cup (1/2, \infty)$. It is easy to see that $S_q(1) = 0$ and $\lim_{x\to 0} S_q(x) = 1$ which give the sign of the function. When $q \geq 1$, we get $S_q(x) = S_{q^{-1}}(x)$. This ends the proof.

3 The main results

In this section, the main results will be provided. At first, we recall that the author in [12] defined the q-analogue of the Euler–Mascheroni constant as

$$\gamma_q = \frac{1-q}{\log q}\psi_q(1), \quad 0 < q < 1, \tag{3.1}$$

and proved the identity

$$\psi_q(x+1) = \psi_q(x) - \frac{q^x \log q}{1-q^x}, \quad x > 0. \tag{3.2}$$

We are now in a position to prove the following:

Theorem 3.1 *Let x and q be positive real numbers. Then the function $F_q(x)$ defined as in (1.4) is strictly increasing on $(0, 1) \cup (1, \infty)$ and has the limits:*

1. $\lim_{x\to 0} F_q(x) = 0$
2. $\lim_{x\to 1} F_q(x) = 1 - \hat{q}^{-1}\gamma_{\hat{q}}$
3. $\lim_{x\to\infty} F_q(x) = 1.$

Proof When $0 < q < 1$, differentiating (1.4) gives

$$\begin{aligned}(x\log[x]_q)^2 F_q'(x) &= x\log[x]_q\psi_q(x+1) - \log[x]_q \log\Gamma_q(x+1) \\ &\quad + \frac{xq^x\log q}{1-q^x}\log\Gamma_q(x+1) \\ &= \log[x]_q h_q(x) + \frac{xq^x\log q}{1-q^x}\log\Gamma_q(x+1) \\ &= h_q(x)H_q(x)\end{aligned}$$

where h_q and H_q are defined as in Lemmas 2.3 and 2.5, respectively. Hence, the monotonicity of F_q follows immediately from Lemmas 2.3 and 2.5. When $q \geq 1$, inserting (1.7) into (1.4) yields $F_q(x) = F_{q^{-1}}(x)$ which concludes that $F_q(x)$ is increasing on $(0, 1) \cup (1, \infty)$ for all $q > 0$.

In order to evaluate the limits, using l'Hôpital's rule to get

$$\lim_{x\to 0} F_q(x) = \lim_{x\to 1} \frac{\psi_q(x+1) - \frac{2x-1}{2}H(q-1)\log q}{\log[x]_q - \frac{xq^x\log q}{1-q^x} - (2x-1)H(q-1)\log q} = 0.$$

Also, when $0 < q < 1$, we get

$$\lim_{x\to 1} F_q(x) = \lim_{x\to 1} \frac{\psi_q(x+1)}{\log[x]_q - \frac{xq^x\log q}{1-q^x}} = \frac{\psi_q(2)}{-\frac{q\log q}{1-q}}.$$

From the relations (3.1) and (3.2), we get

$$\lim_{x\to 1} F_q(x) = 1 - q^{-1}\gamma_q, \quad 0 < q < 1.$$

Since $F_q(x) = F_{q^{-1}}(x)$ when $q \geq 1$, then we get

$$\lim_{x\to 1} F_q(x) = 1 - q\gamma_{q^{-1}}, \quad q \geq 1.$$

The previous two limits lead to the proof of the second statement. Also, by Moak formula (1.8), we have

$$\lim_{x\to\infty} F_q(x) = \lim_{x\to\infty}\left[\frac{(x+\frac{1}{2})\log[x]_{\hat{q}} + \frac{\mathrm{Li}_2(1-\hat{q}^x)}{\log\hat{q}} + C_{\hat{q}}}{x\log[x]_{\hat{q}}} + O\left(\frac{\hat{q}^x\log\hat{q}}{x(1-\hat{q}^x)}\right)\right] = 1,$$
$$q > 0.$$

This ends the proof.

Corollary 3.2 *Let x and q be positive real numbers. Then the q-gamma function satisfies the inequality*

$$q^{\frac{x(1-x)}{2}(2\alpha-1)H(q-1)}[x]_q^{\alpha x-1} \leq \Gamma_q(x) \leq q^{\frac{x(1-x)}{2}H(q-1)}[x]_q^{\beta x-1}, \quad x \in [1,\infty) \tag{3.3}$$

with the best possible constants $\alpha = 1-\hat{q}^{-1}\gamma_{\hat{q}}$ and $\beta = 1$, where γ_q is the q-analogue of the Euler–Mascheroni constant (3.1)*, and the inequality*

$$q^{\frac{x(1-x)}{2}H(q-1)}[x]_q^{\alpha x-1} \leq \Gamma_q(x) \leq q^{\frac{x(x-1)}{2}H(q-1)}[x]_q^{\beta x-1}, \quad x \in (0,1] \tag{3.4}$$

with the best possible constants $\alpha = 1$ and $\beta = 0$.

Proof The proof of this corollary comes immediately from Theorem 3.1.

Corollary 3.3 *Let $y > x > 1$ and q be positive real numbers. Then the q-gamma function satisfies the inequalities*

$$\frac{\log\Gamma_q(y+1) - \frac{y(y-1)}{2}H(q-1)\log q}{\log\Gamma_q(x+1) - \frac{x(x-1)}{2}H(q-1)\log q} > \left(\frac{y\log[y]_q - y(y-1)H(q-1)\log q}{x\log[x]_q - x(x-1)H(q-1)\log q}\right)^{\alpha} \tag{3.5}$$

for all $\alpha \leq 1$ with the best possible constant $\alpha = 1$.

Proof Taking the logarithm of two sides to obtain $\alpha < P(x, y; q)$ where

$$P(x, y; q) = \frac{\log\left(\log\left(q^{\frac{y(1-y)}{2}H(q-1)}\Gamma_q(y+1)\right)\right) - \log\left(\log\left(q^{\frac{x(1-x)}{2}H(q-1)}\Gamma_q(x+1)\right)\right)}{\log\left(\log\left(q^{y(1-y)H(q-1)}[y]_q^y\right)\right) - \log\left(\log\left(q^{x(1-x)H(q-1)}[x]_q^x\right)\right)}.$$

When $0 < q < 1$, using l'Hośpital rule, one gets

$$\begin{aligned}\lim_{y\to\infty} P(x, y; q) &= \lim_{y\to\infty} \frac{\log\left(\log\left(\Gamma_q(y+1)\right)\right) - \log\left(\log\left(\Gamma_q(x+1)\right)\right)}{\log\left(y\log[y]_q\right) - \log\left(x\log[x]_q^x\right)}\\ &= \lim_{y\to\infty} \frac{y\log[y]_q\psi_q(y+1)}{\log\Gamma_q(y+1)(\log[y]_q - \frac{yq^y\log q}{1-q^y})}\\ &= \lim_{y\to\infty} \frac{y\psi_q(y+1)}{\log\Gamma_q(y+1)} \lim_{y\to\infty}\left(1 - \frac{yq^y\log q}{(1-q^y)\log[y]_q}\right)^{-1}\\ &= \lim_{y\to\infty} \frac{\psi_q(y+1) + y\psi_q'(y+1)}{\psi_q(y+1)} \times 1 = 1.\end{aligned}$$

Here, we use $yq^y \to 0$ as $y \to \infty$ and $\lim_{y\to\infty} y\psi_q'(y+1) = 0$ which comes immediately from (2.2). When $q \geq 1$, it is clear that $P(x, y; q) = P(x, y; q^{-1})$.

Theorem 3.4 *Let x and q be positive real numbers. Then the function*

$$G_q(x) = \frac{\log\Gamma_q(1+\frac{x}{2}) - \frac{x(x-2)}{8}H(q-1)\log q}{x\log[x]_q - x(x-1)H(q-1)\log q} \tag{3.6}$$

is strictly increasing on $(0, 1) \cup [2, \infty)$ *and has the values* $G_q(2) = 0;\ \lim_{x\to 0} G_q(x) = 0$ *and* $\lim_{x\to\infty} G_q(x) = \frac{1}{2}$.

Proof The function $G_q(x)$ after replacing x by $2x$ can be read as

$$\begin{aligned}G_q(2x) &= \frac{1}{2}\frac{\log\Gamma_q(x+1) - \frac{x(x-1)}{2}H(q-1)\log q}{x\log[x]_q - x(x-1)H(q-1)\log q}\frac{x\log[x]_q - x(x-1)H(q-1)\log q}{x\log[2x]_q - x(2x-1)H(q-1)\log q}\\ &= \frac{1}{2}F_q(x)S_q(x)\end{aligned}$$

where $F_q(x)$ and $S_q(x)$ defined as in (1.4) and (2.9), respectively. Differentiation gives

$$4G_q'(2x) = F_q'(x)S_q(x) + F_q(x)S_q'(x)$$

It is clear from Theorem 3.1 and Lemma 2.6 that $G_q'(2x) > 0$ for all $x \in (0, 1/2) \cup [1, \infty)$ which lead to the function $G_q(x)$ is increasing on $(0, 1) \cup [2, \infty)$ for all $q > 0$. To obtain $\lim_{x\to\infty} G_q(x) = \frac{1}{2}$, use the l'Hośpital rule and the relations (2.2) and (2.3).

Corollary 3.5 *Let x and q be positive real numbers. Then the q-gamma function satisfies the double inequality*

$$q^{\frac{x(x-2)}{8}H(q-1)}[x]_q^{-1}(1+q^{\frac{x}{2}}) \le \Gamma_q\left(\frac{x}{2}\right) < q^{\frac{x(2-3x)}{8}H(q-1)}[x]_q^{\frac{x}{2}-1}(1+q^{\frac{x}{2}}) \quad (3.7)$$

for all $x \in [2, \infty)$ and satisfies the one-sided inequality

$$\Gamma_q\left(\frac{x}{2}\right) < q^{\frac{x(x-2)}{8}H(q-1)}[x/2]_q^{-1} \quad (3.8)$$

for all $x \in (0, 1)$.

Remark 3.6 The function $G_q(x)$ defined as in (3.6) approaches the function $G(x)$ defined as in (1.3) when letting $q \to 1$ and so the function $G(x)$ is increasing on the interval $(0, 1)$ which is considered an extension of the results obtained for this function by [1].

Conjecture 3.7 *The function $G_q(x)$ defined as in (3.6) is strictly increasing on the interval $(1, 2]$ for all $q > 0$.*

Conjecture 3.8 *The function $F_q(x)$ defined as in (1.4) is concave on the interval $(0, 1) \cup (1, \infty)$ for all $q > 0$.*

Acknowledgments The author is very grateful to anonymous referees for their valued suggestions.

Compliance with ethical standards

Conflict of interest The author declares that there is no conflict of interests regarding the publication of this paper.

References

1. Anderson, G.D., Qiu, S.-L.: A monotonicity property of the gamma function. Proc. Am. Math. Soc. **125**, 3355–3362 (1997)
2. Elbert, Á., Laforgia, A.: On some properties of the gamma function. Proc. Am. Math. Soc. **128**, 2667–2673 (2000)
3. Alzer, H.: Inequalities for the gamma and polygamma functions. Abh. Math. Sem. Univ. Hamg. **68**, 363–372 (1998)
4. Anderson, G.D., Vamanamurthy, M.K., Vuorinen, M.: Special functions of quasiconformal theory. Expo. Math. **7**, 97–136 (1989)
5. Askey, R.: The q-gamma and q-beta functions. Appl. Anal. **8**, 125–141 (1978)
6. Moak, D.S.: The q-gamma function for $q > 1$. Aequ. Math. **20**, 278–285 (1980)
7. Moak, D.S.: The q-analogue of Stirling's formula. Rocky Mt. J. Math. **14**, 403–413 (1984)
8. Olde, A.B.: Daalhuis, asymptotic expansions of q-gamma, q-exponential and q-Bessel functions. J. Math. Anal. Appl. **186**, 896–913 (1994)
9. Salem, A.: A completely monotonic function involving q-gamma and q-digamma functions. J. Approx. Theory **164**(7), 971–980 (2012)
10. Abramowitz, M., Stegun, C.A.: Handbook of Mathematical Functions with Formulas, Graphs, Mathematical Tables 7th Printing, Applied Mathematics Series, vol. 55. Nathional Bureau of Standards, Washington, DC (1964)
11. Krattenthaler, C., Srivastava, H.M.: Summations for basic hypergeometric series involving a q-analogue of the digamma function. Comput. Math. Appl. **32**(2), 73–91 (1996)

12. Salem, A.: Some Properties and expansions associated with the q-digamma function. Quaestiones Mathematicae **36**(1), 67–77 (2013)
13. Batir, N.: Monotonicity properties of q-digamma and q-trigamma functions. J. Approx. Theory **192**(7), 336–346 (2015)
14. Batir, N.: q-Extensions of some estimates associated with the digamma function. J. Approx. Theory **174**, 54–64 (2013)
15. Salem, A., Kamel, E.S.: Some completely monotonic functions associated with the q-gamma and the q-polygamma functions. Acta Math. Sci. **25**(5), 1214–1224 (2015)
16. Salem, A.: Completely monotonic functions related to the q-gamma and the q-trigamma functions. Anal. Appl. **13**(2), 125–134 (2015)
17. Salem, A.: On the q-beta function inequalities. Math. Inequ. Appl. **18**(2), 639–648 (2015)
18. Salem, A.: Two classes of bounds for the q-zeta functions. Banach J. Math. Anal. **8**(1), 109–117 (2014)
19. Salem, A.: Complete monotonicity properties of functions involving q-gamma and q-digamma functions. Math. Inequ. Appl. **17**(3), 801–811 (2014)
20. Salem, A.: An infinite class of completely monotonic functions involving the q-gamma function. J. Math. Anal. Appl. **406**(2), 392–399 (2013)
21. Alzer, H., Grinshpan, A.Z.: Inequalities for the gamma and q-gamma functions. J. Approx. Theory **144**, 67–83 (2007)