Monatsh Math (2016) 180:255–270
DOI 10.1007/s00605-015-0823-7

Nonexistence results for pseudo-parabolic equations in the Heisenberg group

Mohamed Jleli[1] · **Mokhtar Kirane**[2,3] · **Bessem Samet**[1]

Received: 27 December 2014 / Accepted: 22 September 2015 / Published online: 5 October 2015

Abstract We derive sufficient conditions for the nonexistence of global weak solutions to the nonlinear pseudo-parabolic equation

$$u_t - \Delta_{\mathbb{H}} u_t - \Delta_{\mathbb{H}} u = |u|^p + f(t, \vartheta), \quad (t, \vartheta) \in (0, \infty) \times \mathbb{H},$$

where $\Delta_{\mathbb{H}}$ is the Kohn–Laplace operator on the $(2N + 1)$-dimensional Heisenberg group $\mathbb{H}$, $p > 1$ and $f(t, \vartheta)$ is a given function. Next, we extend this result to the case of systems. Our technique of proof is based on the test function method.

Keywords Nonexistence · Nonlinear pseudo-parabolic equation · System · Heisenberg group

Mathematics Subject Classification 47J35 · 34A34 · 35R03

Communicated by J. Escher.

✉ Bessem Samet
bsamet@ksu.edu.sa

Mohamed Jleli
jleli@ksu.edu.sa

Mokhtar Kirane
mkirane@univ-lr.fr

1 Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

2 Laboratoire de Mathématiques, Image et Applications Pôle Sciences et Technologies, Université de La Rochelle, Avenue M. Crépeau, 17042 La Rochelle, France

3 NAAM Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Contents

1 Introduction

In this paper, we are concerned with nonexistence of global weak solutions to the nonlinear pseudo-parabolic equation

$$u_t - \Delta_{\mathbb{H}} u_t - \Delta_{\mathbb{H}} u = |u|^p + f(t, \vartheta), \quad (t, \vartheta) \in (0, \infty) \times \mathbb{H}, \tag{1.1}$$

under the initial condition

$$u(0, \vartheta) = u_0(\vartheta), \quad \vartheta \in \mathbb{H}, \tag{1.2}$$

where $\Delta_{\mathbb{H}}$ is the Kohn–Laplace operator on the $(2N + 1)$-dimensional Heisenberg group $\mathbb{H}$, $p > 1$ and $f(t, \vartheta)$ is a given function. In the Euclidean case, this type of equations models a variety of important physical process, for example, the seepage of homogeneous fluids through a fissured rock [2], the unidirectional propagation of nonlinear dispersive long waves [3] and the aggregation of populations [13]. Furthermore, problem (1.1) and (1.2) can be regarded as a Sobolev type equation or Sobolev–Galpern type equation [16].

The critical Fujita exponent to the pseudo-parabolic equation (1.1) and (1.2) (with $f \equiv 0$) in the Euclidean case was determined as $p^* = 1 + \frac{2}{N}$ in recent years, i.e., by Kaikina et al. [10] for $p > p^*$ and Cao et al. [5] for $p \le p^*$. Yang et al. [17] extended the above results to the case of coupled nonlinear pseudo-parabolic equations.

In this paper, first we provide a sufficient condition for the nonexistence of global weak solutions to the nonlinear problem (1.1) and (1.2). Next, we extend this result to the case of 2×2 systems. More precisely, we consider two kinds of coupled nonlinear pseudo-parabolic equations. First, we consider the system

$$\begin{cases} u_t - \Delta_{\mathbb{H}} u_t - \Delta_{\mathbb{H}} u = |v|^q + f(t, \vartheta), & (t, \vartheta) \in (0, \infty) \times \mathbb{H}, \\ v_t - \Delta_{\mathbb{H}} v_t - \Delta_{\mathbb{H}} v = |u|^p + g(t, \vartheta), & (t, \vartheta) \in (0, \infty) \times \mathbb{H}, \\ u(0, \vartheta) = u_0(\vartheta), \ v(0, \vartheta) = v_0(\vartheta), & \vartheta \in \mathbb{H}, \end{cases} \tag{1.3}$$

where $p, q > 1$ and f, g are given functions, for which we provide a sufficient condition for the nonexistence of global weak solutions. Note that in the Euclidean case, Yang et al. [17] proved that the critical Fujita curve for this system (with $f = g \equiv 0$) is given by $(pq)^* = 1 + \frac{2}{N} \max\{p + 1, q + 1\}$. Next, we consider the system

$$\begin{cases} u_t - \Delta_{\mathbb{H}} u_t - \Delta_{\mathbb{H}} v = |v|^q + f(t,\vartheta), & (t,\vartheta) \in (0,\infty) \times \mathbb{H}, \\ v_t - \Delta_{\mathbb{H}} v_t - \Delta_{\mathbb{H}} u = |u|^p + g(t,\vartheta), & (t,\vartheta) \in (0,\infty) \times \mathbb{H}, \\ u(0,\vartheta) = u_0(\vartheta), \;\; v(0,\vartheta) = v_0(\vartheta), & \vartheta \in \mathbb{H}, \end{cases} \tag{1.4}$$

where $p, q > 1$ and f, g are given functions.

Before stating and proving our main results, let us recall some mathematical preliminaries used here.

The $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$ is the space $\mathbb{R}^{2N+1}$ endowed with the group operation

$$\vartheta \diamond \vartheta' = (x + x', y + y', \tau + \tau' + 2(x \cdot y' - x' \cdot y)),$$

for all $\vartheta = (x, y, \tau)$, $\vartheta' = (x', y', \tau') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^N$. This group operation endows $\mathbb{H}$ with the structure of a Lie group.

The distance from $\vartheta = (x, y, \tau) \in \mathbb{H}$ to the origin is given by

$$|\vartheta|_{\mathbb{H}} = \left(\tau^2 + \left(\sum_{i=1}^{N} \left(x_i^2 + y_i^2 \right) \right)^2 \right)^{1/4},$$

where $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$.

The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ can be defined from the vectors fields

$$X_i = \partial_{x_i} + 2y_i \partial_\tau \quad \text{and} \quad Y_i = \partial_{y_i} - 2x_i \partial_\tau,$$

for $i = 1, \ldots, N$, as follows

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} (X_i^2 + Y_i^2),$$

that is,

$$\Delta_{\mathbb{H}} u = \sum_{i=1}^{N} (\partial^2_{x_i x_i} u + \partial^2_{y_i y_i} u + 4y_i \partial^2_{x_i \tau} u - 4x_i \partial^2_{y_i \tau} u + 4(x_i^2 + y_i)^2 \partial^2_{\tau\tau} u).$$

For all $\vartheta, \vartheta' \in \mathbb{H}$, we have

$$\Delta_{\mathbb{H}}(u(\vartheta \diamond \vartheta')) = \Delta_{\mathbb{H}} u(\vartheta \diamond \vartheta').$$

For $\lambda \in \mathbb{R}$ and $(x, y, \tau) \in \mathbb{H}$, we have

$$\Delta_{\mathbb{H}}(u(\lambda x, \lambda y, \lambda^2 \tau)) = \lambda^2 (\Delta_{\mathbb{H}} u)(\lambda x, \lambda y, \lambda^2 \tau).$$

If $u(\vartheta) = v(|\vartheta|_{\mathbb{H}})$, then

$$\Delta_{\mathbb{H}} v(\rho) = a(\vartheta)\left(\frac{d^2 v}{d\rho^2} + \frac{Q-1}{\rho}\frac{dv}{d\rho}\right),$$

where $\rho = |\vartheta|_{\mathbb{H}}$, $a(\vartheta) = \rho^{-2}\sum_{i=1}^{N}(x_i^2 + y_i^2)$ and $Q = 2N+2$ is the homogeneous dimension of $\mathbb{H}$.

For more details on Heisenberg groups, we refer to [4,8,12]. For other nonexistence results in Heisenberg groups, we refer to [1,4,6,7,9,14,15,18].

2 Results and proofs

Let $\mathcal{H} = (0,\infty)\times\mathbb{H}$. For $R > 0$, let

$$\mathcal{U}_R = \{(t,x,y,\tau)\in\mathcal{H} :\ 0 \le t^2 + |x|^4 + |y|^4 + \tau^2 \le R^4\}.$$

2.1 The case of a single equation

Let $f \in L^1_{loc}(\mathcal{H})$. The definition of solutions we adopt for (1.1) and (1.2) is:

Definition 2.1 We say that u is a global weak solution to (1.1) and (1.2) on $\mathcal{H}$ with initial data $u(0,\cdot) = u_0 \in L^1_{loc}(\mathbb{H})$, if $u \in L^p_{loc}(\mathcal{H})$ and satisfies

$$\begin{aligned}
&\int_{\mathcal{H}} |u|^p \varphi\, dt\,d\vartheta + \int_{\mathbb{H}} u_0(\vartheta)\varphi(0,\vartheta)\,d\vartheta + \int_{\mathcal{H}} f\varphi\,dt\,d\vartheta\\
&\quad = -\int_{\mathcal{H}} u\varphi_t\,dt\,d\vartheta + \int_{\mathcal{H}} u(\Delta_{\mathbb{H}}\varphi)_t\,dt\,d\vartheta - \int_{\mathcal{H}} u\Delta_{\mathbb{H}}\varphi\,dt\,d\vartheta\\
&\qquad + \int_{\mathbb{H}} u_0(\vartheta)\Delta_{\mathbb{H}}\varphi(0,\vartheta)\,d\vartheta,
\end{aligned}$$

for any regular test function φ, $\varphi(\cdot,t) = 0$, $t \ge T$ (t is large enough).

Our first main result is given by the following theorem.

Theorem 2.2 *Let $u_0 \in L^1(\mathbb{H})$ and $f^- \in L^1(\mathcal{H})$, where $f^- = \max\{-f, 0\}$. Suppose that*

$$\int_{\mathbb{H}} u_0\,d\vartheta + \liminf_{R\to\infty}\int_{\mathcal{U}_R} f\,dt\,d\vartheta > 0. \tag{2.1}$$

If

$$1 < p \le p^* = 1 + \frac{2}{Q},$$

then (1.1) and (1.2) does not admit any global weak solution.

Proof Suppose that u is a global weak solution to (1.1) and (1.2). Then for any regular test function φ, we have

$$\begin{aligned}
&\int_{\mathcal{H}} |u|^p \varphi \, dt d\vartheta + \int_{\mathbb{H}} u_0(\vartheta)\varphi(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} f\varphi \, dt d\vartheta \\
&\le \int_{\mathcal{H}} |u||\varphi_t| \, dt d\vartheta + \int_{\mathcal{H}} |u||(\Delta_{\mathbb{H}}\varphi)_t| \, dt d\vartheta + \int_{\mathcal{H}} |u||\Delta_{\mathbb{H}}\varphi| \, dt d\vartheta \\
&\quad + \int_{\mathbb{H}} |u_0(\vartheta)||\Delta_{\mathbb{H}}\varphi(0,\vartheta)| \, d\vartheta.
\end{aligned} \tag{2.2}$$

Using the ε-Young inequality with parameters p and $p/(p-1)$, we obtain

$$\int_{\mathcal{H}} |u||\varphi_t| \, dt d\vartheta \le \varepsilon \int_{\mathcal{H}} |u|^p \varphi \, dt d\vartheta + c_\varepsilon \int_{\mathcal{H}} \varphi^{\frac{-1}{p-1}} |\varphi_t|^{\frac{p}{p-1}} \, dt d\vartheta, \tag{2.3}$$

for some positive constant c_ε.

Similarly, we have

$$\int_{\mathcal{H}} |u||(\Delta_{\mathbb{H}}\varphi)_t| \, dt d\vartheta \le \varepsilon \int_{\mathcal{H}} |u|^p \varphi \, dt d\vartheta + c_\varepsilon \int_{\mathcal{H}} \varphi^{\frac{-1}{p-1}} |(\Delta_{\mathbb{H}}\varphi)_t|^{\frac{p}{p-1}} \, dt d\vartheta \tag{2.4}$$

and

$$\int_{\mathcal{H}} |u||\Delta_{\mathbb{H}}\varphi| \, dt d\vartheta \le \varepsilon \int_{\mathcal{H}} |u|^p \varphi \, dt d\vartheta + c_\varepsilon \int_{\mathcal{H}} \varphi^{\frac{-1}{p-1}} |\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}} \, dt d\vartheta. \tag{2.5}$$

Using (2.2)–(2.5), for $\varepsilon > 0$ small enough, we get

$$\begin{aligned}
&\int_{\mathcal{H}} |u|^p \varphi \, dt d\vartheta + \int_{\mathbb{H}} u_0(\vartheta)\varphi(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} f\varphi \, dt d\vartheta \\
&\le C\left(A_p(\varphi) + B_p(\varphi) + C_p(\varphi) + \int_{\mathbb{H}} |u_0(\vartheta)||\Delta_{\mathbb{H}}\varphi(0,\vartheta)| \, d\vartheta\right),
\end{aligned} \tag{2.6}$$

where

$$A_p(\varphi) = \int_{\mathcal{H}} \varphi^{\frac{-1}{p-1}} |\varphi_t|^{\frac{p}{p-1}} \, dt d\vartheta, \tag{2.7}$$

$$B_p(\varphi) = \int_{\mathcal{H}} \varphi^{\frac{-1}{p-1}} |(\Delta_{\mathbb{H}}\varphi)_t|^{\frac{p}{p-1}} \, dt d\vartheta, \tag{2.8}$$

$$C_p(\varphi) = \int_{\mathcal{H}} \varphi^{\frac{-1}{p-1}} |\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}} \, dt d\vartheta. \tag{2.9}$$

Now, let us consider the test function

$$\varphi_R(t,\vartheta) = \phi^\omega\left(\frac{t^2 + |x|^4 + |y|^4 + \tau^2}{R^4}\right), \qquad R > 0, \ \omega \gg 1, \tag{2.10}$$

where $\phi \in C_0^\infty(\mathbb{R}^+)$ is a decreasing function satisfying

$$\phi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2. \end{cases}$$

Observe that $\mathrm{supp}(\varphi_R)$ is a subset of

$$\Omega_R = \{(t, x, y, \tau) \in \mathcal{H} : \ 0 \le t^2 + |x|^4 + |y|^4 + \tau^2 \le 2R^4\},$$

while $\mathrm{supp}(\varphi_{Rt})$, $\mathrm{supp}(\Delta_{\mathbb{H}}\varphi_R)$ and $\mathrm{supp}((\Delta_{\mathbb{H}}\varphi_R)_t)$ are subsets of

$$\Theta_R = \{(t, x, y, \tau) \in \mathcal{H} : \ R^4 \le t^2 + |x|^4 + |y|^4 + \tau^2 \le 2R^4\}.$$

Let

$$\rho = \frac{t^2 + |x|^4 + |y|^4 + \tau^2}{R^4}.$$

Then we have

$$\begin{aligned} \Delta_{\mathbb{H}}\varphi_R(t, \vartheta) = {} & \frac{4\omega(N+4)}{R^4}(|x|^2 + |y|^2)\phi'(\rho)\phi^{\omega-1}(\rho) \\ & + \frac{16\omega(\omega-1)}{R^8}((|x|^6 + |y|^6) + 2\tau(|x|^2 - |y|^2)x \cdot y + \tau^2(|x|^2 + |y|^2)) \\ & \times \phi'^2(\rho)\phi^{\omega-2}(\rho) \\ & + \frac{16\omega}{R^8}((|x|^6 + |y|^6) + 2\tau(|x|^2 - |y|^2)x \cdot y + \tau^2(|x|^2 + |y|^2)) \\ & \times \phi''(\rho)\phi^{\omega-1}(\rho) \end{aligned}$$

and

$$\begin{aligned} (\Delta_{\mathbb{H}}\varphi_R)_t(t, \vartheta) = {} & \frac{8\omega(N+4)t}{R^8}(|x|^2 + |y|^2)(\phi''(\rho)\phi(\rho) + (\omega-1)\phi'^2(\rho))\phi^{\omega-2}(\rho) \\ & + \frac{32\omega(\omega-1)t}{R^{12}}((|x|^6 + |y|^6) + 2\tau(|x|^2 - |y|^2)x \cdot y + \tau^2(|x|^2 + |y|^2)) \\ & \times (2\phi(\rho)\phi'(\rho)\phi''(\rho) + (\omega-2)\phi'^3(\rho))\phi^{\omega-3}(\rho) \\ & + \frac{32\omega t}{R^{12}}((|x|^6 + |y|^6) + 2\tau(|x|^2 - |y|^2)x \cdot y + \tau^2(|x|^2 + |y|^2)) \\ & \times (\phi(\rho)\phi'''(\rho) + (\omega-1)\phi'(\rho)\phi''(\rho))\phi^{\omega-2}(\rho). \end{aligned}$$

It follows that there is a positive constant $C > 0$, independent of R, such that for all $(t, \vartheta) \in \Omega_R$, we have

$$|\Delta_{\mathbb{H}}\varphi_R(t, \vartheta)| \le CR^{-2}\phi^{\omega-2}(\rho)\chi(\rho), \tag{2.11}$$

where

$$\chi(\rho) = |\phi'(\rho)|\phi(\rho) + \phi'^2(\rho) + |\phi''(\rho)|\phi(\rho),$$

and

$$|(\Delta_{\mathbb{H}}\varphi_R)_t(t,\vartheta)| \leq CR^{-4}\phi^{\omega-3}(\rho)\xi(\rho), \tag{2.12}$$

where

$$\begin{aligned}\xi(\rho) = \phi^2(\rho)|\phi''(\rho)| + \phi'^2(\rho)\phi(\rho) + \phi(\rho)|\phi'(\rho)||\phi''(\rho)| + |\phi'^3(\rho)| \\ + \phi^2(\rho)|\phi'''(\rho)|.\end{aligned}$$

Using (2.11) and (2.12), we get

$$B_p(\varphi_R) \leq CR^{\frac{-4p}{p-1}} \int_{\mathcal{H}} \phi^{\omega-\frac{3p}{p-1}}(\rho)\xi^{\frac{p}{p-1}}(\rho)\, dt d\vartheta, \tag{2.13}$$

$$C_p(\varphi_R) \leq CR^{\frac{-2p}{p-1}} \int_{\mathcal{H}} \phi^{\omega-\frac{2p}{p-1}}(\rho)\chi^{\frac{p}{p-1}}(\rho)\, dt d\vartheta. \tag{2.14}$$

Let us consider now the change of variables

$$(t, x, y, \tau) = (t, \vartheta) \mapsto (\widetilde{t}, \widetilde{v}) = (\widetilde{t}, \widetilde{x}, \widetilde{y}, \widetilde{\tau}), \tag{2.15}$$

where

$$\widetilde{t} = R^{-2}t, \quad \widetilde{x} = R^{-1}x,\ \widetilde{y} = R^{-1}y, \quad \widetilde{\tau} = R^{-2}\tau.$$

Let

$$\begin{aligned}\widetilde{\rho} &= \widetilde{t}^2 + |\widetilde{x}|^4 + |\widetilde{y}|^4 + \widetilde{\tau}^2,\\ \widetilde{\Theta} &= \{(\widetilde{t}, \widetilde{x}, \widetilde{y}, \widetilde{\tau}) \in \mathcal{H} :\ 1 \leq \widetilde{\rho} \leq 2\},\\ \Sigma_R &= \{(x, y, \tau) \in \mathbb{H} :\ R^4 \leq |x|^4 + |y|^4 + \tau^2 \leq 2R^4\},\end{aligned}$$

and

$$\begin{aligned}C_\phi = \max\Bigg\{&\int_{\widetilde{\Theta}} \phi^{\omega-\frac{p}{p-1}}(\widetilde{\rho})|\phi_{\widetilde{t}}^{\frac{p}{p-1}}(\widetilde{\rho})|\, d\widetilde{t} d\widetilde{\vartheta}, \int_{\widetilde{\Theta}} \phi^{\omega-\frac{3p}{p-1}}(\widetilde{\rho})\xi^{\frac{p}{p-1}}(\widetilde{\rho})\, d\widetilde{t} d\widetilde{\vartheta},\\ &\times \int_{\widetilde{\Theta}} \phi^{\omega-\frac{2p}{p-1}}(\widetilde{\rho})\chi^{\frac{p}{p-1}}(\widetilde{\rho})\, d\widetilde{t} d\widetilde{\vartheta}\Bigg\}.\end{aligned}$$

Using (2.6), (2.13) and (2.14), we obtain

$$\begin{aligned}&\int_{\mathcal{H}} |u|^p \varphi_R\, dt d\vartheta + \int_{\mathbb{H}} u_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} f\varphi_R\, dt d\vartheta\\ &\quad \leq CC_\phi\left(R^{\lambda_1} + R^{\lambda_2} + \int_{\Sigma_R} |u_0(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(0,\vartheta)|\, d\vartheta\right),\end{aligned} \tag{2.16}$$

where

$$\lambda_1 = Q + 2 - \frac{2p}{p-1} \quad \text{and} \quad \lambda_2 = Q + 2 - \frac{4p}{p-1}.$$

On the other hand, we have

$$\liminf_{R\to\infty} \left(\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta + \int_{\mathbb{H}} u_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} f\varphi_R \, dt d\vartheta \right)$$
$$\geq \liminf_{R\to\infty} \int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta + \liminf_{R\to\infty} \int_{\mathbb{H}} u_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta + \liminf_{R\to\infty} \int_{\mathcal{H}} f\varphi_R \, dt d\vartheta.$$

Using the monotone convergence theorem, we get

$$\liminf_{R\to\infty} \int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta = \int_{\mathcal{H}} |u|^p \, dt d\vartheta.$$

Since $u_0 \in L^1(\mathbb{H})$, by the dominated convergence theorem, we have

$$\liminf_{R\to\infty} \int_{\mathbb{H}} u_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta = \int_{\mathbb{H}} u_0(\vartheta)\, d\vartheta.$$

Writing $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$, we have

$$\int_{\mathcal{H}} f\varphi_R \, dt d\vartheta = \int_{\mathcal{U}_R} f \, dt d\vartheta + \int_{\Theta_R} f^+\varphi_R \, dt d\vartheta - \int_{\Theta_R} f^-\varphi_R \, dt d\vartheta$$
$$\geq \int_{\mathcal{U}_R} f \, dt d\vartheta - \int_{\Theta_R} f^-\varphi_R \, dt d\vartheta.$$

Since $f^- \in L^1(\mathcal{H})$, by the dominated convergence theorem we have

$$\lim_{R\to\infty} \int_{\Theta_R} f^-\varphi_R \, dt d\vartheta = 0.$$

Then

$$\liminf_{R\to\infty} \int_{\mathcal{H}} f\varphi_R \, dt d\vartheta \geq \liminf_{R\to\infty} \int_{\mathcal{U}_R} f \, dt d\vartheta.$$

Now, we have

$$\liminf_{R\to\infty} \left(\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta + \int_{\mathbb{H}} u_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} f\varphi_R \, dt d\vartheta \right)$$
$$\geq \int_{\mathcal{H}} |u|^p \, dt d\vartheta + \ell,$$

where form (2.1),

$$\ell=\int_{\mathbb{H}} u_0(\vartheta)\, d\vartheta+\liminf_{R\to\infty}\int_{\mathcal{U}_R} f\, dt d\vartheta>0.$$

By the definition of the limit inferior, for every $\varepsilon>0$, there exists $R_0>0$ such that

$$\begin{aligned}
&\int_{\mathcal{H}}|u|^p\varphi_R\, dt d\vartheta+\int_{\mathbb{H}} u_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta+\int_{\mathcal{H}} f\varphi_R\, dt d\vartheta\\
&\quad>\liminf_{R\to\infty}\left(\int_{\mathcal{H}}|u|^p\varphi_R\, dt d\vartheta+\int_{\mathbb{H}} u_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta+\int_{\mathcal{H}} f\varphi_R\, dt d\vartheta\right)-\varepsilon\\
&\quad\geq\int_{\mathcal{H}}|u|^p\, dt d\vartheta+\ell-\varepsilon,
\end{aligned}$$

for every $R\geq R_0$. Taking $\varepsilon=\ell/2$, we obtain

$$\begin{aligned}
&\int_{\mathcal{H}}|u|^p\varphi_R\, dt d\vartheta+\int_{\mathbb{H}} u_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta+\int_{\mathcal{H}} f\varphi_R\, dt d\vartheta\\
&\quad\geq\int_{\mathcal{H}}|u|^p\, dt d\vartheta+\frac{\ell}{2},
\end{aligned}$$

for every $R\geq R_0$. Then from (2.16), we have

$$\int_{\mathcal{H}}|u|^p\, dt d\vartheta+\frac{\ell}{2}\leq CC_\phi\left(R^{\lambda_1}+R^{\lambda_2}+\int_{\Sigma_R}|u_0(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(0,\vartheta)|\, d\vartheta\right), \quad (2.17)$$

for R large enough.

Now, we require that $\lambda_1=\max\{\lambda_1,\lambda_2\}\leq 0$, which is equivalent to $1<p\leq 1+\dfrac{2}{Q}$. We distinguish two cases.

- Case 1. If $1<p<1+\dfrac{2}{Q}$.

  In this case, letting $R\to\infty$ in (2.17) and using the dominated convergence theorem, we obtain

  $$\int_{\mathcal{H}}|u|^p\, dt d\vartheta+\frac{\ell}{2}\leq 0,$$

  which is a contradiction with $\ell>0$.

- Case 2. If $p=1+\dfrac{2}{Q}$.

  In this case, from (2.17), we obtain

  $$\int_{\mathcal{H}}|u|^p\, dt d\vartheta\leq C<\infty. \quad (2.18)$$

Using the Hölder inequality with parameters p and $p/(p-1)$, from (2.2), we obtain

$$\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta + \frac{\ell}{2} \le C \left(\int_{\Theta_R} |u|^p \varphi_R \, dt d\vartheta \right)^{\frac{1}{p}}.$$

Letting $R \to \infty$ in the above inequality and using (2.18), we obtain

$$\int_{\mathcal{H}} |u|^p \, dt d\vartheta + \frac{\ell}{2} = 0.$$

This contradiction completes the proof of the theorem.

□

2.2 The case of 2 × 2 systems

Let $f, g \in L^1_{loc}(\mathcal{H})$.

2.2.1 The case of system (1.3)

The definition of solutions we adopt for (1.3) is:

Definition 2.3 We say that the pair (u, v) is a global weak solution to (1.3) on $\mathcal{H}$ with initial data $(u(0,\cdot), v(0,\cdot)) = (u_0, v_0) \in L^1_{loc}(\mathbb{H}) \times L^1_{loc}(\mathbb{H})$, if $(u, v) \in L^p_{loc}(\mathcal{H}) \times L^q_{loc}(\mathcal{H})$ and satisfies

$$\begin{aligned} &\int_{\mathcal{H}} |v|^q \varphi \, dt d\vartheta + \int_{\mathbb{H}} u_0(\vartheta)\varphi(0,\vartheta) \, d\vartheta + \int_{\mathcal{H}} f\varphi \, dt d\vartheta \\ &\quad = -\int_{\mathcal{H}} u\varphi_t \, dt d\vartheta + \int_{\mathcal{H}} u(\Delta_{\mathbb{H}}\varphi)_t \, dt d\vartheta - \int_{\mathcal{H}} u\Delta_{\mathbb{H}}\varphi \, dt d\vartheta \\ &\qquad + \int_{\mathbb{H}} u_0(\vartheta)\Delta_{\mathbb{H}}\varphi(0,\vartheta) \, d\vartheta \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathcal{H}} |u|^p \varphi \, dt d\vartheta + \int_{\mathbb{H}} v_0(\vartheta)\varphi(0,\vartheta) \, d\vartheta + \int_{\mathcal{H}} g\varphi \, dt d\vartheta \\ &\quad = -\int_{\mathcal{H}} v\varphi_t \, dt d\vartheta + \int_{\mathcal{H}} v(\Delta_{\mathbb{H}}\varphi)_t \, dt d\vartheta - \int_{\mathcal{H}} v\Delta_{\mathbb{H}}\varphi \, dt d\vartheta \\ &\qquad + \int_{\mathbb{H}} v_0(\vartheta)\Delta_{\mathbb{H}}\varphi(0,\vartheta) \, d\vartheta, \end{aligned}$$

for any regular test function φ, $\varphi(\cdot, t) = 0$, $t \ge T$.

Our second main result is given by the following theorem.

Theorem 2.4 *Let* $(u_0, v_0) \in L^1(\mathbb{H}) \times L^1(\mathbb{H})$ *and* $(f^-, g^-) \in L^1(\mathcal{H}) \times L^1(\mathcal{H})$. *Suppose that*

$$\int_{\mathbb{H}} u_0 \, d\vartheta + \liminf_{R\to\infty} \int_{\mathcal{U}_R} f \, dt d\vartheta > 0$$

and

$$\int_{\mathbb{H}} v_0 \, d\vartheta + \liminf_{R\to\infty} \int_{\mathcal{U}_R} g \, dt d\vartheta > 0.$$

If $1 < pq \le (pq)^*$, *where*

$$(pq)^* = 1 + \frac{2}{Q} \max\{p+1, q+1\},$$

then there exists no nontrivial global weak solution to (1.3).

Proof Suppose that (u, v) is a nontrivial global weak solution to (1.3). Then for any regular test function φ, we have

$$\begin{aligned}
&\int_{\mathcal{H}} |v|^q \varphi \, dt d\vartheta + \int_{\mathbb{H}} u_0(\vartheta)\varphi(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} f\varphi \, dt d\vartheta \\
&\quad \le \int_{\mathcal{H}} |u||\varphi_t| \, dt d\vartheta + \int_{\mathcal{H}} |u||(\Delta_{\mathbb{H}}\varphi)_t| \, dt d\vartheta + \int_{\mathcal{H}} |u||\Delta_{\mathbb{H}}\varphi| \, dt d\vartheta \\
&\quad\quad + \int_{\mathbb{H}} |u_0(\vartheta)||\Delta_{\mathbb{H}}\varphi(0,\vartheta)| \, d\vartheta
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathcal{H}} |u|^p \varphi \, dt d\vartheta + \int_{\mathbb{H}} v_0(\vartheta)\varphi(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} g\varphi \, dt d\vartheta \\
&\quad \le \int_{\mathcal{H}} |v||\varphi_t| \, dt d\vartheta + \int_{\mathcal{H}} |v||(\Delta_{\mathbb{H}}\varphi)_t| \, dt d\vartheta + \int_{\mathcal{H}} |v||\Delta_{\mathbb{H}}\varphi| \, dt d\vartheta \\
&\quad\quad + \int_{\mathbb{H}} |v_0(\vartheta)||\Delta_{\mathbb{H}}\varphi(0,\vartheta)| \, d\vartheta.
\end{aligned}$$

Taking $\varphi = \varphi_R$, the test function given by (2.10), using the Hölder inequality with parameters p and $p/(p-1)$, we get

$$\begin{aligned}
&\int_{\mathcal{H}} |v|^q \varphi_R \, dt d\vartheta + \int_{\mathbb{H}} u_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} f\varphi_R \, dt d\vartheta \\
&\quad - \int_{\mathbb{H}} |u_0(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(0,\vartheta)| \, d\vartheta \\
&\quad \le \left(A_p(\varphi_R)^{\frac{p-1}{p}} + B_p(\varphi_R)^{\frac{p-1}{p}} + C_p(\varphi_R)^{\frac{p-1}{p}} \right) \left(\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta \right)^{\frac{1}{p}},
\end{aligned}$$

where $A_p(\varphi)$, $B_p(\varphi)$ and $C_p(\varphi)$ are given respectively by (2.7)–(2.9). Similarly, by the Hölder inequality with parameters q and $q/(q-1)$, we get

$$\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta + \int_{\mathbb{H}} v_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} g\varphi_R \, dt d\vartheta - \int_{\mathbb{H}} |v_0(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(0,\vartheta)|\, d\vartheta \leq \left(A_q(\varphi_R)^{\frac{q-1}{q}} + B_q(\varphi_R)^{\frac{q-1}{q}} + C_q(\varphi_R)^{\frac{q-1}{q}} \right) \left(\int_{\mathcal{H}} |v|^q \varphi_R \, dt d\vartheta \right)^{\frac{1}{q}}.$$

Without restriction of the generality, we may assume that for R large enough, we have

$$\int_{\mathbb{H}} u_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} f\varphi_R \, dt d\vartheta - \int_{\mathbb{H}} |u_0(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(0,\vartheta)|\, d\vartheta \geq 0$$

and

$$\int_{\mathbb{H}} v_0(\vartheta)\varphi_R(0,\vartheta)\, d\vartheta + \int_{\mathcal{H}} g\varphi_R \, dt d\vartheta - \int_{\mathbb{H}} |v_0(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(0,\vartheta)|\, d\vartheta \geq 0.$$

Slight modifications yield the proof in the general case (see the proof of Theorem 2.2). Then for R large enough, we have

$$\int_{\mathcal{H}} |v|^q \varphi_R \, dt d\vartheta \leq (A_p(\varphi_R)^{\frac{p-1}{p}} + B_p(\varphi_R)^{\frac{p-1}{p}} + C_p(\varphi_R)^{\frac{p-1}{p}}) \left(\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta \right)^{\frac{1}{p}} \tag{2.19}$$

and

$$\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta \leq (A_q(\varphi_R)^{\frac{q-1}{q}} + B_q(\varphi_R)^{\frac{q-1}{q}} + C_q(\varphi_R)^{\frac{q-1}{q}}) \left(\int_{\mathcal{H}} |v|^q \varphi_R \, dt d\vartheta \right)^{\frac{1}{q}}. \tag{2.20}$$

Using the change of variables (2.15), from (2.19) and (2.20), we obtain

$$\int_{\mathcal{H}} |v|^q \varphi_R \, dt d\vartheta \leq C R^{\frac{Q(p-1)-2}{p}} \left(\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta \right)^{\frac{1}{p}} \tag{2.21}$$

and

$$\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta \leq C R^{\frac{Q(q-1)-2}{q}} \left(\int_{\mathcal{H}} |v|^q \varphi_R \, dt d\vartheta \right)^{\frac{1}{q}}. \tag{2.22}$$

Combining (2.21) with (2.22), we obtain

$$\left(\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta \right)^{1-\frac{1}{pq}} \leq C R^{\upsilon_1} \tag{2.23}$$

and

$$\left(\int_{\mathcal{H}} |v|^q \varphi_R \, dt d\vartheta\right)^{1-\frac{1}{pq}} \leq C R^{\upsilon_2}, \tag{2.24}$$

where

$$\upsilon_1 = \frac{Q(pq-1) - 2(p+1)}{pq} \quad \text{and} \quad \upsilon_2 = \frac{Q(pq-1) - 2(q+1)}{pq}.$$

We require that $\upsilon_1 \leq 0$ or $\upsilon_2 \leq 0$ which is equivalent to $1 < pq \leq 1 + \frac{2}{Q} \max\{p+1, q+1\}$. We distinguish two case.

- Case 1. If $1 < pq < 1 + \frac{2}{Q} \max\{p+1, q+1\}$.
  Without restriction of the generality, we may suppose that $0 < q \leq p$. In this case, letting $R \to \infty$ in (2.23), we obtain

$$\int_{\mathcal{H}} |u|^p \, dt d\vartheta = 0,$$

  which is a contradiction.
- Case 2. If $pq = 1 + \frac{2}{Q} \max\{p+1, q+1\}$.
  This case can be treated in the same way as in the proof of Theorem 2.2.

Remark 2.5 If $p = q$ and $u = v$ in Theorem 2.4, we obtain the result for a single equation given by Theorem 2.2.

2.2.2 The case of system (1.4)

The definition of solutions we adopt for (1.4) is:

Definition 2.6 We say that the pair (u, v) is a global weak solution to (1.3) on $\mathcal{H}$ with initial data $(u(0, \cdot), v(0, \cdot)) = (u_0, v_0) \in L^1_{loc}(\mathbb{H}) \times L^1_{loc}(\mathbb{H})$, if $(u, v) \in L^p_{loc}(\mathcal{H}) \times L^q_{loc}(\mathcal{H})$ and satisfies

$$\begin{aligned}
&\int_{\mathcal{H}} |v|^q \varphi \, dt d\vartheta + \int_{\mathbb{H}} u_0(\vartheta)\varphi(0, \vartheta) \, d\vartheta + \int_{\mathcal{H}} f\varphi \, dt d\vartheta \\
&\quad = -\int_{\mathcal{H}} u\varphi_t \, dt d\vartheta + \int_{\mathcal{H}} u(\Delta_{\mathbb{H}}\varphi)_t \, dt d\vartheta - \int_{\mathcal{H}} v\Delta_{\mathbb{H}}\varphi \, dt d\vartheta \\
&\qquad + \int_{\mathbb{H}} u_0(\vartheta)\Delta_{\mathbb{H}}\varphi(0, \vartheta) \, d\vartheta
\end{aligned}$$

and

$$\int_{\mathcal{H}} |u|^p \varphi \, dt d\vartheta + \int_{\mathbb{H}} v_0(\vartheta)\varphi(0, \vartheta) \, d\vartheta + \int_{\mathcal{H}} g\varphi \, dt d\vartheta$$

$$= -\int_{\mathcal{H}} v\varphi_t \, dt d\vartheta + \int_{\mathcal{H}} v(\Delta_{\mathbb{H}}\varphi)_t \, dt d\vartheta - \int_{\mathcal{H}} u\Delta_{\mathbb{H}}\varphi \, dt d\vartheta$$
$$+ \int_{\mathbb{H}} v_0(\vartheta)\Delta_{\mathbb{H}}\varphi(0, \vartheta) \, d\vartheta,$$

for any regular test function φ, $\varphi(\cdot, t) = 0$, $t \geq T$.

We have the following result.

Theorem 2.7 *Let* $(u_0, v_0) \in L^1(\mathbb{H}) \times L^1(\mathbb{H})$ *and* $(f^-, g^-) \in L^1(\mathcal{H}) \times L^1(\mathcal{H})$. *Suppose that*

$$\int_{\mathbb{H}} u_0 \, d\vartheta + \liminf_{R\to\infty} \int_{\mathcal{U}_R} f \, dt d\vartheta > 0$$

and

$$\int_{\mathbb{H}} v_0 \, d\vartheta + \liminf_{R\to\infty} \int_{\mathcal{U}_R} g \, dt d\vartheta > 0.$$

If

$$Q \leq 2\max\{Q_1, Q_2\}, \tag{2.25}$$

where

$$Q_1 = \min\left\{\frac{1}{p-1}, \frac{1}{q-1}\left(1 + \frac{q^2}{q+1}\right), \frac{2}{q+1}\left(\frac{q^2}{q-1} + \frac{p}{p-1}\right) - 1\right\},$$
$$Q_2 = \min\left\{\frac{1}{q-1}, \frac{1}{p-1}\left(1 + \frac{p^2}{p+1}\right), \frac{2}{p+1}\left(\frac{p^2}{p-1} + \frac{q}{q-1}\right) - 1\right\},$$

then there exists no nontrivial global weak solution to (1.4).

Proof Suppose that (u, v) is a nontrivial weak solution to (1.4). We continue to use the same notations of the proof of Theorem 2.4. By proceeding in the same manner as in the proof of Theorem 2.4, for R large enough, we obtain

$$X^p \leq C_p(\varphi_R)^{\frac{p-1}{p}} X + D_q(\varphi_R) Y$$

and

$$Y^q \leq D_p(\varphi_R) X + C_q(\varphi_R)^{\frac{q-1}{q}} Y,$$

where

$$X = \left(\int_{\mathcal{H}} |u|^p \varphi_R \, dt d\vartheta\right)^{\frac{1}{p}}, \quad Y = \left(\int_{\mathcal{H}} |v|^q \varphi_R \, dt d\vartheta\right)^{\frac{1}{q}},$$
$$D_p(\varphi_R) = A_p(\varphi_R)^{\frac{p-1}{p}} + B_p(\varphi_R)^{\frac{p-1}{p}}.$$

Using Lemma 3 in [11], we obtain

$$X^{pq} \leq C\left(C_p(\varphi_R)^q + C_q(\varphi_R)D_q(\varphi_R)^q + (D_q(\varphi_R)^q D_p(\varphi_R))^{\frac{pq}{pq-1}}\right)$$

and

$$Y^{pq} \leq C\left(C_q(\varphi_R)^p + C_p(\varphi_R)D_p(\varphi_R)^p + (D_p(\varphi_R)^p D_q(\varphi_R))^{\frac{pq}{pq-1}}\right).$$

Using the change of variables (2.15), we get

$$X^{pq} \leq C(R^{\rho_1} + R^{\rho_2} + R^{\rho_3}),$$

where

$$\rho_1 = q\left(Q + 2 - \frac{2p}{p-1}\right),$$
$$\rho_2 = (Q+2)(q+1) - \frac{2q}{q-1}(2q+1),$$
$$\rho_3 = \frac{pq}{pq-1}\left((Q+2)(q+1) - \frac{4p}{p-1} - \frac{4q^2}{q-1}\right),$$

and

$$Y^{pq} \leq C(R^{\nu_1} + R^{\nu_2} + R^{\nu_3}),$$

where

$$\nu_1 = p\left(Q + 2 - \frac{2q}{q-1}\right),$$
$$\nu_2 = (Q+2)(p+1) - \frac{2p}{p-1}(2p+1),$$
$$\nu_3 = \frac{pq}{pq-1}\left((Q+2)(p+1) - \frac{4q}{q-1} - \frac{4p^2}{p-1}\right).$$

As in the previous proof, to get a contradiction, we have just to take $\max\{\rho_1, \rho_2, \rho_3\} \leq 0$ or $\max\{\nu_1, \nu_2, \nu_3\} \leq 0$, which is equivalent to (2.25). This ends the proof of Theorem 2.7. □

Remark 2.8 If $p = q$ and $u = v$ in Theorem 2.7, we obtain the result for a single equation given by Theorem 2.2.

Acknowledgments M. Jleli and B. Samet would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding of this research through the Research Group Project No RGP-1435-034.

References

1. Azman, I., Jleli, M., Samet, B.: Blow-up of solutions to parabolic inequalities in the Heisenberg group. Electron. J. Differ. Equ. **167**, 1–9 (2015)
2. Barenblatt, G.I., Zheltov, IuP, Kochina, I.N.: Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks. J. Appl. Math. Mech. **24**(5), 1286–1303 (1960)
3. Benjamin, T.B., Bona, J.L., Mahony, J.J.: Model equations for long waves in nonlinear dispersive systems. Philos. Trans. R. Soc. Lond. Ser. A **272**(1220), 47–78 (1972)
4. Birindelli, I., Capuzzo-Dolcetta, I., Cutri, A.: Liouville theorems for semilinear equations on the Heisenberg group. Ann. Inst. Henri Poincaré **14**, 295–308 (1997)
5. Cao, Y., Yin, J., Wang, C.: Cauchy problems of semilinear pseudo-parabolic equations. J. Differ. Equ. **246**, 4568–4590 (2009)
6. D'Ambrosio, L.: Critical degenerate inequalities on the Heisenberg group. Manuscr. Math. **106**(4), 519–536 (2001)
7. El Hamidi, A., Kirane, M.: Nonexistence results of solutions to systems of semilinear differential inequalities on the Heisenberg group. Abstr. Appl. Anal. **2004**(2), 155–164 (2004)
8. Folland, G.B., Stein, E.M.: Estimate for the $\partial_{\mathbb{H}}$ complex and analysis on the Heisenberg group. Commun. Pure Appl. Math. **27**, 492–522 (1974)
9. Jleli, M., Kirane, M., Samet, B.: Nonexistence results for a class of evolution equations in the Heisenberg group. Fract. Calc. Appl. Anal. **18**(3), 717–734 (2015)
10. Kaikina, E.I., Naumkin, P.I., Shishmarev, I.A.: The Cauchy problem for a Sobolev type equation with power like nonlinearity. Izv. Math. **69**, 59–111 (2005)
11. Kirane, M., Qafsaoui, M.: Global nonexistence for the Cauchy problem of some nonlinear reaction–diffusion systems. J. Math. Anal. Appl. **268**, 217–243 (2002)
12. Lanconelli, E., Uguzzoni, F.: Asymptotic behaviour and non existence theorems for semilinear Dirichlet problems involving critical exponent on unbounded domains of the Heisenberg group. Boll. Un. Mat. Ital. **1**(1), 139–168 (1998)
13. Padron, V.: Effect of aggregation on population recovery modeled by a forward–backward pseudoparabolic equation. Trans. Am. Math. Soc. **356**(7), 2739–2756 (2004)
14. Pascucci, A.: Semilinear equations on nilpotent Lie groups: global existence and blow-up of solutions. Matematiche **53**(2), 345–357 (1998)
15. Pohozaev, S.I., Véron, L.: Nonexistence results of solutions of semilinear differential inequalities on the Heisenberg group. Manuscr. Math. **102**, 85–99 (2000)
16. Sobolev, S.L.: On a new problem of mathematical physics. Izv. Akad. Nauk SSSR Ser. Math. **18**, 3–50 (1954)
17. Yang, J.G., Cao, Y., Zheng, S.N.: Fujita phenomena in nonlinear pseudo-parabolic system. Sci China Math. **57**, 555–568 (2014)
18. Zhang, Q.S.: The critical exponent of a reaction diffusion equation on some Lie groups. Math. Z. **228**(1), 51–72 (1998)