Monatsh Math (2016) 180:357–371
DOI 10.1007/s00605-015-0822-8

Annihilator methods for spectral synthesis on locally compact Abelian groups

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Received: 29 December 2014 / Accepted: 21 September 2015 / Published online: 5 October 2015

Abstract Spectral analysis and spectral synthesis study translation invariant linear function spaces on Abelian groups. Basic function classes for this study are the exponential monomials. These function classes have been investigated on discrete Abelian groups successfully using the annihilator method. In this paper we extend this technique to non-discrete locally compact Abelian groups.

Keywords Group algebra · Spectral analysis · Spectral synthesis · Annihilator · Exponential monomial

Mathematics Subject Classification 43A45 · 22D15

1 Introduction

Spectral analysis and spectral synthesis deal with the description of different varieties. One of the fundamental theorems on this field is due to Laurent Schwartz [1]. Recently several new results on spectral analysis and spectral synthesis have been found on discrete Abelian groups (see [2,3]). In [4] the author formulated problems and proved results concerning spectral synthesis on locally compact Abelian groups. In [5] we made an attempt to formulate and study the basic problems of spectral analysis and

Communicated by K. Gröchenig.

The research was partly supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. K111651.

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spectral synthesis in the non-commutative non-discrete setting. In a former paper [6] we introduced a method of studying spectral synthesis problems using annihilators of varieties on discrete Abelian groups (see also [7]). Here we extend this method to non-discrete locally compact Abelian groups.

In this paper $\mathbb{C}$ denotes the set of complex numbers. For a locally compact Abelian group G we denote by $\mathcal{C}(G)$ the locally convex topological vector space of all continuous complex valued functions defined on G, equipped with the pointwise operations and with the topology of uniform convergence on compact sets. For each function f in $\mathcal{C}(G)$ we define $\check{f}$ by $\check{f}(x) = f(-x)$, whenever x is in G. For a subset H in $\mathcal{C}(G)$ we denote by $\check{H}$ the set of all functions $\check{f}$ with f in H. By a *ring* we always mean a commutative ring with unit.

It is known that the dual of $\mathcal{C}(G)$ can be identified with the space $\mathcal{M}_c(G)$ of all compactly supported complex Borel measures on G which is equipped with the pointwise operations and with the weak*-topology. The pairing between $\mathcal{C}(G)$ and $\mathcal{M}_c(G)$ is given by the formula

$$\langle \mu, f \rangle = \int f \, d\mu.$$

The following theorem, describing the dual of $\mathcal{M}_c(G)$ is fundamental. The proof can be found in [8], 17.6, p. 155 (see also [7], Theorem 3.43, p. 48).

Theorem 1 *Let G be a locally compact Abelian group. For every weak*-continuous linear functional $F : \mathcal{M}_c(G) \to \mathbb{C}$ there exists a continuous function $f : G \to \mathbb{C}$ such that $F(\mu) = \mu(f)$ for each μ in $\mathcal{M}_c(G)$.*

In fact, the function f in this theorem is uniquely determined by F, as it is clear from the following result.

Theorem 2 *Let G be a locally compact Abelian group. The finitely supported complex measures form a weak*-closed subspace in $\mathcal{M}_c(G)$.*

Proof Let X be the weak*-closure of the linear space of all finitely supported complex measures in $\mathcal{M}_c(G)$. Assuming that X is a proper subspace, by the Hahn–Banach Theorem, there exists a nonzero weak*-continuous linear functional $F : \mathcal{M}_c \to \mathbb{C}$ vanishing on X. In particular, $F(\delta_y) = 0$ for each y in G. However, by the previous theorem, there exists a continuous function $f : G \to \mathbb{C}$ such that $f(y) = \delta_y(f) = F(\delta_y) = 0$ for each y in G, which is a contradiction, and our theorem is proved. □

For each μ in $\mathcal{M}_c(G)$ we define $\check{\mu}$ by the equation $\check{\mu}(f) = \mu(\check{f})$ whenever f is in $\mathcal{C}(G)$. For every subset K in $\mathcal{M}_c(G)$ the symbol $\check{K}$ denotes the set of all measures of the form $\check{\mu}$ with μ in K. The *orthogonal complement* of the subset H in $\mathcal{C}(G)$ is the set of all measures μ in $\mathcal{M}_c(G)$ satisfying $\mu(f) = 0$ for each f in H, and it is denoted by $H^\perp$.The dual concept is the orthogonal complement of a set K in $\mathcal{M}_c(G)$ of all functions f in $\mathcal{C}(G)$ satisfying $\mu(f) = 0$ for every μ in K, and it is denoted by $K^\perp$. Obviously, $H^\perp$, resp. $K^\perp$ is a closed subspace in $\mathcal{M}_c(G)$, resp. in $\mathcal{C}(G)$.

Convolution on $\mathcal{M}_c(G)$ is defined by

$$\int f\, d(\mu * \nu) = \int f(x+y)\, d\mu(x)\, d\nu(y)$$

for each μ, ν in $\mathcal{M}_c(G)$ and x in G. Convolution converts the linear space $\mathcal{M}_c(G)$ into a commutative topological algebra with unit δ_0, 0 being the zero in G.

We also define convolution of measures in $\mathcal{M}_c(G)$ with arbitrary functions in $\mathcal{C}(G)$ by the similar formula

$$f * \mu(x) = \int f(x-y)\, d\mu(y)$$

for each μ in $\mathcal{M}_c(G)$, f in $\mathcal{C}(G)$ and x in G. The linear operators $f \mapsto \mu * f$ on $\mathcal{C}(G)$ are called *convolution operators*. It is easy to see that equipped with the action $f \mapsto f * \mu$ the space $\mathcal{C}(G)$ is a topological module over $\mathcal{M}_c(G)$. For each subset K in $\mathcal{M}_c(G)$ and H in $\mathcal{C}(G)$ we use the notation

$$KH = \{f * \mu : \mu \in K, f \in H\}.$$

For each subset H in $\mathcal{C}(G)$ the *annihilator* of H in $\mathcal{M}_c(G)$ is the set

$$\operatorname{Ann} H = \{\mu : f * \mu = 0 \quad \text{for each} \quad f \in H\}.$$

We also define the dual concept: for every subset K in $\mathcal{M}_c(G)$ the *annihilator* of K in $\mathcal{C}(G)$ is the set

$$\operatorname{Ann} K = \{f : f * \mu = 0 \quad \text{for each} \quad \mu \in K\}.$$

Translation with the element y in G is the operator mapping the function f in $\mathcal{C}(G)$ onto its *translate* $\tau_y f$ defined by $\tau_y f(x) = f(x+y)$ for each x in G. Clearly, τ_y is a convolution operator, namely, it is the convolution with the measure δ_{-y}: we have $\tau_y f = f * \delta_{-y}$. A subset of $\mathcal{C}(G)$ is called *translation invariant*, if it contains all translates of its elements. A closed linear subspace of $\mathcal{C}(G)$ is called a *variety* on G, if it is translation invariant. Obviously, varieties are exactly the closed submodules in $\mathcal{C}(G)$. As it is easy to see, $\check{V}$ is a variety whenever V is a variety. For each function f the smallest variety containing f is called the *variety generated by* f, or simply the *variety of* f, and it is denoted by $\tau(f)$, which is obviously the intersection of all varieties containing f.

Theorem 3 *For each variety V in $\mathcal{C}(G)$ its annihilator* Ann V *is a closed ideal in $\mathcal{M}_c(G)$, and* $\operatorname{Ann} V = (\check{V})^{\perp}$. *Similarly, for each ideal I in $\mathcal{M}_c(G)$ its annihilator* Ann I *is a variety in $\mathcal{C}(G)$, and* $\operatorname{Ann} I = (\check{I})^{\perp}$.

Proof Clearly, Ann V is a closed subspace in $\mathcal{M}_c(G)$. For each μ in Ann V, ν in $\mathcal{M}_c(G)$ and f in V we have

$$(\nu * \mu) * f(x) = \int f(x-y)\, d(\nu * \mu)(y)$$
$$= \int \int f(x-u-v)\, d\mu(v) d\nu(u) = \int (f * \mu)(x-u)\, d\nu(u) = 0,$$

as $f * \mu = 0$. This means $\nu * \mu$ is in Ann V, and Ann V is a closed ideal in $\mathcal{M}_c(G)$. On the other hand, we have

$$\mu(\check{f}) = f * \mu(0) = 0,$$

hence μ is in $(\check{V})^{\perp}$. Conversely, if ν is in $(\check{V})^{\perp}$, then for each f in V we have

$$f * \nu(x) = \int f(x-y)\, d\nu(y) = \nu(\tau_{-x}\check{f}) = 0,$$

as $\tau_{-x}\check{f}$ is in $\check{V}$. It follows that μ is in Ann V.

For the dual statement it is clear that Ann I is a closed subspace in $\mathcal{C}(G)$. Moreover, if f is in Ann I, y is in G and μ is in I, then $\delta_{-y} * \mu$ is in I, hence we have

$$\tau_y f * \mu = (f * \delta_{-y}) * \mu = f * (\delta_{-y} * \mu) = 0,$$

and we infer that $\tau_y f$ is in Ann I, hence Ann I is a variety. On the other hand, we have

$$\check{\mu}(f) = \int f(y)\, d\check{\mu}(y) = \int f(-y)\, d\mu(y) = f * \mu(0) = 0,$$

as f annihilates μ. This means that Ann $I \subseteq (\check{I})^{\perp}$. Conversely, if g is in $\mathcal{C}(G)$ with the property that $\check{\mu}(g) = 0$ for each μ in I, then $\mu * \check{g}(0) = 0$ for each μ in I. As I is an ideal, this implies $(\delta_{-x} * \mu)(\check{g}) = 0$ for each x in G, hence

$$0 = \int \int \check{g}(t+y)\, d\delta_{-x}(t)\, d\mu(y) = \int g(x-y)\, d\mu(y) = g * \mu(x),$$

that is g is in Ann I, which proves Ann $I = (\check{I})^{\perp}$, and the proof is complete. □

Theorem 4 *For each variety* $V \subseteq W$ *in* $\mathcal{C}(G)$ *we have* Ann $V \supseteq$ Ann W *and for each ideal* $I \subseteq J$ *in* $\mathcal{M}_c(G)$ *we have* Ann $I \supseteq$ Ann J. *In addition, we have* Ann (Ann V) $= V$ *and* Ann (Ann I) $\supseteq I$. *In particular,* $V \neq W$ *implies* Ann $V \neq$ Ann W.

Proof Let $V \subseteq W$ be varieties in $\mathcal{C}(G)$ and let $I \subseteq J$ be ideals in $\mathcal{M}_c(G)$. For every μ in Ann W and for each f in V we have that f is in W, hence $f * \mu = 0$. This proves that μ is in Ann V, and Ann $V \supseteq$ Ann W. Similarly, if f is in Ann J and μ is in I, then μ is in J, hence $f * \mu = 0$, which proves that f is in Ann I, and Ann $I \supseteq$ Ann J.

Assume that f is in V and μ is in Ann V, then, by definition, $f*\mu = 0$, hence f is in Ann (Ann V), which proves Ann (Ann V) $\supseteq V$. Similarly, we have Ann (Ann I) $\supseteq I$.

Suppose now that Ann (Ann V) $\subsetneq V$. Consequently, there is a function f in Ann (Ann V) such that f is not in V. By the Hahn–Banach Theorem, there is a λ in $\mathcal{M}_c(G)$ such that $\check{\lambda}(f) \neq 0$, and $\check{\lambda}$ vanishes on V. This means

$$(\varphi * \lambda)(0) = \int \varphi(-y)\, d\lambda(y) = \lambda(\check{\varphi}) = \check{\lambda}(\varphi) = 0,$$

whenever φ is in V. As V is a variety, this implies, by the previous theorem, that λ is in Ann V, in particular, $f * \lambda = 0$, a contradiction. This proves Ann (Ann V) $= V$, which also implies Ann $V \neq$ Ann W, whenever $V \neq W$. □

We note that for ideals in $\mathcal{M}_c(G)$ the equality Ann (Ann I) $= I$ does not hold in general. For this, by Theorem 3, it is enough to show that $I^{\perp\perp} = I$ does not hold, in general. The following example can be found in [3].

Consider $G = \mathbb{R}$ with the usual topology, and let I denote the ideal generated by the measures $\mu_n = \delta_0 - \delta_{1/n}$ for $n = 1, 2, \ldots$. If f is in $I^{\perp}$, then f is periodic mod $1/n$ for every n, and thus, by continuity, f must be constant. Therefore $\delta_0 - \delta_\alpha$ is in $I^{\perp\perp}$ for each α in $\mathbb{R}$. However, $\delta_0 - \delta_\alpha$ is not in I if α is irrational. Indeed, for every positive integer N there is a continuous function f such that f is *periodic* mod $1/n$ for each $n \leq N$ in $\mathbb{N}$ but f is not periodic mod α. This implies immediately that $\delta_0 - \delta_\alpha$ does not belong to the ideal generated by μ_n for n in $\mathbb{N}$. However, if $\delta_0 - \delta_\alpha$ is I, then $\delta_0 - \delta_\alpha$ belongs to an ideal generated by finitely many of the measures μ_n, which is not the case.

Nevertheless, the following theorem holds true (see [3]).

Theorem 5 *Let G be a discrete Abelian group. Then* Ann (Ann I) $= I$ *holds for every ideal I in* $\mathcal{M}_c(G)$.

We need the following lemma.

Lemma 1 *Let G be a locally compact group, and let I be an ideal in* $\mathcal{M}_c(G)$. *Then* Ann $\big($Ann (Ann I)$\big)$ = Ann I.

Proof Let V = Ann I, then V is a variety on G, hence, by Theorem 4, we have Ann (Ann V) $= V$. It follows Ann $I = V =$ Ann (Ann V) = Ann $\big($Ann (Ann I)$\big)$. □

Now we can prove the following theorem characterizing those ideals in $\mathcal{M}_c(G)$ which coincide with their second annihilator.

Theorem 6 *Let G be a locally compact group, and let I be an ideal in* $\mathcal{M}_c(G)$. *Then we have* Ann (Ann I) $= I$ *if and only if I is closed. Also, we have* $I^{\perp\perp} = I$ *if and only if I is closed.*

Proof By Theorem 3, the annihilator of each variety is closed, in particular, J = Ann (Ann I), as the annihilator of the variety Ann I, is closed, which proves the necessity of our condition.

Conversely, suppose that I is closed, and I is a proper subset of J. By Lemma 1, we have $\mathrm{Ann}\, J = \mathrm{Ann}\, I$. Let μ be in J such that μ is not in I. As the space $\mathcal{M}_c(G)$ with the weak*-topology is locally convex, hence, by the Hahn–Banach Theorem, there is a linear functional ξ in $\mathcal{M}_c(G)^*$, such that $\breve{\xi}$ vanishes on I and $\breve{\xi}(\mu) \neq 0$. It is known (see [8], 17.6, p. 155), that every weak*-continuous linear functional on a dual space arises from an element of the original space, that is, there is an f in $\mathcal{C}(G)$ with $\xi(\nu) = \nu(f)$ for each ν in $\mathcal{M}_c(G)$. We infer $\mu(\breve{f}) = \breve{\xi}(\mu) \neq 0$, and μ is in J, hence f is not in $(\breve{J})^{\perp} = \mathrm{Ann}\, J$. On the other hand, $\breve{\nu}(f) = \nu(\breve{f}) = \breve{\xi}(\nu) = 0$ for each ν in I, as $\breve{\xi}$ vanishes on I, which implies that f is in $(\breve{I})^{\perp} = \mathrm{Ann}\, I = \mathrm{Ann}\, J$, a contradiction.

The second statement is a consequence of Theorem 3. Our theorem is proved. □

Corollary 1 *Let G be a locally compact Abelian group. Then the mappings $V \leftrightarrow \mathrm{Ann}\, V$ and $V \leftrightarrow V^{\perp}$ set up one-to-one inclusion-reversing correspondences between the varieties in $\mathcal{C}(G)$ and the closed ideals in $\mathcal{M}_c(G)$.*

By this corollary, closed ideals have special importance. In particular, the following theorem describes a class of closed ideals.

Theorem 7 *Let G be a locally compact Abelian group. If V is a finite dimensional variety in $\mathcal{C}(G)$, then every ideal including $\mathrm{Ann}\, V$, or $V^{\perp}$ is closed in $\mathcal{M}_c(G)$.*

Proof By Theorem 3, it is enough to proof the statement for $I \supseteq V^{\perp}$. Obviously, $\mathcal{M}_c(G)/V^{\perp}$ can be identified with the dual V^* of V, which is a finite dimensional vector space, hence every subspace of it is closed. In particular, every ideal in $\mathcal{M}_c(G)/V^{\perp}$ is closed. The natural homomorphism F of $\mathcal{M}_c(G)$ onto $\mathcal{M}_c(G)/V^{\perp}$ is continuous and every ideal including $V^{\perp}$ in $\mathcal{M}_c(G)$ is the inverse image of an ideal in $\mathcal{M}_c(G)/V^{\perp}$ by F, hence it is closed. □

Theorem 8 *Let G be a locally compact group.*

1. *For each family $(V_\gamma)_{\gamma\in\Gamma}$ of varieties in $\mathcal{C}(G)$ we have*

$$\mathrm{Ann}\left(\sum_{\gamma\in\Gamma} V_\gamma\right) = \bigcap_{\gamma\in\Gamma} \mathrm{Ann}\, V_\gamma, \quad \left(\sum_{\gamma\in\Gamma} V_\gamma\right)^{\perp} = \bigcap_{\gamma\in\Gamma} V_\gamma^{\perp}.$$

2. *For each family $(I_\gamma)_{\gamma\in\Gamma}$ of ideals in $\mathcal{M}_c(G)$ we have*

$$\mathrm{Ann}\left(\sum_{\gamma\in\Gamma} I_\gamma\right) = \bigcap_{\gamma\in\Gamma} \mathrm{Ann}\, I_\gamma, \quad \left(\sum_{\gamma\in\Gamma} I_\gamma\right)^{\perp} = \bigcap_{\gamma\in\Gamma} I_\gamma^{\perp}.$$

We note that here $\sum_{\gamma\in\Gamma} V_\gamma$ denotes the *topological sum* of the family of varieties $(V_\gamma)_{\gamma\in\Gamma}$, that is, the closure of the union of the sums of finite subfamilies. However, $\sum_{\gamma\in\Gamma} I_\gamma$ denotes the *algebraic sum* of the family of ideals $(I_\gamma)_{\gamma\in\Gamma}$, that is, the ideal generated by the sums of finite subfamilies.

Proof If μ is in $\bigcap_{\gamma\in\Gamma}$ Ann V_γ, then μ annihilates each of the varieties V_γ, hence it annihilates every finite sum of these varieties, and, by continuity, μ annihilates the closure of the sums of finite subfamilies. Hence μ annihilates $\sum_{\gamma\in\Gamma} V_\gamma$.

Conversely, if μ annihilates $\sum_{\gamma\in\Gamma} V_\gamma$, then μ annihilates every subvariety of it, hence it belongs to each Ann V_γ. This proves the first half of the first statement. The second half is the consequence of Theorem 3.

To prove the second statement we take f in $(\sum_{\gamma\in\Gamma} I_\gamma)^\perp$. Then f is in the orthogonal complement of the sum of any finite subfamily of $(I_\gamma)_{\gamma\in\Gamma}$, in particular, it is in the orthogonal complement of each of these ideals. Hence it belongs to $I_\gamma^\perp$ for every γ.

For the reverse inclusion we take an f which is in the orthogonal complement of each ideal I_γ. Then clearly, every measure in the ideal generated by finite sums of these ideals vanishes on f, hence f is in $(\sum_{\gamma\in\Gamma} I_\gamma)^\perp$. This proves the second half of the second statement. The first half is the consequence of Theorem 3. □

2 Exponentials

A basic function class is formed by the joint eigenfunctions of all translation operators, that is, by those nonzero continuous functions $\varphi : G \to \mathbb{C}$ satisfying

$$\tau_y\varphi = m(y)\cdot\varphi \tag{1}$$

with some $m : G \to \mathbb{C}$, that is

$$\varphi(x * y) = m(y)\varphi(x) \tag{2}$$

for all x, y in G. It follows $\varphi(y) = \varphi(0)\cdot m(y)$ which implies that $\varphi(0) \neq 0$ and, by (2),

$$m(x+y) = m(x)m(y) \tag{3}$$

for each x, y in G. Nonzero continuous functions $m : G \to \mathbb{C}$ satisfying (3) for each x, y in G are called *exponentials*. Clearly, every exponential generates a one dimensional variety, and conversely, every one dimensional variety is generated by an exponential. Sometimes exponentials are called *generalized characters*.

Using translation one introduces *modified differences* in the following manner: for each continuous function f in $\mathcal{C}(G)$ and y in G we let

$$\Delta_{f;y} = \delta_{-y} - f(y)\delta_0.$$

Hence $\Delta_{f;y}$ is an element of $\mathcal{M}_c(G)$. Products of modified differences will be denoted in the following way: for each f in $\mathcal{C}(G)$, for every natural number n and for arbitrary $y_1, y_2, \ldots, y_{n+1}$ in G we let

$$\Delta_{f;y_1,y_2,\ldots,y_{n+1}} = \Pi_{i=1}^{n+1}\big(\delta_{-y_i} - f(y_i)\delta_0\big),$$

where Π denotes convolution. In the case $f \equiv 1$ we use the simplified notation Δ_y for $\Delta_{1;y}$ and we call it *difference*. Accordingly, we write $\Delta_{y_1,y_2,\ldots,y_{n+1}}$ for $\Delta_{1;y_1,y_2,\ldots,y_{n+1}}$

For a given f in $\mathcal{C}(G)$ the closed ideal generated by all modified differences of the form $\Delta_{f;y}$ with y in G is denoted by M_f. We have the following theorem.

Theorem 9 *Let G be a locally compact Abelian group and $f : G \to \mathbb{C}$ a continuous function. The ideal M_f is proper if and only if f is an exponential. In this case $M_f = \operatorname{Ann} \tau(f)$ is maximal, and $\mathcal{M}_c(G)/M_f$ is topologically isomorphic to the complex field.*

Proof As M_f is closed, by Theorem 6, we have $\operatorname{Ann}(\operatorname{Ann} M_f) = M_f$ and $M_f^{\perp\perp} = M_f$.

Suppose that f is an exponential. Then $f \neq 0$, and

$$\Delta_{f;y} * f(x) = f(x+y) - f(y)f(x) = 0$$

for each x, y in G, hence f is in $\operatorname{Ann} M_f$. As $\tau(f)$ consists of all constant multiples of f, we infer that $\tau(f) \subseteq \operatorname{Ann} M_f$. Moreover, if φ is in $\operatorname{Ann} M_f$, then we have

$$0 = \Delta_{f;y} * \varphi(x) = \varphi(x+y) - f(y)\varphi(x)$$

for each x, y in G. It follows $\varphi = \varphi(0) \cdot f$, hence φ is in $\tau(f)$. We conclude that $\tau(f) = \operatorname{Ann} M_f$, and $M_f = \operatorname{Ann} \tau(f)$.

We define the mapping $\Phi_f : \mathcal{M}_c(G) \to \mathbb{C}$ by

$$\Phi_f(\mu) = \mu(\check{f}) = \int f(-y)\, d\mu(y)$$

for each μ in $\mathcal{M}_c(G)$. Then Φ_f is a linear mapping, $\Phi_f(\delta_0) = 1$, and for each μ, ν in $\mathcal{M}_c(G)$ we have

$$\begin{aligned}\Phi_f(\mu * \nu) &= \int f(-x-y) d\mu(x)\, d\nu(y) \\ &= \int f(-x)\, d\mu(x) \int f(-y)\, d\nu(y) = \Phi_f(\mu) \cdot \Phi_f(\nu),\end{aligned}$$

hence Φ_f is an algebra homomorphism. Obviously, Φ_f maps $\mathcal{M}_c(G)$ onto $\mathbb{C}$, hence it is a multiplicative linear functional. We infer that $\operatorname{Ker} \Phi_f$ is a maximal ideal and $\mathcal{M}_c(G)/\operatorname{Ker} \Phi_f$ is isomorphic to the complex field $\mathbb{C}$. For each μ in $\operatorname{Ker} \Phi_f$ we have $\mu(\check{f}) = 0$, hence for each complex number c we have

$$cf * \mu(x) = c \int f(x-y)\, d\mu(y) = cf(x)\mu(\check{f}) = 0,$$

consequently μ is in $\operatorname{Ann} \tau(f) = M_f$. It follows $\operatorname{Ker} \Phi_f \subseteq M_f$, which implies that M_f is a maximal ideal. We also have that $\operatorname{Ker} \Phi_f$ is closed, hence Φ_f is continuous. As Φ_f is also open, we have that $\mathcal{M}_c(G)/M_f$ is topologically isomorphic to the complex field.

Finally, if M_f is proper, then $\operatorname{Ann} M_f$ is nonzero. Let $\varphi \neq 0$ be a function in $\operatorname{Ann} M_f$, then we have

$$0 = \Delta_{f;y} * \varphi(x) = \varphi(x + y) - f(y)\varphi(x),$$

and in the same way like above we conclude that f is an exponential. The theorem is proved. □

Given a ring R we call a maximal ideal M in R an *exponential maximal ideal*, if the residue ring R/M is isomorphic to the complex field. If R is a topological ring, then we require the isomorphism to be topological. From the above proof it is clear that if G is a locally compact Abelian group, then each exponential maximal ideal in $\mathcal{M}_c(G)$ is of the form $M_m = \operatorname{Ann} \tau(m)$ with some exponential m.

3 Fourier–Laplace transformation

Given the locally compact Abelian group G let $\widetilde{G}$ denote the set of all exponentials on G. Obviously, $\widetilde{G}$ is an Abelian group with respect to pointwise multiplication. We equip $\widetilde{G}$ with the compact-open topology which makes $\widetilde{G}$ a topological Abelian group. For every μ in $\mathcal{M}_c(G)$ we define the function $\widehat{\mu} : \widetilde{G} \to \mathbb{C}$ by

$$\widehat{\mu}(m) = \mu(\check{m}) = \int m(-y)\, d\mu(y)$$

whenever m is in $\widetilde{G}$. Obviously, $\widehat{\mu}(m) = m * \mu(0)$. Also we have $\widehat{\mu}(m) = \Phi_m(\mu)$, where Φ_m is defined in Theorem 9 with $m = f$. The function $\widehat{\mu}$ is called the *Fourier–Laplace transform* of μ and the mapping $\mu \mapsto \widehat{\mu}$ is called the *Fourier–Laplace transformation*.

Theorem 10 *Let G be a locally compact Abelian group. Then for each measure μ in $\mathcal{M}_c(G)$ its Fourier–Laplace transform $\widehat{\mu}$ is a continuous function on $\widetilde{G}$.*

Proof Let $(m_i)_{i\in I}$ be a generalized sequence in $\widetilde{G}$ converging to the exponential m in $\widetilde{G}$. Then $\check{\mu}_i \to \check{\mu}$ uniformly on the compact set $\operatorname{supp} \mu$, hence we have $\widehat{\mu}_i(m) \to \widehat{\mu}$, which proves that $\widehat{\mu}$ is continuous. □

Theorem 11 *Let G be a locally compact Abelian group. The Fourier–Laplace transformation $\mu \to \widehat{\mu}$ is a continuous injective algebra homomorphism of $\mathcal{M}_c(G)$ into $\mathcal{C}(\widetilde{G})$, the latter equipped with the pointwise linear operations and multiplication, and with the topology of pointwise convergence.*

Proof We use the notation

$$\mathcal{F}(\mu) = \widehat{\mu}$$

for each μ in $\mathcal{M}_c(G)$. Obviously, $\mathcal{F} : \mathcal{M}_c(G) \to \mathcal{C}(\widetilde{G})$ is a linear mapping. Suppose that $(\mu_\alpha)_{\alpha\in A}$ is a generalized sequence in $\mathcal{M}_c(G)$ converging to μ in the weak*-topology. Then for each m in $\widetilde{G}$ we have $\mu_\alpha(\check{m}) \to \mu(\check{m})$, that is $\widehat{\mu}_\alpha(m) \to \widehat{\mu}(m)$, which gives the continuity of $\mathcal{F}$. Finally, for μ, ν in $\mathcal{M}_c(G)$ and m in $\widetilde{G}$ we have

$$\begin{aligned}\mathcal{F}(\mu * \nu)(m) &= (\mu * \nu)(\check{m}) \\ &= \int m(-x-y)\, d\mu(x)\, d\nu(y) \\ &= \int m(-x)\, d\mu(x) \int m(-y)\, d\nu(y) = \mu(\check{m}) \cdot \nu(\check{m}) \\ &= \mathcal{F}(\mu)(m) \cdot \mathcal{F}(\nu)(m),\end{aligned}$$

hence $\mathcal{F}$ is an algebra homomorphism.

The injectivity of the Fourier–Laplace transformation follows from the injectivity of the Fourier transform (see e.g. [9, Section 1.5]). □

The range of the Fourier–Laplace transformation in $\mathcal{C}(\widetilde{G})$, that is the set of all Fourier–Laplace transforms will be denoted by $\mathcal{A}(G)$. This is a subalgebra of $\mathcal{C}(\widetilde{G})$, isomorphic to $\mathcal{M}_c(G)$, sometimes called the *Fourier algebra* of G.

4 Exponential monomials

Another important function class is the one formed by the solutions of the equation

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f = 0, \tag{4}$$

where m is an exponential, n is a natural number, $f : G \to \mathbb{C}$ is a continuous function and the equation is supposed to hold for every $y_1, y_2, \dots, y_{n+1}$ in G. The function f is called a *generalized exponential monomial*. In the case $n = 0$ we have that f is a constant multiple of the exponential m. We have the following result.

Theorem 12 *Let G be a locally compact Abelian group and $m : G \to \mathbb{C}$ an exponential. The continuous function $f : G \to \mathbb{C}$ satisfies Eq. (4) if and only if $M_m^{n+1} \subseteq \operatorname{Ann} \tau(f)$.*

Proof Obvious, as the modified differences $\Delta_{m;y_1,y_2,\dots,y_{n+1}}$ generate an ideal which is dense in M_m^{n+1}. □

Lemma 2 *Let G be a locally compact Abelian group and $f : G \to \mathbb{C}$ a nonzero continuous function. Then there exists at most one exponential m such that $M_m^{n+1} \subseteq \operatorname{Ann} \tau(f)$ holds for some natural number n.*

Proof As f is nonzero, $\operatorname{Ann} \tau(f)$ is a proper ideal. There is a maximal ideal M in $\mathcal{M}_c(G)$ such that $\operatorname{Ann} \tau(f) \subseteq M$. Suppose that $M_m^{n+1} \subseteq \operatorname{Ann} \tau(f)$ holds for some exponential m and natural number n. Then $M_m^{n+1} \subseteq M$. As M is maximal, it is also prime, hence we have $M_m \subseteq M$. By Theorem 9, M_m is also maximal, which implies $M_m = M$. It follows that M_m and M are unique with the given properties. □

By the previous lemma, for a given nonzero generalized exponential monomial f there is a unique exponential m such that Eq. (4) holds. We say that f is *associated* to the exponential m, and the smallest natural number n in (4) is called the *degree* of f.

Theorem 13 *Let G be a locally compact Abelian group. The continuous function $f : G \to \mathbb{C}$ is a generalized exponential monomial if and only if $\mathcal{M}_c(G)/\text{Ann}\,\tau(f)$ is a local ring with a nilpotent exponential maximal ideal.*

Proof Let $F : \mathcal{M}_c(G) \to \mathcal{M}_c(G)/\text{Ann}\,\tau(f)$ denote the natural homomorphism. If $f \neq 0$ is a generalized exponential monomial, then, by Theorem 12, we have

$$F(M_m)^{n+1} = F(M_m^{n+1}) \subseteq F(\text{Ann}\,\tau(f)) = 0,$$

hence the ideal $F(M_m)$ is nilpotent, and it is obviously maximal. Clearly, it is the unique maximal ideal in the residue ring $R = \mathcal{M}_c(G)/\text{Ann}\,\tau(f)$, by Lemma 2. Finally, we have

$$R/F(M_m) \cong \mathcal{M}_c(G)/M_m,$$

hence $F(M_m)$ is an exponential maximal ideal. The converse statement follows in the same way. □

We shall call a generalized exponential monomial simply an *exponential monomial*, if its variety is finite dimensional. We shall need the following lemma.

Lemma 3 *Let G be a locally compact Abelian group, $f : G \to \mathbb{C}$ a continuous function, m an exponential, and k a natural number. Then for each φ in $M_m^k \tau(f)$ the $\mathcal{M}_c(G)$-module generated by $\varphi + M_m^{k+1}$ in $M_m^k \tau(f)/M_m^{k+1}\tau(f)$ is one dimensional.*

Proof Let $\Phi : \mathcal{M}_c(G) \to \mathbb{C}$ be the multiplicative functional with the property $\text{Ker}\,\Phi = \text{Ann}\,\tau(m)$. For each y in G we have

$$\begin{aligned}\delta_{-y} * \big(\varphi + M_m^{k+1}\tau(f)\big) &= \delta_{-y} * \varphi + M_m^{k+1}\tau(f)\\ &= (\delta_{-y} - m(y)\delta_0)\varphi + m(y)\varphi + M_m^{k+1}\tau(f)\\ &= \Phi(\delta_{-y})\varphi + M_m^{k+1}\tau(f),\end{aligned}$$

as $\delta_{-y} - m(y)\delta_0$ is in M_m, hence $(\delta_{-y} - m(y)\delta_0)\varphi$ is in $M_m^{k+1}\tau(f)$. As each μ in $\mathcal{M}_c(G)$ is a weak*-limit of linear combinations of measures δ_{-y}, by continuity and linearity, we have

$$\mu * (\varphi + M_m^{k+1}\tau(f)) = \Phi(\mu)\cdot\varphi + M_m^{k+1}\tau(f) = \Phi(\mu)\big(\varphi + M_m^{k+1}\tau(f)\big).$$

which proves our statement. □

Theorem 14 *Let G be a locally compact Abelian group. The continuous function $f : G \to \mathbb{C}$ is an exponential monomial if and only if $\mathcal{M}_c(G)/\text{Ann}\,\tau(f)$ is a local Artin ring with exponential maximal ideal.*

Proof Suppose that $f \neq 0$ is an exponential monomial. Then there exists a unique exponential m such that the only maximal ideal containing $\text{Ann}\,\tau(f)$ is M_m. It follows that $\mathcal{M}_c(G)/\mathrm{Ann}\,\tau(f)$ is a local ring with the exponential maximal ideal $F(M_m)$, where $F : \mathcal{M}_c(G) \to \mathcal{M}_c(G)/\mathrm{Ann}\,\tau(f)$ is the natural homomorphism.

By Theorem 7, every ideal in $\mathcal{M}_c(G)/\mathrm{Ann}\,\tau(f)$ is closed. It follows that every strictly descending chain of ideals in $\mathcal{M}_c(G)/\mathrm{Ann}\,\tau(f)$ arises from a strictly descending chain of closed ideals including $\mathrm{Ann}\,\tau(f)$ in $\mathcal{M}_c(G)$, and the annihilators of the ideals in this chain form a strictly ascending chain of subvarieties in $\tau(f)$. By finite dimensionality such a chain must terminate, which implies that $\mathcal{M}_c(G)/\mathrm{Ann}\,\tau(f)$ is an Artin ring.

Now we assume that $f \neq 0$, and $R = \mathcal{M}_c(G)/\mathrm{Ann}\,\tau(f)$ is a local Artin ring with exponential maximal ideal $F(M)$, where $F : \mathcal{M}_c(G) \to \mathcal{M}_c(G)/\mathrm{Ann}\,\tau(f)$ is the natural homomorphism, and M is a maximal ideal in $\mathcal{M}_c(G)$. By the isomorphism

$$R/F(M) \cong \mathcal{M}_c(G)/M \cong \mathbb{C}$$

we have that $M = M_m$ is an exponential maximal ideal with some exponential m. It is well-known that the maximal ideal in a local Artin ring is nilpotent (see e.g. [10], Theorem 7.15, p. 426). Hence, by Theorem 13, we have that f is a generalized exponential monomial associated with the exponential m. It is enough to show that $\tau(f)$ is finite dimensional. Let n be the degree of f, which implies that $M_m^n\tau(f) \neq \{0\}$. Let $\varphi \neq 0$ be in $M_m^n\tau(f)$, then we have for each x, y in G

$$0 = (\delta_{-y} - m(y)\delta_0) * \varphi(x) = \varphi(x+y) - m(y)\varphi(x).$$

Putting $x = 0$ we have $\varphi = \varphi(0) \cdot m$, which means that $M_m\tau(f)$ is one dimensional. We consider the chain of $\mathcal{M}_c(G)$-modules

$$\tau(f), \tau(f)/M_m\tau(f), \ldots, M_m^n\tau(f)/M^{n+1} = M_m^n\tau(f), \{0\}.$$

Suppose that $\tau(f)$ is infinite dimensional. Then there exists a natural number k with $0 \le k \le n-1$ such that $M_m^k\tau(f)$ is infinite dimensional and $M_m^{k+1}\tau(f)$ is finite dimensional. It follows that $M_m^k/M_m^{k+1}\tau(f)$ is infinite dimensional. Then there exists a sequence $\varphi_1, \varphi_2, \ldots, \varphi_l, \ldots$ in $M_m^k\tau(f)$ such that the coset $\varphi_{l+1} + M_m^{k+1}\tau(f)$ is not included in the linear span of the elements $\varphi_j + M_m^{k+1}\tau(f)$ for $j = 1, 2, \ldots, l$ and $l = 1, 2, \ldots$. However, by Lemma 3, the linear span of the elements $\varphi_j + M_m^{k+1}\tau(f)$ for $j = 1, 2, \ldots, l$ coincides with the submodule generated by these elements in $M_m^k\tau(f)/M_m^{k+1}\tau(f)$. Consequently, φ_{l+1} is not included in the subvariety generated by the functions φ_j with $1 \le j \le l$ in $\tau(f)$, which means that these subvarieties form a strictly ascending chain, and their annihilators generate a strictly descending chain of ideals in $\mathcal{M}_c(G)/\mathrm{Ann}\,\tau(f)$. This contradicts the Artin property. □

5 Spectral analysis

Let G be a locally compact Abelian group and V a variety in $\mathcal{C}(G)$. We say that *spectral analysis* holds for the variety V if every nonzero subvariety of V contains an exponential. We say that *spectral analysis holds on* G, if spectral analysis holds for

$\mathcal{C}(G)$, that is, every nonzero variety on G contains an exponential. Clearly, if spectral analysis holds for a variety, then it holds for every subvariety of it, too.

Lemma 4 *Let G be a locally compact Abelian group. If a nonzero generalized exponential monomial associated to the exponential m belongs to a variety, then m belongs to the same variety, too.*

Proof Let $f \neq 0$ be a generalized exponential monomial associated to the exponential m. Then $\operatorname{Ann} \tau(f)$ is a proper ideal, hence there is a maximal ideal M containing $\operatorname{Ann} \tau(f)$. On the other hand, we have $M_m^{n+1} \subseteq \operatorname{Ann} \tau(f)$ for some natural number n. It follows $M_m^{n+1} \subseteq M$. As M is maximal, it is also prime, and we infer $M_m \subseteq M$, which implies, by maximality, that $M_m = M$, and $\operatorname{Ann} \tau(f) \subseteq M_m$. Finally, we have

$$\operatorname{Ann} M_m \subseteq \operatorname{Ann} \operatorname{Ann} \tau(f) = \tau(f),$$

and obviously m is in $\operatorname{Ann} M_m$, which proves our statement. □

Theorem 15 *Let G be a locally compact Abelian group. Spectral analysis holds for a variety if and only if every nonzero subvariety of it contains a nonzero generalized exponential monomial. In particular, spectral analysis holds on G if and only if every nonzero variety on G contains a nonzero generalized exponential monomial.*

As the variety of a generalized exponential monomial consists of generalized exponentials, hence spectral analysis holds for the variety of every generalized exponential monomial.

Theorem 16 *Let G be a locally compact Abelian group. Spectral analysis holds for a nonzero variety V on G if and only if every maximal ideal in $\mathcal{M}_c(G)$ which contains* Ann *V is exponential. In other words, spectral analysis holds for $V \neq \{0\}$ if and only if every maximal ideal in the ring $\mathcal{M}_c(G)/\operatorname{Ann} V$ is exponential. In particular, spectral analysis holds on G if and only if every maximal ideal in $\mathcal{M}_c(G)$ is exponential.*

Proof Indeed, the given condition is clearly necessary. Conversely, if every maximal ideal in $\mathcal{M}_c(G)$ which contains $\operatorname{Ann} V$ is exponential, and $W \subseteq V$ is a nonzero subvariety, then $\operatorname{Ann} W \supseteq \operatorname{Ann} V$, hence every maximal ideal which contains $\operatorname{Ann} W$ also contains $\operatorname{Ann} V$. The other statements are obvious. □

6 Spectral synthesis

Let G be a locally compact Abelian group and V a variety in $\mathcal{C}(G)$. We say that the variety V is *synthesizable*, if the exponential monomials in V span a dense subspace. We say that *spectral synthesis* holds for V, if every subvariety of V is synthesizable. We say that *spectral synthesis holds on* G, if spectral synthesis holds for every variety on G. Clearly, if spectral synthesis holds for a variety, then spectral synthesis and spectral analysis holds for every subvariety of it.

Given a variety V in $\mathcal{C}(G)$ let $\mathcal{I}(V)$ denote the set of all closed ideals I in $\mathcal{M}_c(G)$ such that $\operatorname{Ann} V \subseteq I$ and $\mathcal{M}_c(G)/I$ is a local Artin ring.

Theorem 17 *Let G be a locally compact Abelian group. The variety V in $\mathcal{C}(G)$ is synthesizable if and only if*

$$\operatorname{Ann} V = \bigcap \mathcal{I}(V). \tag{5}$$

Proof Suppose that V is synthesizable. Then

$$V = \sum_{\varphi \in V} \tau(\varphi),$$

where the summation is extended for all exponential monomials φ in V. By Theorem 8, it follows

$$\operatorname{Ann} V = \bigcap_{\varphi \in V} \operatorname{Ann} \tau(\varphi) = \bigcap_{\operatorname{Ann} V \subseteq \operatorname{Ann} \tau(\varphi)} \operatorname{Ann} \tau(\varphi).$$

By Theorem 14, the set of the annihilators $\operatorname{Ann} \tau(\varphi)$ where φ is an exponential monomial in V is identical with the set $\mathcal{I}(V)$, which proves the theorem. □

Theorem 18 *Let G be a locally compact Abelian group and $f : G \to \mathbb{C}$ a generalized exponential monomial, which is not an exponential monomial. Then $\tau(f)$ is non-synthesizable.*

Proof We show that if

$$\operatorname{Ann} \tau(f) = \bigcap \mathcal{F},$$

where $\mathcal{F}$ is a family of closed ideals, then there is an I in $\mathcal{F}$ with $I = \operatorname{Ann} \tau(f)$.

Suppose that $f \neq 0$ is a generalized exponential monomial of degree $n \geq 1$ associated to the exponential m. Then $\operatorname{Ann} I$ is a subvariety of $\tau(f)$, hence it consists of generalized exponential monomials of degree at most n, which are associated to m, too. We also have that $M_m \tau(\varphi)$ consists of generalized exponential monomials of degree at most $n-1$, which are associated to m. For each y in G we have

$$\tau_y f = \delta_{-y} * f = (\delta_{-y} - m(y)\delta_0) * f + m(y) f,$$

hence $\tau_y f$ is in the linear space $X = M_m \tau(f) + \mathbb{C} f$, which is closed in $\mathcal{C}(G)$. Indeed, if $(\varphi_\alpha + c_\alpha f)_{\alpha \in A}$ is a generalized sequence in X, with φ_α in $M_m \tau(f)$ and c_α in $\mathbb{C}$, which converges to ψ in $\mathcal{C}(G)$, then for each μ in M_m^n the generalized sequence $(\mu * \varphi_\alpha + c_\alpha\, \mu * f)_{\alpha \in A}$ converges to $\mu * \psi$. The function $\mu * f$ is in $M_m^n \tau(f)$, which is different from $\{0\}$, as the degree of f is n. This means that we can choose μ such that $\mu * f \neq 0$. On the other hand, as φ_α is in $M_m \tau(f)$, we have that $\mu * \varphi_\alpha$ is in $M_m^{n+1} \tau(f) = \{0\}$, that is $\mu * \varphi_\alpha = 0$. It follows that $(c_\alpha)_{\alpha \in A}$ converges to some c in $\mathbb{C}$, which implies that $(\varphi_\alpha)_{\alpha \in A}$ converges to a function in $M_m \tau(f)$, and ψ is in X.

As X is closed, we have $\tau(f) \subseteq M_m \tau(f) + \mathbb{C} f$, in fact $\tau(f) = M_m \tau(f) + \mathbb{C} f$. On the other hand, $M_m \tau(f) \cap \mathbb{C} f = \{0\}$, hence $\tau(f)$ is the direct sum of the closed subspaces $M_m \tau(f)$ and $\mathbb{C} f$.

Now suppose that

$$\operatorname{Ann}\tau(f) = \bigcap \mathcal{F},$$

where $\mathcal{F}$ is a family of closed ideals, then

$$\tau(f) = \sum_{I\in\mathcal{F}} \operatorname{Ann} I.$$

As f is of degree n, there must be an I in $\mathcal{F}$ such that $\operatorname{Ann} I$ includes a generalized monomial of degree n. By the direct decomposition of $\tau(f)$ we have

$$\operatorname{Ann} I = (\operatorname{Ann} I \cap M_m\tau(f)) + (\operatorname{Ann} I \cap \mathbb{C}f).$$

If $\operatorname{Ann} I \cap \mathbb{C}f \neq \{0\}$, then f is in $\operatorname{Ann} I$ and $\operatorname{Ann} I = \tau(f)$, and $I = \operatorname{Ann}\tau(f)$. However, $\operatorname{Ann} I \cap \mathbb{C}f = \{0\}$ is impossible, because in this case $\operatorname{Ann} I \subseteq M_m\tau(f)$, which consist of generalized exponential monomials of degree at most $n-1$, a contradiction. This proves that there is an I in $\mathcal{F}$ such that $\operatorname{Ann}\tau(f) = I$. Assume now that $\tau(f)$ is synthesizable. By Theorem 17, we have a representation of $\tau(f)$ as the intersection of the ideals in the family $\mathcal{F} = \mathcal{I}(\tau(f))$. We have seen above that in this case $\operatorname{Ann}\tau(f)$ must be in $\mathcal{F}$, but, by Theorem 14, this is impossible, as $\mathcal{M}_c(G)/\operatorname{Ann}\tau(f)$ is a not local Artin ring with exponential maximal ideal. □

Corollary 2 *Let G be a locally compact Abelian group. If a variety in $\mathcal{C}(G)$ contains a generalized exponential monomial, which is not an exponential monomial, then spectral synthesis fails to hold for the variety.*

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