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Strongly Asplund generated and strongly conditionally weakly compactly generated Banach spaces

Sebastián Lajara[1] · José Rodríguez[2]

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Abstract We study strongly Asplund generated (SAG) and strongly conditionally weakly compactly generated (SCWCG) Banach spaces. These spaces are defined like the strongly weakly compactly generated (SWCG) Banach spaces of Schlüchtermann and Wheeler, but replacing weakly compact sets by Asplund sets and conditionally weakly compact sets, respectively. We show that every SAG space is SCWCG and that a Banach space is SWCG if and only if it is SAG/SCWCG and weakly sequentially complete. We also prove that the notions of SAG and SCWCG space coincide for Banach lattices. Some related results on Lebesgue–Bochner spaces are also given. We prove that if the norm of the Banach space X is weakly uniformly rotund (WUR) and μ is any probability measure, then $L^1(\mu, X)$ admits an equivalent norm which is WUR when restricted to any Asplund subspace of $L^1(\mu, X)$.

Keywords Asplund set · Conditionally weakly compact set · Strongly generated Banach space · Lebesgue–Bochner space · Weakly uniformly rotund norm

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✉ José Rodríguez
joserr@um.es

Sebastián Lajara
sebastian.lajara@uclm.es

1 Departamento de Matemáticas, Escuela de Ingenieros Industriales, Universidad de Castilla-La Mancha, 02071 Albacete, Spain

2 Departamento de Matemática Aplicada, Facultad de Informática, Universidad de Murcia, 30100 Espinardo, Murcia, Spain

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1 Introduction

In this paper we study strongly Asplund generated (SAG) and strongly conditionally weakly compactly generated Banach spaces. These classes of spaces were introduced by Kunze and Schlüchtermann [22], inspired by the strongly weakly compactly generated Banach spaces of Schlüchtermann and Wheeler [27]. To recall the definition we need some terminology. Given a Banach space Z, a set $A \subseteq Z$ is said to be *conditionally weakly compact* if every sequence in A admits a weakly Cauchy subsequence. A set $A \subseteq Z$ is said to be *Asplund* if there exist an Asplund Banach space Y and an operator $T : Y \to Z$ such that $A \subseteq T(B_Y)$. We denote by $\mathcal{WC}(Z)$, $\mathcal{A}(Z)$ and $\mathcal{CWC}(Z)$ the families of all weakly compact, Asplund and conditionally weakly compact subsets of Z, respectively. In general, we have $\mathcal{WC}(Z) \subseteq \mathcal{A}(Z) \subseteq \mathcal{CWC}(Z)$.

Definition 1.1 Let Z be a Banach space, $\mathcal{H}$ a family of subsets of Z and $G \subseteq Z$. We say that $\mathcal{H}$ is *dominated* by G (or that G *dominates* $\mathcal{H}$) if for every $H \in \mathcal{H}$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that $H \subseteq nG + \varepsilon B_Z$.

Definition 1.2 A Banach space Z is called:

(i) strongly *weakly compactly* generated (*SWCG*) if $\mathcal{WC}(Z)$ is dominated by some $G \in \mathcal{WC}(Z)$;
(ii) strongly *Asplund* generated (*SAG*) if $\mathcal{A}(Z)$ is dominated by some $G \in \mathcal{A}(Z)$;
(iii) strongly *conditionally weakly compactly* generated (*SCWCG*) if $\mathcal{CWC}(Z)$ is dominated by some $G \in \mathcal{CWC}(Z)$.

The class of SWCG spaces has been studied thoroughly in [15,21,25,27,28] (see also [20, Section 6.4]). In this paper we shall focus on SAG and SCWCG spaces. The basic properties of such spaces were discussed in [22]. Clearly, every Asplund space is SAG and every Banach space not containing ℓ^1 is SCWCG.

In Sect. 2 we discuss the connection between these classes of Banach spaces. In general, the implications "SWCG $\Longrightarrow$ SAG $\Longrightarrow$ SCWCG" hold (Theorems 2.1, 2.2). On the other hand, a Banach space is SWCG if and only if it is both SCWCG and weakly sequentially complete (Theorem 2.2). The stability of "being a subspace of a SAG/SCWCG space" under countable ℓ^p-sums ($1 < p < \infty$) and c_0-sums is discussed in Theorem 2.6. We finish the section by proving that the notions of SAG and SCWCG space coincide for Banach lattices (Corollary 2.11).

In Sect. 3 we study these properties in Lebesgue–Bochner function spaces $L^p(\mu, X)$ (where $1 \leq p < \infty$, μ is a probability measure and X a Banach space). The space $L^p(\mu, X)$ is SAG (resp. SCWCG) if X is Asplund (resp. $X \not\supseteq \ell^1$), see Example 3.5. The converse holds true for $1 < p < \infty$ whenever μ is non-trivial (Proposition 3.1). We finish the paper by proving that if the norm of X is weakly uniformly rotund (WUR), then $L^1(\mu, X)$ admits an equivalent norm which is WUR when restricted to any Asplund subspace of $L^1(\mu, X)$ (Theorem 3.9). We should stress here that, for $1 < p < \infty$, the canonical norm of $L^p(\mu, X)$ is WUR if the norm of X is WUR, thanks to a result of [29] and the fact that every Banach space admitting a WUR equivalent norm is Asplund (see [19]).

We use standard Banach space terminology as can be found in [1,12]. Our Banach spaces are real. The closed unit ball of a Banach space Z is denoted by B_Z. The norm of Z is denoted by $\|\cdot\|_Z$ or simply $\|\cdot\|$. By an *operator* we mean a linear continuous map between Banach spaces. By a *subspace* of a Banach space we mean a closed linear subspace. Given two Banach spaces Z and Y, we write $Z \not\supseteq Y$ if Z contains no subspace isomorphic to Y, and we write $Z \supseteq Y$ if Y is isomorphic to a subspace of Z (in this case, we just say that Y is a subspace of Z). For complete information on Asplund sets we refer the reader to [5, Chapter 5] and [10, Chapter 1].

2 SAG and SCWCG spaces

We begin this section by showing the general relationships between SWCG, SAG and SCWCG Banach spaces.

Theorem 2.1 *Let Z be a SAG Banach space. Then*:

(i) *Z is SCWCG.*
(ii) $\mathcal{A}(Z) = \mathcal{CWC}(Z)$.
(iii) *Every subspace of Z not containing ℓ^1 is Asplund.*

Theorem 2.2 *Let Z be a Banach space. The following statements are equivalent*:

(i) *Z is SWCG*;
(ii) *Z is SAG and weakly sequentially complete*;
(iii) *Z is SCWCG and weakly sequentially complete.*

In view of Theorem 2.1, any Banach space not containing ℓ^1 which is not Asplund (like the James tree space) is SCWCG but not SAG.

On the other hand, c_0 is an Asplund (hence SAG) space which is not weakly sequentially complete, and so it fails to be SWCG. Spaces like ℓ^1 and $L^1[0,1]$ are SWCG (hence SAG) but not Asplund.

For the proofs of Theorems 2.1 and 2.2 we will need the following lemma, which is based on the argument given in [27, Theorem 2.5] to prove that every SWCG space is weakly sequentially complete.

Lemma 2.3 *Let Z be a Banach space and $G \subseteq Z$ a balanced set. The following statements are equivalent*:

(i) *$\mathcal{WC}(Z)$ is dominated by G;*
(ii) *$\mathcal{CWC}(Z)$ is dominated by G.*

Proof (ii) $\Rightarrow$ (i) is obvious (and does not require that G is balanced). (i) $\Rightarrow$ (ii) Our proof is by contradiction. Suppose there exist $H \in \mathcal{CWC}(Z)$ and $\varepsilon > 0$ such that $H \not\subseteq nG + \varepsilon B_Z$ for all $n \in \mathbb{N}$. Let (z_n) be a sequence in H such that $z_n \notin nG + \varepsilon B_Z$ for all $n \in \mathbb{N}$. Since G is balanced, we have:

($\star$) for every $m \in \mathbb{N}$ the set $\{n \in \mathbb{N} \colon z_n \in mG + \varepsilon B_Z\}$ is finite.

Since H is conditionally weakly compact and ($\star$) holds for any subsequence of (z_n), by passing to a further subsequence we can assume that (z_n) is weakly Cauchy.

For each $n \in \mathbb{N}$ and $i \in \{1, 2\}$ we define

$$m_i(n) := \min \left\{ m \in \mathbb{N} \colon z_n \in mG + \frac{\varepsilon}{i} B_Z \right\}.$$

Clearly, $z_n \in mG + \frac{\varepsilon}{i} B_Z$ if and only if $m \geq m_i(n)$ (because G is balanced). Let $\psi \colon \mathbb{N} \to \mathbb{N}$ be any function such that $\lim_{n\to\infty} \frac{n}{\psi(n)} = 0$ and $n \leq \psi(n)$ for all $n \in \mathbb{N}$. We claim that there is a subsequence (z_{n_k}) such that

$$\psi(m_2(n_k)) < m_1(n_{k+1}) \quad \text{for all } k \in \mathbb{N}. \tag{2.1}$$

Indeed, set $n_1 = 1$ and suppose that $n_k \in \mathbb{N}$ has already been chosen. Then ($\star$) ensures the existence of $n_{k+1} \in \mathbb{N}$ with $n_{k+1} > n_k$ such that

$$z_{n_{k+1}} \notin \psi(m_2(n_k))G + \varepsilon B_Z,$$

hence $\psi(m_2(n_k)) < m_1(n_{k+1})$. This proves the claim. On the other hand, since

$$m_2(n_k) \leq \psi(m_2(n_k)) \overset{(2.1)}{<} m_1(n_{k+1}) \leq m_2(n_{k+1}) \quad \text{for all } k \in \mathbb{N},$$

the sequence $(m_2(n_k))$ is strictly increasing and we have

$$\lim_{k\to\infty} \frac{m_2(n_k)}{\psi(m_2(n_k))} = 0. \tag{2.2}$$

Define $h_k := z_{n_{k+1}} - z_{n_k}$ for all $k \in \mathbb{N}$, so that (h_k) is a weakly null sequence in Z. Since $\mathcal{WC}(Z)$ is dominated by G, there is $m_0 \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ we have $h_k \in m_0 G + \frac{\varepsilon}{2} B_Z$. Take any $k \in \mathbb{N}$. Then $z_{n_k} \in m_2(n_k)G + \frac{\varepsilon}{2} B_Z$, hence

$$z_{n_{k+1}} = h_k + z_{n_k} \in (m_0 + m_2(n_k))G + \varepsilon B_Z$$

and we get

$$\psi(m_2(n_k)) \overset{(2.1)}{<} m_1(n_{k+1}) \leq m_0 + m_2(n_k).$$

As $k \in \mathbb{N}$ is arbitrary, this contradicts (2.2). The proof is finished. □

The following stability property of Asplund sets (see e.g. [10, Lemma 1.4.3]) will be used several times in the paper.

Fact 2.4 *Let Z be a Banach space and $A \subseteq Z$. If for every $\varepsilon > 0$ there is $B \in \mathcal{A}(Z)$ such that $A \subseteq B + \varepsilon B_Z$, then $A \in \mathcal{A}(Z)$.*

We are now ready to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1 Let $G \in \mathcal{A}(Z)$ which dominates $\mathcal{A}(Z)$. We can assume without loss of generality that G is balanced. Since $\mathcal{A}(Z) \supseteq \mathcal{WC}(Z)$, Lemma 2.3 ensures that G also dominates $\mathcal{CWC}(Z)$. Since G is conditionally weakly compact, Z is SCWCG. Bearing in mind that G is Asplund and dominates $\mathcal{CWC}(Z)$, from Fact 2.4 it follows that every conditionally weakly compact subset of Z is Asplund. Finally, (iii) is immediate from (ii) applied to the closed unit ball of the subspace (a Banach space is Asplund if and only if its closed unit ball is an Asplund set). □

Proof of Theorem 2.2 (i) ⇒ (ii) Let $G \in \mathcal{WC}(Z)$ which dominates $\mathcal{WC}(Z)$. By the Krein–Smulyan theorem (see e.g. [9, p. 51, Theorem 11]), we can assume that G is absolutely convex and, in particular, balanced. Then G dominates $\mathcal{CWC}(Z)$ (by Lemma 2.3). Since $\mathcal{CWC}(Z) \supseteq \mathcal{A}(Z)$ and G is Asplund, we deduce that Z is SAG. Bearing in mind that the weakly compact set G dominates $\mathcal{CWC}(Z)$, Grothendieck's test (see e.g. [12, Lemma 13.32]) ensures that every conditionally weakly compact subset of Z is relatively weakly compact, that is, Z is weakly sequentially complete.

(ii) ⇒ (iii) follows from Theorem 2.1.

(iii) ⇒ (i) This is immediate since conditional weak compactness and relative weak compactness coincide in any weakly sequentially complete Banach space. □

Remark 2.5 In general, subspaces of SWCG/SAG/SCWCG spaces need not be SWCG/SAG/SCWCG. Indeed, in [25, Section 3] there is an example of a subspace $Z \subseteq L^1[0, 1]$ which is not SWCG. Since $L^1[0, 1]$ is weakly sequentially complete, Z cannot be SAG or SCWCG.

The following result extends [22, Theorem 4.5]. Its proof uses some ideas from [27, Theorem 3.2] and [26].

Theorem 2.6 *Let (X_n) be a sequence of Banach spaces and let Y be either $(\bigoplus_{n\in\mathbb{N}} X_n)_{\ell_p}$ for $1 < p < \infty$ or $(\bigoplus_{n\in\mathbb{N}} X_n)_{c_0}$. If Y is a subspace of a SAG (resp. SCWCG) space, then X_n is Asplund (resp. $X_n \not\supseteq \ell^1$) for all but finitely many $n \in \mathbb{N}$.*

Proof We divide the proof into several steps.

Step 1 It suffices to prove the SCWCG case. Indeed, if Y is a subspace of a SAG Banach space Z, then so is each X_n, hence $X_n \not\supseteq \ell^1$ if and only if X_n is Asplund (by Theorem 2.1). Bearing in mind that Z is SCWCG (Theorem 2.1), it is clear that the SAG case follows from the SCWCG case.

Step 2 We shall prove that if $X_n \supseteq \ell^1$ for all $n \in \mathbb{N}$, then Y is not a subspace of a SCWCG space. By James' ℓ^1 distortion theorem (see e.g. [1, Theorem 10.3.1]), for each $n \in \mathbb{N}$ there is a normalized sequence (x_k^n) in X_n such that

$$\left\| \sum_{k\in\mathbb{N}} a_k x_k^n \right\|_{X_n} \geq \frac{1}{2} \sum_{k\in\mathbb{N}} |a_k| \quad \text{for all } (a_k) \in \ell^1. \tag{2.3}$$

Let $\Lambda \subseteq \mathbb{N}^{\mathbb{N}}$ be the set of all strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$. Fix $\varphi \in \Lambda$. For each $j \in \mathbb{N}$, let $f_{\varphi,j} \in Y$ be defined as

$$\pi_n(f_{\varphi,j}) := \begin{cases} x^j_{\varphi(j)} & \text{if } n = j, \\ 0 & \text{if } n \neq j, \end{cases}$$

where $\pi_n : Y \to X_n$ denotes the nth-coordinate projection for all $n \in \mathbb{N}$. The sequence $(f_{\varphi,j})$ is weakly null in Y, because it is bounded and for every $n \in \mathbb{N}$ we have $\pi_n(f_{\varphi,j}) = 0$ whenever $j > n$. Therefore, the set $K_\varphi := \{f_{\varphi,j} : j \in \mathbb{N}\} \cup \{0\}$ is weakly compact in Y.

Step 3 By contradiction, suppose that there exists an isomorphic embedding $T : Y \to Z$, where Z is a SCWCG Banach space. Fix $G \in \mathcal{CWC}(Z)$ which dominates $\mathcal{CWC}(Z)$ and fix $0 < \varepsilon < c := \frac{1}{2}\|T^{-1}|_{T(Y)}\|^{-1}$. For each $\varphi \in \Lambda$ we choose $m(\varphi) \in \mathbb{N}$ with the property that $T(K_\varphi) \subseteq m(\varphi)G + \varepsilon B_Z$. Then $\Lambda = \bigcup_{m\in\mathbb{N}} \mathcal{B}_m$, where $\mathcal{B}_m := \{\varphi \in \Lambda : m(\varphi) = m\}$ for every $m \in \mathbb{N}$. It is easy to check that the equality $\Lambda = \bigcup_{m\in\mathbb{N}} \mathcal{B}_m$ implies that there is $m \in \mathbb{N}$ such that $\{\varphi(m) : \varphi \in \mathcal{B}_m\}$ is infinite. Notice that

$$\bigcup_{\varphi\in\mathcal{B}_m} T(K_\varphi) \subseteq mG + \varepsilon B_Z. \tag{2.4}$$

Enumerate $\{\varphi(m) : \varphi \in \mathcal{B}_m\} = \{\varphi_1(m) < \varphi_2(m) < \dots\}$ for some sequence (φ_k) in $\mathcal{B}_m$. Define $g_k := f_{\varphi_k,m} \in K_{\varphi_k} \subseteq Y$ for all $k \in \mathbb{N}$. Observe that

$$\Big\|\sum_{k\in\mathbb{N}} b_k g_k\Big\|_Y = \Big\|\sum_{k\in\mathbb{N}} b_k x^m_{\varphi_k(m)}\Big\|_{X_m} \overset{(2.3)}{\geq} \frac{1}{2}\sum_{k\in\mathbb{N}} |b_k| \quad \text{for all } (b_k) \in \ell^1,$$

hence

$$\Big\|\sum_{k\in\mathbb{N}} b_k T(g_k)\Big\|_Z \geq c \sum_{k\in\mathbb{N}} |b_k| \quad \text{for all } (b_k) \in \ell^1. \tag{2.5}$$

Step 4 By (2.4), for each $k \in \mathbb{N}$ there is $h_k \in mG$ such that $\|T(g_k) - h_k\|_Z \leq \varepsilon$. For every $(b_k) \in \ell^1$ we have

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{N}} b_k h_k\Big\|_Z &= \Big\|\sum_{k\in\mathbb{N}} b_k T(g_k) - \sum_{k\in\mathbb{N}} b_k\big(T(g_k) - h_k\big)\Big\|_Z \\ &\geq \Big\|\sum_{k\in\mathbb{N}} b_k T(g_k)\Big\|_Z - \sum_{k\in\mathbb{N}} |b_k|\,\|T(g_k) - h_k\|_Z \\ &\geq \Big\|\sum_{k\in\mathbb{N}} b_k T(g_k)\Big\|_Z - \varepsilon\sum_{k\in\mathbb{N}} |b_k| \overset{(2.5)}{\geq} (c-\varepsilon)\sum_{k\in\mathbb{N}} |b_k|.\end{aligned}$$

Thus, (h_k) is an ℓ^1-sequence contained in $mG \in \mathcal{CWC}(Z)$, a contradiction which finishes the proof. □

Since $C[0,1]$ contains any separable Banach space (see e.g. [1, Theorem 1.4.3]), the previous theorem applied to the space $\ell^2(\ell^1)$ yields:

Corollary 2.7 *$C[0,1]$ is not a subspace of a SCWCG space.*

Given a compact Hausdorff topological space K, the Banach space $C(K)$ is Asplund if and only if K is scattered (see e.g. [12, Theorem 14.25]). On the other hand, if K is not scattered, then $C[0, 1]$ is a subspace of $C(K)$ (see e.g. the proof of [12, Theorem 14.26(v)]). These facts and Corollary 2.7 allow us to deduce:

Corollary 2.8 *Let K be a compact Hausdorff topological space. Then $C(K)$ is a subspace of a SCWCG space if and only if $C(K)$ is Asplund.*

The following result was proved in [21, Corollary 2.29]. The particular case not involving subspaces was first noticed in [28, Theorem 5.10].

Corollary 2.9 (Kampoukos–Mercourakis) *Let (X_n) be a sequence of Banach spaces and let $1 < p < \infty$. If $(\bigoplus_{n\in\mathbb{N}} X_n)_{\ell_p}$ is a subspace of a SWCG space, then X_n is reflexive for all but finitely many $n \in \mathbb{N}$.*

Proof By Theorems 2.2 and 2.6, we have $X_n \not\supseteq \ell^1$ for all but finitely many $n \in \mathbb{N}$. On the other hand, bearing in mind that every SWCG space is weakly sequentially complete (see [27, Theorem 2.5], cf. Theorem 2.2) and that weak sequential completeness is inherited by subspaces, we get that each X_n is weakly sequentially complete. From Rosenthal's ℓ^1 theorem (see e.g. [1, Theorem 10.2.1]) it follows at once that X_n is reflexive for all but finitely many $n \in \mathbb{N}$. □

It is known that a Banach *lattice* is Asplund if (and only if) it does not contain ℓ_1 (see [9, p. 95] and [17, Theorem 7]). We finish this section by proving that the properties of being SAG and SCWCG are also equivalent in Banach lattices.

Theorem 2.10 *If Z is a Banach lattice and $Z \not\supseteq C[0, 1]$, then $\mathcal{A}(Z) = \mathcal{CWC}(Z)$.*

Proof Let $H \in \mathcal{CWC}(Z)$. Since $Z \not\supseteq C[0, 1]$, the convex solid hull $\tilde{H}$ of H is conditionally weakly compact as well (see [16, Corollary II.4]). Let Y be the interpolation Banach space obtained from $\tilde{H}$ by applying the Davis–Figiel–Johnson–Pełczyński method and let $T : Y \to X$ be its associated operator (see e.g. [2, Theorem 5.37]). Since $\tilde{H}$ is conditionally weakly compact, we have $Y \not\supseteq \ell^1$ (see e.g. [18, Theorem 5.3.6]). Since $\tilde{H}$ is solid, Y is a Banach lattice (see e.g. [2, Theorem 5.41]). According to the comments preceding the theorem, Y is Asplund and so $H \subseteq T(B_Y)$ is an Asplund set. □

Corollary 2.11 *A Banach lattice is SCWCG if and only if it is SAG.*

Proof Combine Theorem 2.10 and Corollary 2.7. □

3 Lebesgue–Bochner spaces

Throughout this section X is a Banach space, (Ω, Σ, μ) a probability space and, for $1 \leq p < \infty$, we consider the Banach space $L^p(\mu, X)$ of all (equivalence classes) of strongly measurable functions $f : \Omega \to X$ such that

$$\|f\|_{L^p(\mu,X)} = \left(\int_\Omega \|f(\omega)\|^p \, d\mu(\omega)\right)^{\frac{1}{p}} < \infty.$$

A set $C \subseteq L^1(\mu, X)$ is called *uniformly integrable* if it is bounded and for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f 1_A\|_{L^1(\mu,X)} \le \varepsilon$ for every $A \in \Sigma$ with $\mu(A) \le \delta$ and every $f \in C$. (Here 1_A denotes the characteristic function of A.) It is known that every conditionally weakly compact subset of $L^1(\mu, X)$ is uniformly integrable (see e.g. [9, p. 104, Theorem 4]). Conversely, a result of Bourgain, Maurey and Pisier (see e.g. [7, Theorem 2.2.1]) states that every uniformly integrable subset of $L^1(\mu, X)$ is conditionally weakly compact if and only if $X \not\supseteq \ell^1$.

For $1 < p < \infty$, the space $L^p(\mu, X)$ is Asplund (resp. $L^p(\mu, X) \not\supseteq \ell^1$) if and only if X is Asplund (resp. $X \not\supseteq \ell^1$), see e.g. [9, IV.1] (resp. [7, Theorem 2.2.2]). We say that μ is *non-trivial* if $L^1(\mu)$ is infinite dimensional or, equivalently, there is an infinite sequence of pairwise disjoint elements of Σ with positive measure. In this case, $\ell^p(X)$ is a subspace of $L^p(\mu, X)$. Thus, from Theorem 2.6 we get:

Proposition 3.1 *Suppose μ is non-trivial and let $1 < p < \infty$. The following statements are equivalent*:

(i) *$L^p(\mu, X)$ is a subspace of a SAG (resp. SCWCG) space;*
(ii) *$L^p(\mu, X)$ is Asplund (resp. $L^p(\mu, X) \not\supseteq \ell^1$);*
(iii) *X is Asplund (resp. $X \not\supseteq \ell^1$).*

Note that X is a complemented subspace of $L^1(\mu, X)$, hence X is SWCG, SAG or SCWCG whenever $L^1(\mu, X)$ is. Schlüchtermann and Wheeler [27] asked whether $L^1(\mu, X)$ is SWCG whenever X is SWCG. Some partial answers have been given in [24,26,27], but the general question still remains open in full generality. In the same way, we might ask:

Question 3.2 *Is $L^1(\mu, X)$ SAG if X is SAG?*

Question 3.3 *Is $L^1(\mu, X)$ SCWCG if X is SCWCG?*

Remark 3.4 If the answer to either Question 3.2 or 3.3 were affirmative, then the problem of Schlüchtermann and Wheeler would have positive solution as well. Indeed, this follows from Theorem 2.2 and Talagrand's striking result that $L^1(\mu, X)$ is weakly sequentially complete if X is (see [30, Theorem 11]).

Part (i) of the following example should be compared with Theorem 2.1.

Example 3.5 (i) If X is Asplund, then:
(a) $L^1(\mu, X)$ is SAG.
(b) Every uniformly integrable subset of $L^1(\mu, X)$ is an Asplund set.
(c) A subspace $Y \subseteq L^1(\mu, X)$ is Asplund if and only if B_Y is uniformly integrable.
(ii) If $X \not\supseteq \ell^1$, then $L^1(\mu, X)$ is SCWCG.

Proof (i) Let $i\colon L^2(\mu, X) \to L^1(\mu, X)$ be the identity operator. It is easy to check that the family of all uniformly integrable subsets of $L^1(\mu, X)$ is dominated by $i(B_{L^2(\mu,X)})$ (just adapt the proof of [20, Proposition 6.41] to the vector-valued case). In particular, $\mathcal{A}(L^1(\mu, X))$ is dominated by $i(B_{L^2(\mu,X)})$. Since $L^2(\mu, X)$ is Asplund, the set $i(B_{L^2(\mu,X)})$ is Asplund and therefore $L^1(\mu, X)$ is SAG. Statement (b) follows from Fact 2.4 and (c) is immediate from (b) applied to B_Y. The proof of part (ii) is similar to that of (i)(a). □

We next consider some special subfamilies of $\mathcal{A}(L^1(\mu, X))$ and $\mathcal{CWC}(L^1(\mu, X))$.

Definition 3.6 $H \subseteq L^1(\mu, X)$ is said to be a $\delta\mathcal{A}$-set (resp. $\delta\mathcal{C}$-*set*) if it is uniformly integrable and for every $\delta > 0$ there exists $W_\delta \in \mathcal{A}(X)$ (resp. $W_\delta \in \mathcal{CWC}(X)$) such that: for every $f \in H$ there is $A \in \Sigma$ such that $\mu(\Omega\backslash A) \leq \delta$ and $f(A) \subseteq W_\delta$.

Of course, the typical example of $\delta\mathcal{A}$-set (resp. $\delta\mathcal{C}$-set) is

$$L(C) := \{f \in L^1(\mu, X) : f(\Omega) \subseteq C\},$$

where $C \in \mathcal{A}(X)$ (resp. $C \in \mathcal{CWC}(X)$). The $\delta\mathcal{C}$-sets were studied in [3,4]. It was shown in [4, Proposition 13] (cf. [7, Theorem 2.2.1]) that every $\delta\mathcal{C}$-set of $L^1(\mu, X)$ is conditionally weakly compact.

Proposition 3.7 *Every $\delta\mathcal{A}$-set of $L^1(\mu, X)$ is Asplund.*

Proof We divide the proof into several cases.

Case 1. $L(C)$ is Asplund whenever $C \subseteq X$ is countable and Asplund. To prove this, let Y be an Asplund Banach space and $T : Y \to X$ an operator such that $C \subseteq T(B_Y)$. Then $L^2(\mu, Y)$ is Asplund and we can consider the operator

$$\tilde{T} : L^2(\mu, Y) \to L^1(\mu, X), \quad \tilde{T}(f) := T \circ f.$$

We claim that $L(C) \subseteq \tilde{T}(B_{L^2(\mu,Y)})$. Indeed, the fact that C is countable ensures that for every $g \in L(C)$ there is a countably-valued strongly measurable function $f : \Omega \to B_Y$ such that $T \circ f = g$, hence $f \in B_{L^2(\mu,Y)}$ and $\tilde{T}(f) = g$.

Case 2. $L(C)$ is Asplund whenever $C \subseteq X$ is Asplund. It suffices to prove that every countable subset of $L(C)$ is Asplund (see e.g. [10, Theorem 1.4.5]). Fix a sequence (g_n) in $L(C)$ and take any $\varepsilon > 0$. For each $n \in \mathbb{N}$ we choose a countably-valued function $\tilde{g}_n \in L(C)$ such that $\|g_n(\omega) - \tilde{g}_n(\omega)\| \leq \varepsilon$ for μ-a.e. $\omega \in \Omega$. There is a countable set $C_0 \subseteq C$ such that $\tilde{g}_n \in L(C_0)$ for all $n \in \mathbb{N}$, hence $\{g_n : n \in \mathbb{N}\} \subseteq L(C_0) + \varepsilon B_{L^1(\mu,X)}$. Since C_0 is Asplund, so is $L(C_0)$ (by Case 1). As $\varepsilon > 0$ is arbitrary, from Fact 2.4 it follows that $\{g_n : n \in \mathbb{N}\}$ is Asplund.

General case. Let $H \subseteq L^1(\mu, X)$ be any $\delta\mathcal{A}$-set and fix $\varepsilon > 0$. Since H is uniformly integrable, there is $\delta > 0$ such that

$$\sup_{f \in H} \|f 1_B\|_{L^1(\mu,X)} \leq \varepsilon$$

for every $B \in \Sigma$ with $\mu(B) \leq \delta$. Let $W_\delta \in \mathcal{A}(X)$ be as in Definition 3.6 and set $C := W_\delta \cup \{0\} \in \mathcal{A}(X)$. Given any $f \in H$, there is $A_f \in \Sigma$ such that $\mu(\Omega\backslash A_f) \leq \delta$ and $f(A_f) \subseteq W_\delta$, hence $\|f 1_{\Omega\backslash A_f}\|_{L^1(\mu,X)} \leq \varepsilon$ and $f 1_{A_f} \in L(C)$. Therefore, $H \subseteq L(C) + \varepsilon B_{L^1(\mu,X)}$. Another appeal to Fact 2.4 ensures that H is Asplund. The proof is finished. □

In [3, Section 4] there is an example of a Banach space X and a weakly compact subset of $L^1([0, 1], X)$ which is not a $\delta\mathcal{C}$-set.

The following result can be obtained similarly as in [26, Theorem 3.7]. The proof is omitted.

Proposition 3.8 *The following statements are equivalent*:

(i) *X is SAG* (*resp. SCWCG*);
(ii) *the family of all $\delta\mathcal{A}$-sets* (*resp. $\delta\mathcal{C}$-sets*) *of $L^1(\mu, X)$ is dominated by some $\delta\mathcal{A}$-set* (*resp. $\delta\mathcal{C}$-set*).

We finish the paper by studying renorming properties of the space $L^1(\mu, X)$. Some ideas from [23] were adapted to the vector-valued case in [13,14] to show that some convexity and smoothness properties of X lift to $L^1(\mu, X)$ when equipped with the Orlicz-type equivalent norm $|||\cdot|||$ defined below.

From now on, $M: \mathbb{R} \to [0,\infty)$ is a fixed Orlicz function with some additional properties: M is Lipschitz and twice differentiable, M'' is decreasing and $\lim_{t\to\infty} t^2 M''(t) \in (0,\infty)$. An example of such a function is $M(t) = |t| - \log(1+|t|)$. We consider the equivalent norm $|||\cdot|||$ on $L^1(\mu, X)$ defined by

$$|||f||| := \inf\left\{\rho > 0\colon \int_\Omega M\left(\frac{\|f(\omega)\|}{\rho}\right) d\mu(\omega) \leq 1\right\}. \tag{3.1}$$

Here we focus on a property of the norm (WUR) which is close to the space being Asplund. The norm $\|\cdot\|$ of a Banach space Z is said to be WUR if for every two sequences (z_n) and (z'_n) in the unit sphere $S_Z := \{z \in Z : \|z\| = 1\}$ satisfying $\|z_n + z'_n\| \to 2$, we have $z_n - z'_n \to 0$ weakly in Z. Every Banach space admitting a WUR equivalent norm is Asplund (see [19, Theorem 1], cf. [11, Appendix]). Conversely, every Banach space with separable dual admits a WUR equivalent norm (see e.g. [8, Ch. II, Corollary 6.9(ii)]).

We shall prove the following:

Theorem 3.9 *If $(X, \|\cdot\|)$ is WUR and $Y \subseteq L^1(\mu, X)$ is an Asplund subspace, then $(Y, |||\cdot|||)$ is WUR.*

Since the Banach space X of Theorem 3.9 is necessarily Asplund, the subspaces for which the result applies are exactly those with uniformly integrable closed unit ball [Example 3.5(i)]. Our proof of Theorem 3.9 follows the ideas of [14]. We isolate some steps as auxiliary lemmas for the convenience of the reader.

Lemma 3.10 *For every $f \in L^1(\mu, X)$ we have*

$$\frac{1}{c}|||f||| \leq \|f\|_{L^1(\mu,X)} \leq c'|||f|||$$

where c is the Lipschitz constant of M and $c' := 2 + \frac{1}{M'(1)}$.

Proof See [13, p. 250]. □

Definition 3.11 Let $f, g \in L^1(\mu, X)$. We define two non-negative functions $\varphi_{(f,g)}$ and $\psi_{(f,g)}$ belonging to $L^1(\mu)$ by

$$\varphi_{(f,g)}(\omega) := M''(\max\{\|f(\omega)\|, \|g(\omega)\|\}) \cdot (\|f(\omega)\| - \|g(\omega)\|)^2$$

and

$$\psi_{(f,g)}(\omega) := M\left(\frac{\|f(\omega)\| + \|g(\omega)\|}{2}\right) - M\left(\frac{\|f(\omega) + g(\omega)\|}{2}\right).$$

Lemma 3.12 *Let (f_n) and (g_n) be sequences in $L^1(\mu, X)$ such that*

$$|||f_n||| = |||g_n||| = 1 \quad \textit{for all } n \in \mathbb{N} \quad \textit{and} \quad \lim_{n\to\infty} |||f_n + g_n||| = 2.$$

Then

$$\lim_{n\to\infty} \int_\Omega \varphi_{(f_n,g_n)}\, d\mu = \lim_{n\to\infty} \int_\Omega \psi_{(f_n,g_n)}\, d\mu = 0.$$

Proof See the proof of Theorem 1 in [14]. □

The symbol $\sigma' := \sigma(L^1(\mu, X), L^\infty(\mu, X^*))$ denotes the topology on $L^1(\mu, X)$ of pointwise convergence on $L^\infty(\mu, X^*)$, the duality being given by

$$\langle f, g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle\, d\mu(\omega), \quad f \in L^1(\mu, X), \quad g \in L^\infty(\mu, X^*).$$

Note that $L^\infty(\mu, X^*)$ embeds isometrically into $L^1(\mu, X)^*$ and σ' is weaker than the weak topology of $L^1(\mu, X)$. It is known that $L^1(\mu, X)^* = L^\infty(\mu, X^*)$ whenever X is Asplund, see e.g. [9, p. 98, Theorem 1].

Proof of Theorem 3.9 It suffices to check that if (f_n) and (g_n) are sequences in Y such that

$$|||f_n||| = |||g_n||| = 1 \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} |||f_n + g_n||| = 2,$$

then for every $\varepsilon > 0$ there exist a subsequence (n_k) of $\mathbb{N}$ and a weakly null sequence (h_k) in $L^1(\mu, X)$ such that $\|(f_{n_k} - g_{n_k}) - h_k\|_{L^1(\mu,X)} \le \varepsilon$ for all $k \in \mathbb{N}$.

Since any bounded subset of Y is uniformly integrable, there is $\delta > 0$ such that

$$\sup_{n\in\mathbb{N}} \int_A \|f_n(\omega) - g_n(\omega)\|\, d\mu(\omega) \le \varepsilon \quad \text{for every } A \in \Sigma \text{ with } \mu(A) \le \delta. \tag{3.2}$$

Define $A_n := \{\omega \in \Omega\colon \|f_n(\omega)\| + \|g_n(\omega)\| > \frac{2c'}{\delta}\} \in \Sigma$,

$$\tilde{f}_n := f_n 1_{\Omega\setminus A_n} \quad \text{and} \quad \tilde{g}_n := g_n 1_{\Omega\setminus A_n}$$

for all $n \in \mathbb{N}$. Note that $(\tilde{f}_n)$ and $(\tilde{g}_n)$ are uniformly bounded. Bearing in mind that the inequalities

$$0 \le \varphi_{(\tilde{f}_n,\tilde{g}_n)} \le \varphi_{(f_n,g_n)} \quad \text{and} \quad 0 \le \psi_{(\tilde{f}_n,\tilde{g}_n)} \le \psi_{(f_n,g_n)}$$

hold μ-a.e. for every $n \in \mathbb{N}$, an appeal to Lemma 3.12 yields

$$\lim_{n\to\infty} \|\varphi_{(\tilde{f}_n,\tilde{g}_n)}\|_{L^1(\mu)} = \lim_{n\to\infty} \|\psi_{(\tilde{f}_n,\tilde{g}_n)}\|_{L^1(\mu)} = 0.$$

Therefore, there is a subsequence (n_k) of $\mathbb{N}$ such that

$$\lim_{k\to\infty} \varphi_{(\tilde{f}_{n_k},\tilde{g}_{n_k})} = \lim_{k\to\infty} \psi_{(\tilde{f}_{n_k},\tilde{g}_{n_k})} = 0 \quad \mu\text{-a.e.} \tag{3.3}$$

Given any $k \in \mathbb{N}$, we define $h_k := \tilde{f}_{n_k} - \tilde{g}_{n_k}$. By Chebyshev's inequality and Lemma 3.10, we have

$$\begin{aligned}\mu(A_{n_k}) &\le \frac{\delta}{2c'} \int_\Omega (\|f_{n_k}(\omega)\| + \|g_{n_k}(\omega)\|)\, d\mu(\omega) \\ &= \frac{\delta}{2c'} (\|f_{n_k}\|_{L^1(\mu,X)} + \|g_{n_k}\|_{L^1(\mu,X)}) \le \delta,\end{aligned}$$

hence (3.2) yields

$$\|(f_{n_k} - g_{n_k}) - h_k\|_{L^1(\mu,X)} = \|(f_{n_k} - g_{n_k})1_{A_{n_k}}\|_{L^1(\mu,X)} \le \varepsilon.$$

Therefore, to finish the proof it remains to check that (h_k) is weakly null in $L^1(\mu, X)$.

Claim *The sequence $(h_k(\omega))$ is weakly null in X for μ-a.e. $\omega \in \Omega$.* Indeed, since $\|\cdot\|$ is WUR, the claim will be established with the help of [8, Ch. II, Proposition 6.2(i)] by checking the following statements (a) and (b):

(a) $\|\tilde{f}_{n_k}(\omega)\| - \|\tilde{g}_{n_k}(\omega)\| \to 0$ μ-a.e. as $k \to \infty$. Indeed, since M'' is decreasing, for every $k \in \mathbb{N}$ and every $\omega \in \Omega$ we have

$$0 \le M''\left(\frac{2c'}{\delta}\right) \cdot (\|\tilde{f}_{n_k}(\omega)\| - \|\tilde{g}_{n_k}(\omega)\|)^2 \le \varphi_{(\tilde{f}_{n_k},\tilde{g}_{n_k})}(\omega).$$

The previous inequalities, the fact that M'' does not vanish on $(0, \infty)$ and (3.3) yield $\|\tilde{f}_{n_k}(\omega)\| - \|\tilde{g}_{n_k}(\omega)\| \to 0$ μ-*a.e.* as $k \to \infty$.

(b) $\|\tilde{f}_{n_k}(\omega) + \tilde{g}_{n_k}(\omega)\| - 2\|\tilde{f}_{n_k}(\omega)\| \to 0$ μ-a.e. as $k \to \infty$. Indeed, by (3.3) and (a) we have

$$M\left(\frac{\|\tilde{f}_{n_k}(\omega) + \tilde{g}_{n_k}(\omega)\|}{2}\right) - M\big(\|\tilde{f}_{n_k}(\omega)\|\big) \to 0 \quad \mu\text{-a.e. as } k \to \infty. \tag{3.4}$$

Since M is strictly increasing and continuous on $[0, \infty)$, it is a homeomorphism between $[0, \frac{2c'}{\delta}]$ and $[0, M(\frac{2c'}{\delta})]$. The uniform continuity of M^{-1} on $[0, M(\frac{2c'}{\delta})]$ and (3.4) imply that $\frac{1}{2}\|\tilde{f}_{n_k}(\omega) + \tilde{g}_{n_k}(\omega)\| - \|\tilde{f}_{n_k}(\omega)\| \to 0$ μ-a.e. as $k \to \infty$. This finishes the proof of the claim.

Note that the sequence (h_k) is uniformly bounded. Given any $x^* \in X^*$, the previous *claim* and Lebesgue's dominated convergence theorem applied to the sequence of compositions $(x^* \circ h_k)$ yield

$$\lim_{k\to\infty} x^*\left(\int_A h_k\,d\mu\right) = \lim_{k\to\infty}\int_A (x^* \circ h_k)\,d\mu = 0 \quad \text{for every } A \in \Sigma.$$

Hence the sequence $(\int_A h_k\,d\mu)$ is weakly null in X for every $A \in \Sigma$. This fact and the uniform integrability of (h_k) imply that (h_k) is σ'-convergent to 0 (see [6, Theorem 4]). Since σ' and the weak topology of $L^1(\mu, X)$ coincide (because X is Asplund), the sequence (h_k) is weakly null and the proof is finished. □

Remark 3.13 In [13, Theorem 3.1] it is proved that *if the norm* $\|\cdot\|$ *of* X *is Fréchet smooth, then* $L^1(\mu, X)$ *admits an equivalent norm which is Fréchet smooth when restricted to any* reflexive *subspace* $Y \subseteq L^1(\mu, X)$. Such a norm is obtained by the formula (3.1) applied to an Orlicz function satisfying certain properties like, for instance, $M(t) = |t| - \log(1 + |t|)$. Actually, the proof of [13, Theorem 3.1] only uses the uniform integrability of B_Y. On the other hand, any Banach space admitting a Fréchet smooth equivalent norm is Asplund (see e.g. [8, Ch. II, Corollary 3.3]). It follows that the aforementioned result in [13] applies exactly to *Asplund* subspaces $Y \subseteq L^1(\mu, X)$. Bearing in mind that $M(t) = |t| - \log(1 + |t|)$ also fulfills the requirements of Theorem 3.9, we deduce that *if the norm* $\|\cdot\|$ *of* X *is Fréchet smooth and WUR, then* $L^1(\mu, X)$ *admits an equivalent norm which is Fréchet smooth and WUR when restricted to any Asplund subspace* $Y \subseteq L^1(\mu, X)$.

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