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# Derivations on group algebras of a locally compact abelian group

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**Abstract** Let $G$ be a locally compact abelian group. In this paper, we study derivations on the Banach algebra $L_0^\infty(G)^*$. We prove that any derivation on $L_0^\infty(G)^*$ maps it into its radical and a derivation on $L_0^\infty(G)^*$ is continuous if and only if its restriction to the right annihilator of $L_0^\infty(G)^*$ is continuous. We also show that the composition of two derivations on $L_0^\infty(G)^*$ is always a derivation on it and the zero map is the only centralizing derivation on $L_0^\infty(G)^*$. Finally, we characterize the space of inner derivations of $L_0^\infty(G)^*$ and show that $G$ is discrete if and only if there exist $i, j, k \in \mathbb{N}$ such that $[d(m), n]_i^j = [m, n]_k$ for all $m, n \in L_0^\infty(G)^*$; or equivalently, any inner derivation on $L_0^\infty(G)^*$ is zero.

**Keywords** Locally compact abelian group · Derivation · Inner derivation · $k$-centralizing

**Mathematics Subject Classification** 43A15 · 47B47 · 46H40 · 16W25

## 1 Introduction

Let $G$ be a locally compact abelian group with a fixed left Haar measure and let $L^1(G)$ be the group algebra of $G$ defined as in [4] equipped with the convolution product $*$

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and the norm $\|.\|_1$. We denote by $L_0^\infty(G)$ the subspace of all functions $f \in L^\infty(G)$, the usual Lebesgue space as defined in [4] equipped with the essential supremum norm $\|.\|_\infty$, that for each $\varepsilon > 0$, there is a compact subset $K$ of $G$ for which

$$\|f\chi_{G\backslash K}\|_\infty < \varepsilon,$$

where $\chi_{G\backslash K}$ denotes the characteristic function $G\backslash K$ on $G$. It is well-known from [6] that the subspace $L_0^\infty(G)$ is a topologically introverted subspace of $L^\infty(G)$, that is, for each $n \in L_0^\infty(G)^*$ and $f \in L_0^\infty(G)$, the function $nf \in L_0^\infty(G)$, where

$$\langle nf, \phi\rangle = \langle n, f\phi\rangle, \quad \text{in which} \quad \langle f\phi, \psi\rangle = \langle f, \phi * \psi\rangle$$

for all $\phi, \psi \in L^1(G)$. Hence $L_0^\infty(G)^*$ is a Banach algebra with the first Arens product "·" defined by the formula

$$\langle m \cdot n, f\rangle = \langle m, nf\rangle$$

for all $m, n \in L_0^\infty(G)^*$ and $f \in L_0^\infty(G)$. Note that $L^1(G)$ may be regarded as a subspace of $L_0^\infty(G)^*$ and then $L^1(G)$ is a closed ideal in $L_0^\infty(G)^*$ with a bounded approximate identity [6]. Let $\Lambda_0(G)$ denote the set of all weak*-cluster points of an approximate identity in $L^1(G)$ bounded by one. It is easy to see that if $u \in \Lambda_0(G)$, then for every $m \in L_0^\infty(G)^*$ and $\phi \in L^1(G)$

$$m \cdot u = m \quad \text{and} \quad u \cdot \phi = \phi.$$

Let $\pi$ denote the natural continuous operator that associates to any functional in $L_0^\infty(G)^*$ its restriction to $C_0(G)$, the space of all continuous functions on $G$ vanishing at infinity. Then the restriction map $\pi$ from $L_0^\infty(G)^*$ into $M(G)$, the measure algebra of $G$ as defined in [4] endowed with the convolution product $*$ and the total variation norm, is a homomorphism and

$$\pi_u := \pi|_{u \cdot L_0^\infty(G)^*}$$

is an isomorphism for all $u \in \Lambda_0(G)$. Note that, for every $f \in L_0^\infty(G)$ and $\phi \in L^1(G)$, we have $f\phi \in C_0(G)$. Hence for every $n \in L_0^\infty(G)^*$ and $f \in L_0^\infty(G)$, we may define the function $\pi(n)f \in L^\infty(G)$ by

$$\langle \pi(n)f, \phi\rangle = \langle \pi(n), f\phi\rangle.$$

Then

$$\pi(n)f = nf \in L_0^\infty(G).$$

This enable us to define the functional $m \cdot \pi(n) \in L_0^\infty(G)^*$ by

$$\langle m \cdot \pi(n), f\rangle = \langle m, \pi(n) f\rangle.$$

It follows that

$$m \cdot \pi(n) = m \cdot n$$

for all $m, n \in L_0^\infty(G)^*$; see [6]. Let $\mathrm{Ann}_r(L_0^\infty(G)^*)$ denote the right annihilator of $L_0^\infty(G)^*$; i.e. the set of all $r \in L_0^\infty(G)^*$ such that $m \cdot r = 0$ for all $m \in L_0^\infty(G)^*$. one can easily prove that

$$\mathrm{Ann}_r(L_0^\infty(G)^*) = \ker(\pi).$$

Furthermore, an easy application of the Hahn-Banach theorem shows that $G$ is discrete if and only if

$$\mathrm{Ann}_r(L_0^\infty(G)^*) = \{0\}.$$

Let $\mathfrak{A}$ be a Banach algebra; a linear mapping $d : \mathfrak{A} \to \mathfrak{A}$ is called a *derivation* if

$$d(ab) = d(a)b + ad(b).$$

A fundamental question for derivations concerns their image. Singer and Wermer [12] showed that the range of a continuous derivation on a commutative Banach algebra is contained in the radical of algebra. They conjectured that this result holds for discontinuous derivations. Thomas [13] proved this conjecture. Posner [10] gave a noncommutative version of the Singer-Wermer theorem for prime rings. He proved that the zero map is the only centralizing derivation on a noncommutative prime ring (Posner's second theorem). These results have been extended in various directions by several authors; see for instance [1,3,5,7,8,11,14].

Can we apply the well-known results concerning derivations of commutative Banach algebras and derivations of prime rings to $L_0^\infty(G)^*$? This question seems natural, because $L_0^\infty(G)^*$ is neither a commutative Banach algebra nor a prime ring, when $G$ is a non-discrete group. In this paper, we investigate the truth of these results for $L_0^\infty(G)^*$.

This paper is organized as follows: In Sect. 2, we investigate the Singer- Wermer conjecture and automatic continuity for $L_0^\infty(G)^*$. We prove that the range of a derivation on the noncommutative Banach algebra $L_0^\infty(G)^*$ is contained in the radical of $L_0^\infty(G)^*$ and a derivation on $L_0^\infty(G)^*$ is continuous if and only if its restriction to $\mathrm{Ann}_r(L_0^\infty(G)^*)$ is continuous. In Sect. 3, we investigate Posner's second theorem and show that the zero map is the only centralizing derivation on $L_0^\infty(G)^*$. In Sect. 4, we characterize the space of all inner derivations of $L_0^\infty(G)^*$ and prove that $G$ is discrete if and only if any inner derivation on $L_0^\infty(G)^*$ is zero.

## 2 The Singer-Wermer conjecture for $L_0^\infty(G)^*$

We commence this section with the following result.

**Theorem 1** *Let $G$ be a locally compact abelian group and $d$ be a derivation on $L_0^\infty(G)^*$. Then $d$ has its image in the right annihilator of $L_0^\infty(G)^*$.*

*Proof* Let $u \in \Lambda_0(G)$. Define the function $D : M(G) \to M(G)$ by

$$D(\mu) = \pi \circ \tilde{d} \circ \pi_u^{-1}(\mu),$$

where $\tilde{d} = d|_{u \cdot L_0^\infty(G)^*}$. It is routine to check that $D$ is derivation on the commutative semisimple Banach algebra $M(G)$. Hence $D$ is zero. It follows that

$$\tilde{d} \circ \pi_u^{-1}(M(G)) \subseteq \ker(\pi) = \text{Ann}_r(L_0^\infty(G)^*).$$

Since $\pi_u$ maps $u \cdot L_0^\infty(G)^*$ onto $M(G)$, we have

$$d(u \cdot L_0^\infty(G)^*) \subseteq \text{Ann}_r(L_0^\infty(G)^*).$$

On the one hand,

$$m \cdot d(r) = d(m \cdot r) - d(m) \cdot r = 0$$

for all $m \in L_0^\infty(G)^*$ and $r \in \text{Ann}_r(L_0^\infty(G)^*)$. So

$$d(\text{Ann}_r(L_0^\infty(G)^*)) \subseteq \text{Ann}_r(L_0^\infty(G)^*).$$

Now, we only need to recall that $L_0^\infty(G)^*$ is the Banach space direct sum of $u \cdot L_0^\infty(G)^*$ and $\text{Ann}_r(L_0^\infty(G)^*)$. □

Before we give the following consequence of Theorem 1, let us recall that a linear mapping $T$ on $L_0^\infty(G)^*$ is called *spectrally bounded* if there is a non-negative number $\alpha$ such that $r(T(m)) \leq \alpha r(m)$ for all $m \in L_0^\infty(G)^*$, where $r(\cdot)$ stands for the spectral radius.

**Corollary 1** *Let $G$ be a locally compact abelian group. Then the following statements hold.*

*(i) Every derivation on $L_0^\infty(G)^*$ maps it into its radical.*
*(ii) Primitive ideals of $L_0^\infty(G)^*$ are invariant under derivations on $L_0^\infty(G)^*$.*
*(iii) Every derivation on $L_0^\infty(G)^*$ is spectrally bounded.*
*(iv) The composition of two derivations on $L_0^\infty(G)^*$ is always a derivation on $L_0^\infty(G)^*$.*

*Proof* The statement (i) follows from Theorem 1 together with the fact that the set of nilpotent elements is contained in the radical of the algebra. The statement (ii) follows immediately from (i). For (iii), note that if $d$ is a derivation on $L_0^\infty(G)^*$, then

$$d(m)^i = 0$$

for all $m \in L_0^\infty(G)^*$ and $i \geq 2$. Finally, the statement (iv) follows from Theorem 1. □

As an another consequence of Theorem 1, we have the following result.

**Corollary 2** *Let G be a locally compact abelian group. Then the following statements hold.*

*(i) If $d$ is a derivation on $L_0^\infty(G)^*$, then $d|_{L^1(G)}$ is zero.*
*(ii) The zero map is the only weak$^*$ − weak$^*$ continuous derivation on $L_0^\infty(G)^*$.*

*Proof* First note that

$$r \cdot \phi = \phi \cdot r = 0$$

for all $r \in \mathrm{Ann}_r(L_0^\infty(G)^*)$ and $\phi \in L^1(G)$. So if $d$ is a derivation on $L_0^\infty(G)^*$, then

$$d(\phi_1 * \phi_2) = d(\phi_1) \cdot \phi_2 + \phi_1 \cdot d(\phi_2) = 0$$

for all $\phi_1, \phi_2 \in L^1(G)$. In view of Cohen's factorization theorem, $d = 0$ on $L^1(G)$. So (i) holds. The statement (ii) follows from Goldstein's theorem (see e.g. [2, chapter 5, Proposition 4.1]) and (i). □

**Theorem 2** *Let G be a locally compact abelian group and d be a derivation on $L_0^\infty(G)^*$. Then the following statements hold.*

*(i) For every $u \in \Lambda_0(G)$, $d|_{u \cdot L_0^\infty(G)^*}$ is always continuous.*
*(ii) d is continuous if and only if $d|_{Ann_r(L_0^\infty(G)^*)}$ is continuous.*

*Proof* (i) Let $u \in \Lambda_0(G)$ and $(u \cdot m_\alpha)_{\alpha \in A}$ be a net in $L_0^\infty(G)^*$ such that $u \cdot m_\alpha \to 0$. It follows from Theorem 1 that

$$\begin{aligned}\|d(u \cdot m_\alpha)\| &= \|d(u \cdot u \cdot m_\alpha)\| \\ &= \|d(u) \cdot u \cdot m_\alpha\| \leq \|d(u)\| \, \|u \cdot m_\alpha\|\end{aligned}$$

for all $\alpha \in A$. Hence $d(u \cdot m_\alpha) \to 0$. This shows that $d|_{u \cdot L_0^\infty(G)^*}$ is continuous. (ii) Let $m \in L_0^\infty(G)^*$ and $u \in \Lambda_0(G)$. Then $m - u \cdot m$ is an element of $\mathrm{Ann}_r(L_0^\infty(G)^*)$. If $d|_{\mathrm{Ann}_r(L_0^\infty(G)^*)}$ is continuous, then for some $\alpha > 0$

$$\|d(m - u \cdot m)\| \leq \alpha \|m - u \cdot m\| \leq 2\,\alpha \|m\|.$$

By (i) there exists $\beta > 0$ such that

$$\|d(u \cdot m)\| \leq \beta \, \|u \cdot m\| \leq \beta \, \|m\|.$$

Thus

$$\|d(m)\| = \|d(u \cdot m) + d(m - u \cdot m)\| \leq (2\alpha + \beta)\|m\|.$$

It follows that $d$ is continuous. □

Our last result of this section is an immediate consequence of Theorem 2(ii).

**Corollary 3** *Let G be a discrete abelian locally compact group.Then every derivation on $L_0^\infty(G)^*$ is continuous.*

## 3 Posner's second theorem for $L_0^\infty(G)^*$

Let $\mathrm{Z}(L_0^\infty(G)^*)$ denote the center of $L_0^\infty(G)^*$; that is, the set of all $m \in L_0^\infty(G)^*$ such that $m \cdot n = n \cdot m$ for all $n \in L_0^\infty(G)^*$.

**Proposition 1** *Let G be a locally compact abelian group. Then*

$$\mathrm{Z}(L_0^\infty(G)^*) = L^1(G).$$

*Proof* Let $u \in \Lambda_0(G)$. Since $L^1(G)$ is an ideal in $L_0^\infty(G)^*$ and $\pi$ is identity on $L^1(G)$, we have

$$\begin{aligned}\phi \cdot m &= \pi(\phi \cdot m) = \pi(\phi) * \pi(m)\\ &= \pi(m) * \pi(\phi) = \pi(m \cdot \phi) = m \cdot \phi\end{aligned}$$

for all $\phi \in L^1(G)$ and $m \in L_0^\infty(G)^*$. So $L^1(G)$ is contained in $\mathrm{Z}(L_0^\infty(G)^*)$. For $m \in \mathrm{Z}(L_0^\infty(G)^*)$, we have

$$m = m \cdot u = u \cdot m.$$

This shows that

$$m \in \cap_{u \in \Lambda_0(G)} u \cdot L_0^\infty(G)^*.$$

Hence $\mathrm{Z}(L_0^\infty(G)^*)$ is contained in $L^1(G)$; see Theorem 2.11 of [6]. □

For any positive integer $k$, a mapping $T : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is called $k$-centralizing if

$$[T(m), m^k] \in Z(L_0^\infty(G)^*)$$

for all $m \in L_0^\infty(G)^*$; in a special case when $[T(m), m^k] = 0$ for all $m \in L_0^\infty(G)^*$, $T$ is called *$k$-commuting*, where $[m, n] := m \cdot n - n \cdot m$ for all $m, n \in L_0^\infty(G)^*$.

**Theorem 3** *Let $G$ be a locally compact abelian group, $d$ be a derivation on $L_0^\infty(G)^*$ and $k$ be a positive integer. Then the following assertions are equivalent.*

*(a)* $d = 0$.
*(b)* *$d$ is $k$-centralizing.*
*(c)* *$d$ is $k$-commuting.*

*Proof* It is clear that (a) implies (b). If (b) holds, then by Theorem 1 and Proposition 1, we obtain

$$\begin{aligned}[d(m), m^k] &= d(m) \cdot m^k \\ &= d(m^{k+1}) \in \mathrm{Ann}_r(L_0^\infty(G)^*) \cap L^1(G) = \{0\}\end{aligned}$$

for all $m \in L_0^\infty(G)^*$. Thus (c) holds. Now, let $d$ be $k$-commuting. Choose $u \in \Lambda_0(G)$. Then

$$d(u) = [d(u), u] = [d(u), u^k] = 0. \tag{1}$$

For every $r \in \mathrm{Ann}_r(L_0^\infty(G)^*)$, we have $(r + u) = (r + u)^k$. Hence

$$d(r) = [d(r + u), (r + u)] = [d(r + u), (r + u)^k] = 0. \tag{2}$$

From (1) and (2) we infer that

$$\begin{aligned}d(m) &= d(u \cdot m) + d(m - u \cdot m) \\ &= d(u) \cdot m + d(m - u \cdot m) \\ &= 0\end{aligned}$$

for all $m \in L_0^\infty(G)^*$. Thus (c) implies that (a). □

As an immediate consequence from Theorem 3, we have the following result.

**Corollary 4** *Let $G$ be a locally compact abelian group. Then the zero map is the only centralizing derivation on $L_0^\infty(G)^*$.*

Let $[m, n]_1 = [m, n]$ and $[m, n]_k = [[m, n]_{k-1}, n]$ for all $m, n \in L_0^\infty(G)^*$ and all positive integers $k > 1$.

**Corollary 5** *Let $G$ be a locally compact abelian group and $d$ be a derivation on $L_0^\infty(G)^*$. Then the following assertions are equivalent.*

*(a)* $d = 0$.
*(b)* *$d$ is centralizing.*
*(c)* *For every $k \in \mathbb{N}$, $d$ is $k$-centralizing.*
*(d)* *There exists $k \in \mathbb{N}$ such that $d$ is $k$-centralizing.*
*(e)* *There exist positive integers $k, l$ such that $l \geq 2$ and $[d(m), n]_k = [m, n]^l$ for all $m, n \in L_0^\infty(G)^*$*

*Proof* This follows from Theorem 3 with the observation that for every $m, n \in L_0^\infty(G)^*$, we have $[m, n] \in \mathrm{Ann}_r(L_0^\infty(G)^*)$ and so $[m, n]^l = 0$ for all $l \geq 2$. □

We conclude the section with the following result.

**Theorem 4** *Let G be a locally compact abelian group and d be a derivation on $L_0^\infty(G)^*$. Then the following assertions are equivalent.*

*(a) G is discrete.*
*(b) $L_0^\infty(G)^*$ is commutative.*
*(c) There exist $i, j, k \in \mathbb{N}$ such that $[d(m), n]_i^j = [m, n]_k$ for all $m, n \in L_0^\infty(G)^*$.*

*In this case, $d = 0$.*

*Proof* If $G$ is discrete, then by Proposition 3.1 of [9], we have $L_0^\infty(G)^* = L^1(G)$. Since $G$ is an abelian, $L_0^\infty(G)^*$ is commutative. Thus (a) implies (b). It is clear that (b) implies (c) and (d). Now, let $i, j, k \in \mathbb{N}$ and

$$d(m \cdot n^i)^j = [m, n] \cdot n^k.$$

Then for every $u \in \Lambda_0(G)$, we have

$$d(u)^j = d(u \cdot u^i)^j = [u, u] \cdot u^k = 0.$$

On the one hand, for every $r \in \mathrm{Ann}_r(L_0^\infty(G)^*)$, we get

$$\begin{aligned} d(u)^j &= d(u \cdot (u + r))^j = d(u \cdot (u + r)^i)^j \\ &= [u, u + r] \cdot (u + r)^k = [u, u + r] \cdot (u + r) = -r. \end{aligned}$$

Hence

$$\mathrm{Ann}_r(L_0^\infty(G)^*) = \{0\},$$

which implies that $G$ is discrete. To complete the proof, it suffices to notice that the assertion (c) implies that $d(m \cdot n^i)^j = [m, n] \cdot n^{k-1}$. □

## 4 Inner derivations of $L_0^\infty(G)^*$

A derivation $d$ on $L_0^\infty(G)^*$ is said to be *inner* if there exists $n_0 \in L_0^\infty(G)^*$ such that $d(m) = [m, n_0]$ for all $m \in L_0^\infty(G)^*$.

**Proposition 2** *Let G be a locally compact abelian group and d be a derivation on $L_0^\infty(G)^*$. Then the following assertions are equivalent.*

*(a) d is inner.*
*(b) There exists $n_0 \in L_0^\infty(G)^*$ such that for each $k \in \mathbb{N}$ the mapping $m \mapsto d(m) + n_0 \cdot m$ is k-commuting.*

(c) *There exists* $n_0 \in L_0^\infty(G)^*$ *and* $k \in \mathbb{N}$ *such that the mapping* $m \mapsto d(m) + n_0 \cdot m$ *is* $k$*-commuting.*
(d) *There exists* $n_0 \in L_0^\infty(G)^*$ *and* $k \in \mathbb{N}$ *such that the mapping* $m \mapsto d(m) + n_0 \cdot m$ *is* $k$*-centralizing.*

*Proof* Let there exist $n_0 \in L_0^\infty(G)^*$ such that $d(m) = [m, n_0]$ for all $m \in L_0^\infty(G)^*$. For $k \in \mathbb{N}$ and $m \in L_0^\infty(G)^*$, we obtain

$$\begin{aligned} m^k \cdot n_0 \cdot m &= m^k \cdot \pi(n_0 \cdot m) = m^k \cdot \pi(n_0) * \pi(m) \\ &= m^k \cdot \pi(m) * \pi(n_0) = m^{k+1} \cdot n_0. \end{aligned}$$

It follows that

$$\begin{aligned} [d(m) + n_0 \cdot m, m^k] &= d(m) \cdot m^k + n_0 \cdot m^{k+1} - m^k \cdot n_0 \cdot m \\ &= d(m^{k+1}) + n_0 \cdot m^{k+1} - m^{k+1} \cdot n_0 \\ &= 0. \end{aligned}$$

Hence (a) implies (b). It is obvious that (b)⇒(c)⇒ (d). To complete the proof, let (d) hold. Define the function $D : L_0^\infty(G)^* \to L_0^\infty(G)^*$ by

$$D(m) = d(m) - [m, n_0].$$

It is clear that $D$ is a derivation on $L_0^\infty(G)^*$. So

$$[D(m), m^k] = D(m) \cdot m^k = [d(m) + n_0 m, m^k] \in \mathrm{Z}(L_0^\infty(G)^*).$$

We now invoke Corollary 4 to conclude that $D = 0$. So, we obtain (a). □

In the sequel, let $\mathrm{InnD}(L_0^\infty(G)^*)$ be the space of all inner derivations on $L_0^\infty(G)^*$.

**Theorem 5** *Let $G$ be a locally compact abelian group. Then* $\mathrm{InnD}(L_0^\infty(G)^*)$ *is continuously linearly isomorphic to* $L_0^\infty(G)^*/L^1(G)$.

*Proof* We define the mapping $\mathfrak{I}$ from $L_0^\infty(G)^*/L^1(G)$ into $\mathrm{InnD}(L_0^\infty(G)^*)$ by

$$\mathfrak{I}(m + L^1(G)) = \mathfrak{I}_m,$$

where $\mathfrak{I}_m(n) = [n, m]$ for all $n \in L_0^\infty(G)^*$. By Proposition 1, the mapping $\mathfrak{I}$ is well defined. Obviously, $\mathfrak{I}$ is a linear map from $L_0^\infty(G)^*/L^1(G)$ onto $\mathrm{InnD}(L_0^\infty(G)^*)$. To see that $\mathfrak{I}$ is injective, let $m \in L_0^\infty(G)^*$ and

$$\mathfrak{I}(m + L^1(G)) = 0.$$

Then

$$\mathfrak{I}_m(n) = n \cdot m - m \cdot n = 0$$

for all $n \in L_0^\infty(G)^*$. It follows that

$$m \in \mathrm{Z}(L_0^\infty(G)^*) = L^1(G).$$

Hence $m + L^1(G) = L^1(G)$. Consequently, $\mathfrak{I}$ is an isomorphism. Now, let $n \in L_0^\infty(G)^*$ and $\phi \in L^1(G)$. Then

$$\begin{aligned}\|\mathfrak{I}_m(n)\| &= \|n \cdot m - m \cdot n\| \\ &\leq \|n \cdot m - \phi \cdot n\| + \|\phi \cdot n - m \cdot n\| \\ &\leq \|n\|\ \|m - \phi\| + \|\phi - m\|\ \|n\| \\ &= 2\|n\|\ \|m - \phi\|\end{aligned}$$

for all $m \in L_0^\infty(G)^*$. This implies that

$$\|\mathfrak{I}(m + L^1(G))\| = \|\mathfrak{I}_m\| \leq 2\|m - \phi\|$$

for all $m \in L_0^\infty(G)^*$ and $\phi \in L^1(G)$. Hence

$$\begin{aligned}\|\mathfrak{I}(m + L^1(G))\| &\leq 2\inf\{\|m - \phi\| : \phi \in L^1(G)\} \\ &= 2\inf\{\|m + \phi\| : \phi \in L^1(G)\} = 2\ \|m + L^1(G)\|.\end{aligned}$$

Therefore, $\mathfrak{I}$ is continuous. □

We finish the paper with following result.

**Theorem 6** *Let $G$ be a locally compact abelian group. Then the following assertions are equivalent.*

*(a) $G$ is discrete.*
*(b) Any derivation on $L_0^\infty(G)^*$ is zero.*
*(c) Any inner derivation on $L_0^\infty(G)^*$ is zero.*

*Proof* If $G$ is discrete, then $\mathrm{Ann}_r(L_0^\infty(G)^*) = \{0\}$. By Theorem 1,

$$d(L_0^\infty(G)^*) \subseteq \mathrm{Ann}_r(L_0^\infty(G)^*) = \{0\}.$$

Hence (a) implies (b). It is plain that (b) implies (c). Finally, if (c) holds, then $[m, n] = 0$ for all $m, n \in L_0^\infty(G)^*$. This implies that

$$\mathrm{Z}(L_0^\infty(G)^*) = L_0^\infty(G)^*.$$

So $L^1(G) = L_0^\infty(G)^*$. This shows that $G$ is discrete; see Proposition 3.1 of [9]. □

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