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Maximizing mean exit-time of the Brownian motion on Riemannian manifolds

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Abstract We study the functional $\Omega \mapsto \mathcal{E}(\Omega)$, where Ω runs in the set of all compact domains of fixed volume v in any Riemannian manifold (M, g) and where $\mathcal{E}(\Omega)$ is the *mean exit-time of the Brownian motion* (also called *torsional rigidity*) of Ω. We first prove that, when (M, g) is strictly isoperimetric at one of its points, the maximum of this functional is realized by the geodesic ball centered at this point. When (M, g) is *any* Riemannian manifold, for every domain Ω in M, we prove that $\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*)$, where Ω^* is the corresponding symmetrized domain on a model-space (M^*, g^*). We also consider the functional $\Omega \mapsto \mathcal{E}(\Omega)$, when Ω runs in the set of all compact domains, with smooth boundary in the class of all Riemannian manifolds with "bounded" geometry. We prove two results in this direction. In the first one (Theorem 1.9) we prove that for every complete, connected Riemannian manifold (M, g) whose Ricci curvature satisfies $\mathrm{Ric}_g \geq (n-1)g$ and for every compact domain with smooth boundary Ω in M one has $\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*)$, where Ω^* is a geodesic ball of the canonical

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sphere $(\mathbb{S}^n, g_0)$ such that $\frac{\mathrm{Vol}(\Omega^*, g_0)}{\mathrm{Vol}(\mathbb{S}^n, g_0)} = \frac{\mathrm{Vol}(\Omega, g)}{\mathrm{Vol}(M, g)}$. Morever, if there exists some domain $\Omega \subset M$ such that $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ then (M, g) is isometric to $(\mathbb{S}^n, g_0)$ and Ω is isometric to Ω^*. The second result (Theorem 1.10) shows that if (M, g) is any compact Riemannian manifold and Ω is any compact domain with smooth boundary in M such that $\mathrm{Vol}(\Omega) \leq \frac{1}{2}\mathrm{Vol}(M)$, then $\mathcal{E}(\Omega) \leq \frac{1}{H(M, g)^2}$, where $H(M, g)$ is Cheeger's isoperimetric constant.

Keywords Brownian motion · Harmonic domain · Harmonic manifold · Isoperimetric manifold at a point

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1 Introduction

Let (M, g) be a n-dimensional Riemannian manifold[1] (compact or not) and dv_g the associated Riemannian measure. Let Ω be any compact connected domain in M, with *smooth boundary* $\partial\Omega$ (by this, in the case where M is compact, we also intend that the interior of $M\backslash\Omega$ is a non empty open set). Let us denote by Δ the Laplacian[2] on M associated to the Riemannian metric g, and let f_Ω be the solution of the following Dirichlet problem

$$\begin{cases} \Delta f = 1 & \text{on} \quad \Omega \\ \quad f = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Let $C_c^\infty(\Omega)$ be the space of C^∞ functions with compact support in the interior of Ω and let $H^2_{1,c}(\Omega)$ be its completion with respect to the Solobev norm $\|f\|_{H_1^2(\Omega)} = (\|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)})^{\frac{1}{2}}$. As f_Ω is regular and vanishes on $\partial\Omega$, then $f_\Omega \in H^2_{1,c}(\Omega)$. Moreover, $f_\Omega(x) > 0$ for any $x \in \mathring{\Omega}$.

On the space $H^2_{1,c}(\Omega)$ let us consider the functional E_Ω defined by

$$E_\Omega(f) = \frac{1}{\mathrm{Vol}(\Omega)}\left(2\int_\Omega f\, dv_g - \int_\Omega |\nabla f|^2\, dv_g\right). \tag{2}$$

Computing the first variation of E_Ω at the point f_Ω, using Green's formula and the strict concavity of E_Ω, we get that f_Ω is the (unique) critical point of E_Ω and that it is a maximum. This justifies the following definition (see [18] and references therein for more details).

1 In the sequel, except the hemisphere, all the Riemannian manifolds under consideration will be complete.

2 By convention, the sign of Δ is given by $\Delta f = -\mathrm{Trace}(\nabla df)$ for any $f \in C^\infty(M)$.

Definition 1.1 Let $\Omega \subset M$ be as above. The *mean exit-time from* Ω *of the Brownian motion*[3] (that we shall denote by *mean exit-time from* Ω for the sake of simplicity) is the value

$$\begin{aligned}\mathcal{E}(\Omega) &= \max_{f \in H^2_{1,c}(\Omega)} (E_\Omega(f)) = E_\Omega(f_\Omega) = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega f_\Omega dv_g \\ &= \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega |\nabla f_\Omega|^2 dv_g,\end{aligned}$$

where the two last equalities are deduced from (1) and from Green's formula.

Remark 1.2 Notice that $\widetilde{\mathcal{E}}(\Omega) = \mathrm{Vol}(\Omega)\mathcal{E}(\Omega)$ is the (so called) "torsional rigidity of Ω", whose name comes from the fact that, when Ω is a domain of the Euclidean plane, $\widetilde{\mathcal{E}}(\Omega)$ is the torsional rigidity of a beam whose cross-section is Ω. However we have preferred to consider the invariant $\mathcal{E}(\Omega)$ instead of $\widetilde{\mathcal{E}}(\Omega)$ because its physical meaning "mean exit-time from Ω of the Brownian motion" remains valid on any Riemannian manifold of any dimension, and also because it has the same homogeneity as the Riemannian metric itself, i.e.

$$\mathcal{E}(\Omega, \lambda^2 g) = \lambda^2 \mathcal{E}(\Omega, g), \tag{3}$$

which will simplify the comparison of the "mean exit-times" from two domains in two different compact Riemannian manifolds (see further results).

On any Riemannian manifold (M, g), let us consider the functional $\Omega \mapsto \mathcal{E}(\Omega)$, where Ω runs in the set of all compact domains with smooth boundary and prescribed volume v. It is known that its critical points are the *harmonic domains*, namely those domains $\Omega \subset M$ such that the function $\|\nabla f_\Omega(x)\|$ is constant on the boundary $\partial\Omega$ (see for instance [18], Proposition 2.1). In a Riemannian manifold (M, g) which is harmonic at one of its point x_0 (see next section and [5] for the definition of harmonic manifolds), it is well known that every geodesic ball centered at x_0 is a harmonic domain. Conversely, one of the fundamental questions of this field is:

Question 1.3 On a Riemannian manifold (M, g) which is harmonic at some of its points x_0 is every harmonic domain a geodesic ball centered at x_0?

Positive answers to this question were given by Serrin [21] when (M, g) is the Euclidean space and by Kumaresan and Prajapat [15] when (M, g) is the hyperbolic space or the canonical hemisphere. On the contrary, tubular neighbourhoods in S^3, of some geodesic circle S^1 (and, more generally, domains with isoparametric boundary in $\mathbb{S}^n$) are examples of harmonic domains of $\mathbb{S}^n$ which are not geodesic balls.

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[3] The reason why $\mathcal{E}(\Omega)$ is called "mean exit-time from Ω of the Brownian motion" is the following : for every $x \in \Omega$, $f_\Omega(x)$ is the expectation of the first time exit from Ω of the Brownian motion starting from the point x; thus $\mathcal{E}(\Omega) = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega f_\Omega \, dv_g$ is the mean value of this expectation with respect to all the initial points $x \in \Omega$. See for instance [18] for more explanations about the meaning of this definition.

As the answer to Question 1.3 is negative in a general harmonic manifold (and, a fortiori, in non harmonic Riemannian manifolds), it makes sense to study the *maxima* (instead of the critical points) of the functional $\Omega \mapsto \mathcal{E}(\Omega)$ among all domains of prescribed volume v (obviously such maxima are harmonic domains). This study is one of the aims of the present paper.

The following result gives a positive answer to the analogous of Question 1.3 in the case where (M, g) belongs to the class of Riemannian manifolds which are *strictly isoperimetric* at one of their points (see Definition 2.3 in Sect. 2 below); this class of manifolds contains the euclidean and hyperbolic spaces and the hemisphere as before (see [7] and [18] for previous proofs of Proposition 1.4 in these three cases), and also the whole sphere and a lot of spaces of revolution (see Example 2.4).

Proposition 1.4 *Let (M, g) be a Riemannian manifold which is isoperimetric at some point $x_0 \in M$, for every $v \in]0, \mathrm{Vol}(M, g)[$, the functional $\Omega \mapsto \mathcal{E}(\Omega)$ (where Ω runs in the set of all compact domains in M, with smooth boundary and prescribed volume v) attains its maximum when Ω is the geodesic ball Ω^* of volume v centered at x_0 [i.e., $\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*)$]. Moreover, if (M, g) is strictly isoperimetric at x_0 then this maximum is unique, i.e., the equality $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ is realized if and only if Ω is isometric to Ω^*.*

Proposition 1.4 is a particular case of the following general lemma, which allows to compare the mean exit-times from domains in two different manifolds. The reader is referred to Sect. 3 below for the definitions (Definition 3.1) of the *symmetrized domain* Ω^* of a given domain Ω and of a *pointed isoperimetric model space* (M^*, g^*, x^*) associated to a Riemannian manifold (M, g) (denoted by PIMS in the sequel).

Lemma 1.5 *Let (M, g) be a Riemannian manifold and let (M^*, g^*, x^*) be a PIMS for (M, g). Let Ω be any compact domain with smooth boundary in M and let Ω^* be its symmetrized domain. Then*

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*).$$

Moreover, if (M^*, g^*, x^*) is a strict PIMS for (M, g), then the equality $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ is realized if and only if Ω is isometric to Ω^*.

To prove this lemma, the main tool is a version of Schwarz's symmetrization valid for manifolds, which is described in Sect. 3. Lemma 1.5 is the first step in order to answer the following extension of Question 1.3:

Questions 1.6 [4] Let us consider a class $\mathcal{M}$ of Riemannian manifolds whose geometry is "bounded" (the meaning of this notion will be made precise in each result). Does there exist some Riemannian manifold $(M^*, g^*) \in \mathcal{M}$ and some point $x^* \in M^*$ such that the functional $\Omega \mapsto \mathcal{E}(\Omega)$ (where Ω runs in the set of all compact domains,

[4] A positive answer to this question would provide sharp universal upper bounds C(v) for the mean exit-time $\mathcal{E}(\Omega)$ which are independent on the geometry of (M, g), except for some a priori bounds (on curvature or Cheeger's isoperimetric constant for example), and on the geometry of the domain $\Omega \subset M$ (provided that this domain has prescribed volume v).

with smooth boundary and prescribed volume v, in all the Riemannian manifolds $(M, g) \in \mathcal{M}$) attains its maximum when Ω is the geodesic ball Ω^* of volume v in (M^*, g^*), centered at x^*? When another domain $\Omega \subset (M, g)$ realizes this maximum, is Ω isometric to Ω^*? Is (M, g) isometric to (M^*, g^*)? When there exists some domain $\Omega \subset M$ such that $\mathcal{E}(\Omega)$ is not far from this maximal value, is the ambient manifold M diffeomorphic to M^*?

The second step in order to answer this question is to find a "universal" PIMS which is valid for all the Riemannian manifolds $(M, g) \in \mathcal{M}$.

For example, in the noncompact case, one has the following conjecture, known as Cartan–Hadamard's conjecture (or Aubin's conjecture) in the literature. We recall that a Cartan–Hadamard manifold is a complete simply connected Riemannian manifold with non positive sectional curvature.

Conjecture 1.7 The Euclidean n-dimensional space E^n, pointed at any point $x^* \in E^n$, is a strict PIMS for every Cartan-Hadamard manifold of the same dimension.

This conjecture is known to be true when the dimension n is equal to 2 (it is a classical fact, using the Gauss–Bonnet formula, proved for the first time by A. Weil in [23]), in dimension 4 (it was proved by Croke [8], using Santalo's formula) and in dimension 3 (it is a more recent proof by Kleiner [14]). In higher dimensions, Conjecture 1.7 is still open. Using these results we immediately get the following corollary of Lemma 1.5 which provides a first answer to Questions 1.6 when $\mathcal{M}$ is the class of Cartan–Hadamard manifolds of dimension at most 4:

Corollary 1.8 *Let (M, g) be a Cartan–Hadamard manifold of dimension $n \leq 4$. For every compact domain $\Omega \subset M$ with smooth boundary, one has*

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*),$$

where Ω^* is the Euclidean n-ball with the same volume as Ω. Moreover, the equality $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ is realized if and only if Ω is isometric to an Euclidean ball.

Notice that, if Conjecture 1.7 were true in every dimension n then Corollary 1.8 would be automatically true in any dimension.

In the compact case, the main tools are the celebrated Gromov's isoperimetric inequality (Theorem 4.1) and its generalization due to P. Bérard, G. Besson and S. Gallot (Theorem 4.6). Using these results and a Theorem of G. Perelman (Theorem 4.9) we obtain the following result, which gives a positive answer to Questions 1.6 when the class $\mathcal{M}$ under consideration is the class of Riemannian manifolds whose Ricci curvature is bounded from below by the Ricci curvature of the canonical sphere (see Remarks 4.5).

Theorem 1.9 *For every complete, connected Riemannian manifold (M, g) whose Ricci curvature satisfies $\mathrm{Ric}_g \geq (n-1)g$, for every compact domain with smooth* *boundary* Ω *in* M*, let* Ω^* *be a geodesic ball of the canonical sphere* $(\mathbb{S}^n, g_0)$ *such that* $\frac{\mathrm{Vol}(\Omega^*, g_0)}{\mathrm{Vol}(\mathbb{S}^n, g_0)} = \frac{\mathrm{Vol}(\Omega, g)}{\mathrm{Vol}(M, g)}$*, then* $\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*)$*. Morever,*

(i) *If there exists some domain* $\Omega \subset M$ *such that* $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ *then* (M, g) *is isometric to* $(\mathbb{S}^n, g_0)$ *and* Ω *is isometric to* Ω^**.*

(ii) *If there exists some domain* $\Omega \subset M$ *such that*

$$\mathcal{E}(\Omega) > (1 - \delta(n, \kappa))^{\frac{2}{n}}\, \mathcal{E}(\Omega^*) \qquad \text{with} \qquad \delta(n, \kappa) = \frac{\int_0^{\frac{\varepsilon(n,\kappa)}{2}} (\sin t)^{n-1} dt}{\int_0^{\frac{\pi}{2}} (\sin t)^{n-1} dt}$$

[*where* $-\kappa^2$ *is a lower bound for the sectional curvature of* (M, g) *and where* $\varepsilon(n, \kappa)$ *is the Perelman constant described in Theorem 4.9*] *then* M *is diffeomorphic to* $\mathbb{S}^n$*.*

In the case where the Ricci curvature is bounded below by a nonpositive constant we generalize the inequality of Theorem 1.9 (see Corollary 4.8).

In the general compact case (where Ricci curvature is no longer assumed to be bounded from below), the information on (M, g) that we need is its *Cheeger's isoperimetric constant* $H(M, g)$, which is defined by

$$H(M, g) = \inf_{\Omega} \left(\frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\min\left[\mathrm{Vol}(\Omega), \mathrm{Vol}(M \setminus \Omega)\right)} \right],$$

where Ω runs in the set of all domains with smooth boundary in M.

For any $H > 0$, let us denote by $\mathcal{M}_H$ the set of all Riemannian manifolds (M, g) whose Cheeger's isoperimetric constant is bounded from below by H, i.e., the set of the (M, g)'s which verify the following isoperimetric inequality for every domain with smooth boundary $\Omega \subset M$:

$$\mathrm{Vol}_{n-1}(\partial\Omega) \geq H \cdot \min\left(\mathrm{Vol}(\Omega), \mathrm{Vol}(M \setminus \Omega)\right). \tag{4}$$

In [2] (sections IV.B.13 and IV.B.22) and in [10] (sections 5.B and Appendix A.4), P. Bérard and S. Gallot introduce the "*double of the hyperbolic cusp*" (M^*, g^*_ε), constructed by endowing the manifold $M^* := \mathbb{R} \times \mathbb{S}^{n-1}$ with the Riemannian metric g^*_ε defined, at any point $(t, v) \in M^*$, by $g^*_\varepsilon := (dt)^2 \oplus \varepsilon^2 e^{-2\frac{H}{n-1}|t|} g_0$, where g_0 is the canonical metric of $\mathbb{S}^{n-1}$. They remark that (M^*, g^*_ε) is a (generalized[5]) PIMS, which is valid for all the manifolds $(M, g) \in \mathcal{M}_H$, because the symmetric domains $\Omega^*_r := [r, +\infty[\times \mathbb{S}^{n-1} \subset M^*$ (i.e., the "balls" centered at the pole at infinity) realize

[5] In [2] and [10], the authors view (M^*, g^*_ε) as a " manifold of revolution modeled on the isoperimetric inequality (4)", i.e., as a notion of PIMS which is generalized in the sense that it is allowed to be non compact with finite volume, to admit poles at infinity and to be pointed at one of these poles, the "balls" centered at this pole being the symmetric domains $[r, +\infty[\times \mathbb{S}^{n-1}$, moreover they allow the metric g^*_ε to be piecewise C^1. They prove that the symmetrization method also works in this case.

the equality[6] in the isoperimetric inequality[7] (4). This symmetrization by (M^*, g^*_ε) will be our implicit guide in order to prove the following result.

Theorem 1.10 *Let (M, g) be any compact Riemannian manifold and let Ω be any compact domain with smooth boundary in M such that $Vol(\Omega) \leq \frac{1}{2}$ Vol(M). Then* $\mathcal{E}(\Omega) \leq \dfrac{1}{H(M, g)^2}$.

In Sect. 4.1, we shall see that, as the non compactness of the domains Ω^*_r would technically complicate the arguments, the proof of Theorem 1.10 does not make an explicit use of the symmetrization of (M, g) by (M^*, g^*_ε), but is written more simply as a direct consequence of the isoperimetric inequality (4).

However the symmetrization of (M, g) by (M^*, g^*_ε) turns to be important because a direct computation proves that,[8] for every $r > 0, \mathcal{E}(\Omega^*_r) = \frac{1}{H^2}$ and thus the Theorem 1.10 may be rewritten as follows: *the functional $\Omega \mapsto \mathcal{E}(\Omega)$ (where Ω runs in the set of all domains, with smooth boundary and prescribed volume $v \leq \frac{1}{2}$ Vol(M, g), in all the Riemannian manifolds $(M, g) \in \mathcal{M}_H$) attains its maximum when $(M, g) = (M^*, g^*_\varepsilon)$ and when $\Omega = \Omega^*_r$.* This seems to answer quite easily the first of Questions 1.6 for the class $\mathcal{M}_H$ of Riemannian manifolds, but there are two objections to this assertion: the first one (only technical and thus quite easily solvable as it is done in [2, Sections IV.B.22-28] and in [10, Théorème 5.4 and Appendix A.4]) is that the domains Ω^*_r are not compact, we should thus have to extend the functional $\Omega \mapsto \mathcal{E}(\Omega)$ to non compact domains Ω with finite volume, the second (and deeper) objection is that $(M^*, g^*_\varepsilon) \notin \mathcal{M}_H$ because (as noticed in [10]) there exists non symmetric domains $\Omega' \subset M^*$, such that Vol$(\Omega', g^*_\varepsilon) =$ Vol$(\Omega^*_r, g^*_\varepsilon)$ and Vol$_{n-1}(\partial\Omega') <$ Vol$_{n-1}(\partial\Omega^*_r)$. The authors were not able to overcome this second objection even if they believe that the first of Questions 1.6 has a positive answer for the class $\mathcal{M}_H$ of Riemannian manifolds (and thus the sharpness of Theorem 1.10 would follow).

The paper is organized as follows. In Sect. 2 we recall the definition of manifolds which are harmonic and isoperimetric at one point and provide examples of non standard isoperimetric Riemannian manifolds. In Sect. 3 we give the definition of PIMS for a given manifold (M, g) and we prove Proposition 1.4 and Lemma 1.5. The main tool in the proof of this lemma is the Theorem of symmetrization (Theorem 3.3) which gives precise relationships between the integrals which appear in the definition of $\mathcal{E}(\Omega)$ when calculated on Ω and on its symmetrized domain Ω^*. In Sect. 4 we investigate how to compare mean exit-times from domains in two different compact

[6] In fact, a direct computation gives: $\forall r \in \mathbb{R}$ $\quad \dfrac{\text{Vol}_{n-1}(\partial\Omega^*_r, g^*_\varepsilon)}{\min\left[\text{Vol}(\Omega^*_r), \text{Vol}(M^*\backslash\Omega^*_r)\right]} = H.$

[7] This equality property in (4) corresponds to our definition of a PIMS (see Sect. 3) because, for any $(M, g) \in \mathcal{M}_H$ and for every domain $\Omega \subset M$, one has $\dfrac{\text{Vol}_{n-1}(\partial\Omega, g)}{\text{Vol}(M, g)} \geq \dfrac{\text{Vol}_{n-1}(\partial\Omega^*_r, g^*_\varepsilon)}{\text{Vol}(M^*, g^*_\varepsilon)}$ when the relative volumes of Ω^*_r and Ω are equal, i.e., when $\dfrac{\text{Vol}(\Omega^*_r, g^*_\varepsilon)}{\text{Vol}(M^*, g^*_\varepsilon)} = \dfrac{\text{Vol}(\Omega, g)}{\text{Vol}(M, g)}$.

[8] Just verify that the function $f_{\Omega^*_r}(t, v) = \dfrac{t - r}{H}$ (defined on $M^* = \mathbb{R} \times \mathbb{S}^{n-1}$) lies in $H^2_{1,c}(\Omega^*_r, g^*_\varepsilon)$ and satisfies $\Delta f_{\Omega^*_r} = 1$, which implies, by Green's formula, that $f_{\Omega^*_r}$ is a critical point (and thus a maximum) of the concave functional $E_{\Omega^*_r}$. We thus get $\mathcal{E}(\Omega^*_r) = E_{\Omega^*_r}(f_{\Omega^*_r}) = \dfrac{1}{H^2}$.

manifolds. This will allow us to prove Theorems 1.9 and 1.10. The paper ends with an appendix (Sect. 5) with the proof of Theorem 3.3.

2 Harmonic and isoperimetric manifolds at one point

We briefly recall the definition of harmonic manifolds (the reader is referred to [5] and references therein for details). In any (complete) Riemannian manifold (M, g), for any point $x_0 \in M$, let $\mathbb{S}_{x_0}$ be the unit sphere of the Euclidean space $\left(T_{x_0}M, g_{x_0}\right)$. For any $v \in \mathbb{S}_{x_0}$, we denote by $\mathrm{Cut}(v)$ the maximum of the T's such that the geodesic $c_v : [0, T] \to M$ (with initial speed $\dot{c}_v(0) = v$) is minimizing.

Definition 2.1 (M, g), is said to be *harmonic at* x_0 if $v \mapsto \mathrm{Cut}(v)$ is a (possibly infinite) constant on $\mathbb{S}_{x_0}$ and if any geodesic sphere centered at x_0 of radius $r < \mathrm{Cut}(v)$ is a smooth hypersurface with constant[9] mean curvature.

Definition 2.2 A Riemannian manifold (M, g) is said to be *harmonic* iff it is harmonic at each of its points.

For example, spaces of revolution are harmonic at their pole(s), but they are generally not harmonic in the sense of the Definition 2.2 (see Example 2.4 and Theorem 2.5 below).

Definition 2.3 Let (M, g) be a Riemannian manifold and x_0 a point of M. The manifold (M, g) is said to be *isoperimetric at* x_0 if it is harmonic at x_0 and if, for any compact domain $\Omega \subset M$ with smooth boundary, the geodesic ball Ω^* centered at x_0 with the same volume as Ω satisfies $\mathrm{Vol}_{n-1}(\partial\Omega^*) \leq \mathrm{Vol}_{n-1}(\partial\Omega)$; the same manifold is said to be *strictly isoperimetric at* x_0 if, moreover, the equality occurs iff Ω is isometric to Ω^*.

The Euclidean space, the hyperbolic space and the sphere are strictly isoperimetric at every point (see [6, Sections 10 and 8.6] for proofs and their history). These are the only known examples (up to homotheties) of Riemannian manifolds which are isoperimetric at every point. If we only require the Riemannian manifolds to be isoperimetric at (at least) one point, we get much more examples.

Example 2.4 The first example is given by a 2-dimensional cylinder $[0, +\infty[\times\mathbb{S}^1$ (resp. $[0, L]\times\mathbb{S}^1$) with 1 hemisphere glued to the boundary $\{0\}\times\mathbb{S}^1$ (resp. with 2 hemispheres respectively glued to the boundaries $\{0\}\times\mathbb{S}^1$ and $\{L\}\times\mathbb{S}^1$). Other examples are given by the paraboloid of revolution $z = x^2 + y^2$ or the hyperboloid of equation $x^2 + y^2 - z^2 = -1,\ z > 0$ in $\mathbb{R}^3$ (isoperimetric at their pole).

More generally, a large class of nonstandard examples is given by the following result.

Theorem 2.5 ([19, Theorem 1.2]) *Consider the plane* $\mathbb{R}^2$ *equipped with a complete and rotationally invariant Riemannian metric* g *such that the Gauss curvature is positive and a strictly decreasing function of the distance from the origin. Then* $(\mathbb{R}^2, g)$ *is isoperimetric at the origin.*

[9] Evidently, this constant depends on the radius r.

However, it is not true that every space of revolution is isoperimetric at its pole: a counter-example is given by the hypersurface of revolution S in $\mathbb{R}^3$ of equation $x^2 + y^2 + (|z| + \cos R)^2 = 1$, whose poles are $x_0 = (0, 0, 1 - \cos R)$ and $x_1 = -x_0$. The plane $y = 0$ separates S in two symmetric domains, which have the same area as the geodesic ball $B(x_0, R)$ and whose boundary is shorter than $\partial B(x_0, R)$.

3 The Theorem of symmetrization and the proofs of Proposition 1.4 and Lemma 1.5

Let (M, g) and (M^*, g^*) be two Riemannian manifolds such that $\mathrm{Vol}(M, g)$ and $\mathrm{Vol}(M^*, g^*)$ are both infinite or both finite. Let us define the constant $\alpha(M, M^*)$ by

$$\alpha(M, M^*) = \begin{cases} 1 & \text{if } \mathrm{Vol}(M, g) \text{ and } \mathrm{Vol}(M^*, g^*) \text{ are both infinite,} \\ \dfrac{\mathrm{Vol}(M, g)}{\mathrm{Vol}(M^*, g^*)} & \text{if } \mathrm{Vol}(M, g) \text{ and } \mathrm{Vol}(M^*, g^*) \text{ are both finite.} \end{cases}$$

Definition 3.1 Let x^* be a fixed point of M^*.

(a) For any compact domain $\Omega \subset M$ with smooth boundary, one defines its *symmetrized domain* Ω^* (around x^*) as the geodesic ball $B_{M^*}(x^*, R_0)$ of (M^*, g^*), such that $\mathrm{Vol}(B_{M^*}(x^*, R_0)) = \alpha(M, M^*)^{-1}\mathrm{Vol}(\Omega)$.[10]

(b) (M^*, g^*, x^*) is said to be a *pointed isoperimetric model space* (*PIMS*) *for* (M, g) if, for any compact domain $\Omega \subset M$, with smooth boundary, the symmetrized domain Ω^* satisfies the *isoperimetric inequality* $\mathrm{Vol}_{n-1}(\partial\Omega) \geq \alpha(M, M^*)\mathrm{Vol}_{n-1}(\partial\Omega^*)$; the same manifold is said to be a *strict PIMS* if, moreover, the equality occurs iff Ω is *isometric* to (Ω^*, g^*).

Remark 3.2 When the two manifolds have different finite volumes, the assumption $\mathrm{Vol}(\Omega^*) = \alpha(M, M^*)^{-1}\mathrm{Vol}(\Omega)$ means that the relative volumes $\mathrm{Vol}(\Omega)/\mathrm{Vol}(M, g)$ and $\mathrm{Vol}(\Omega^*)/\mathrm{Vol}(M^*, g^*)$ are equal. We are compelled to make this relative assumption because, under the usual assumption $\mathrm{Vol}(\Omega^*) = \mathrm{Vol}(\Omega)$, it is hopeless to expect some bound from below for $\mathrm{Vol}_{n-1}(\partial\Omega)$ in terms of $\mathrm{Vol}_{n-1}(\partial\Omega^*)$.[11]

Let (M, g) be a Riemannian manifold and (M^*, g^*, x^*) be a PIMS for (M, g). Let $\Omega \subset M$ be any compact domain with smooth boundary. Let $\Omega^* = B_{M^*}(x^*, R_0)$ be its *symmetrized domain* in the sense of Definition 3.1. Let f be any smooth nonnegative function on Ω which vanishes on $\partial\Omega$. We denote by Ω_t (or equivalently by $\{f > t\}$) the set of points $x \in \Omega$ such that $f(x) > t$ and set $A(t) := \mathrm{Vol}_g(\Omega_t)$. For every $t \in [0, \sup f[$, one associates to Ω_t its symmetrized domain Ω_t^* which is the geodesic

[10] By the continuity and the monotonicity of $r \mapsto \mathrm{Vol}\left(B_{M^*}(x^*, r)\right)$, this equation always admit a solution R_0.

[11] When $\mathrm{Vol}(M, g) \neq \mathrm{Vol}(M^*, g^*)$ one generally would not have $\mathrm{Vol}_{n-1}(\partial\Omega) \geq \mathrm{Vol}_{n-1}(\partial\Omega^*)$ if the symmetrization assumption was $\mathrm{Vol}(\Omega^*) = \mathrm{Vol}(\Omega^*)$: in fact, this symmetrization would imply that, when $\mathrm{Vol}(M, g) > \mathrm{Vol}(M^*, g^*)$ and $\mathrm{Vol}(M^*, g^*) < \mathrm{Vol}(\Omega) < \mathrm{Vol}(M, g)$, the symmetrized domain Ω^* does not exist; in the case where $\mathrm{Vol}(M, g) < \mathrm{Vol}(M^*, g^*)$ this symmetrization would imply that the domain $\Omega = M\backslash B_M(x, \varepsilon)$ verifies $\mathrm{Vol}_{n-1}(\partial\Omega) \leq C.\varepsilon^{n-1} << \mathrm{Vol}_{n-1}(\partial\Omega^*)$.

ball $B_{M^*}(x^*, R(t))$ of (M^*, g^*) whose volume is equal to $\alpha(M, M^*)^{-1}A(t)$. As $\Omega_{\sup f}$ is empty, we set $R(\sup f) = 0$.

Since the function $t \mapsto A(t)$ is strictly decreasing, the function $t \mapsto R(t)$ is strictly decreasing too, and thus it is well defined and injective; however R is generally not continuous and the measure of the set $[0, R_0]\backslash\mathrm{Image}(R)$ is generally not zero. This is one of the main problems when studying the regularity of the following "symmetrization" f^* of f. Let us first define the 1−dimensional symmetrization $\bar{f} : [0, R_0] \to [0, \sup f]$ of f by

$$\bar{f}(r) = \inf(R^{-1}([0, r])) = \inf\{t : A(t) \le \alpha(M, M^*)\mathrm{Vol}B_{M^*}(x^*, r)\} ;$$

the *symmetrization of f* is then the function $f^* := \bar{f} \circ \rho : \Omega^* \to \mathbb{R}^+$, where $\rho = d_{M^*}(x^*, \cdot)$ (d_{M^*} being the Riemannian distance in (M^*, g^*)). Notice that $\{f^* > t\} = \Omega^*_t$ [see Property 5.1 (vi) in Appendix for a proof].

We now state the Theorem of symmetrization which (when coupled with sharp isoperimetric inequalities) represents the main tool in our comparison results. Symmetrization methods have their origin in J. Steiner's works. The following application to functional analysis (also called rearrangement) generalizes to Riemannian manifolds ideas of G. Talenti [22].

Theorem 3.3 *Let (M, g) be a Riemannian manifold and (M^*, g^*, x^*) be a PIMS for (M, g). Let $\Omega \subset M$ be a compact domain with smooth boundary and f be any smooth nonnegative function on Ω which vanishes on its boundary. Let f^* be the symmetrization of f, constructed as above on the symmetrized geodesic ball Ω^* of (M^*, g^*), centered at the point x^*. Then*

(i) *f^* is Lipschitz (with Lipschitz constant $\|\nabla f\|_{L^\infty}$) and thus f^* lies in $H^2_{1,c}(\Omega^*, g^*)$;*

(ii) $\dfrac{1}{\mathrm{Vol}(\Omega)}\int_\Omega f(x)^p dv_g(x) = \dfrac{1}{\mathrm{Vol}(\Omega^*)}\int_{\Omega^*}(f^*(x))^p\, dv_{g^*}(x)$ *for every* $p \in [1, +\infty[$;

(iii) $\dfrac{1}{\mathrm{Vol}(\Omega)}\int_\Omega \|\nabla f(x)\|^2 dv_g(x) \ge \dfrac{1}{\mathrm{Vol}(\Omega^*)}\int_{\Omega^*}\|\nabla f^*(x)\|^2 dv_{g^*}(x)$. *If, moreover, (M^*, g^*, x^*) is a strict PIMS for (M, g) then equality holds iff the set $\{f > 0\} \subset (\Omega, g)$ is isometric to the set $\{f^* > 0\} \subset (\Omega^*, g^*)$.*

On the one hand, the proof of this theorem is classical when Ω is a domain of $\mathbb{R}^n$ endowed with the Euclidean metric (see e.g. [1, pp. 47–56], [22] and [13]), and it is natural to believe that Theorem 3.3 can be proved by following the same lines. On the other hand, there exist proofs of Theorem 3.3 on Riemannian manifolds (see [2, Chapters IV.A.4–5], [10, Chapter 5, Lemme 5.7] and [17, pp. 176–178]) which are valid only when f is a Morse function. Specialists of the field are convinced that this is sufficient (using an approximation of smooth functions by Morse functions and an analytic compactness argument) to conclude in the case of smooth functions. However we have not found any place where the above "convincing" arguments are clarified and where the necessary adaptations of the classical proofs are written: it is the reason why, in the Appendix (Sect. 5) we provide a proof of Theorem 3.3 which can be viewed as an adaptation and a simplification of the proof given in [1] for the Euclidean case.

Proof of Lemma 1.5: Let f_Ω be the unique solution of (1) on the domain Ω and let $(f_\Omega)^*$ be its symmetrization. By (ii) and (iii) of Theorem 3.3 we get

$$\begin{aligned}\mathcal{E}(\Omega) = E_\Omega(f_\Omega) &= \frac{1}{\mathrm{Vol}(\Omega)}\left(2\int_\Omega f_\Omega\, dv_g - \int_\Omega |\nabla f_\Omega|^2\, dv_g\right)\\ &\leq \frac{1}{\mathrm{Vol}(\Omega^*)}\left(2\int_{\Omega^*} (f_\Omega)^*\, dv_{g^*} - \int_{\Omega^*} |\nabla (f_\Omega)^*|^2\, dv_{g^*}\right) = E_{\Omega^*}\left((f_\Omega)^*\right).\end{aligned}$$

Let us recall that the *mean exit-time* from the domain Ω^* is the value $\mathcal{E}(\Omega^*) = \max_{u\in H^2_{1,c}(\Omega^*)}(E_\Omega(u))$. Since by (i) of Theorem 3.3 $(f_\Omega)^* \in H^2_{1,c}(\Omega^*, g^*)$ it follows that

$$\mathcal{E}(\Omega^*) \geq E_{\Omega^*}\left((f_\Omega)^*\right) \geq \mathcal{E}(\Omega).$$

Let us suppose that $\mathcal{E}(\Omega^*) = \mathcal{E}(\Omega)$, then all the inequalities are equalities, in particular

$$\int_\Omega |\nabla f_\Omega|^2\, dv_g = \alpha(M, M^*)\int_{\Omega^*} |\nabla (f_\Omega)^*|^2\, dv_{g^*}$$

and $E_{\Omega^*}((f_\Omega)^*) = \mathcal{E}(\Omega^*)$. Thus, since the set $\{f_\Omega > 0\}$ coincides with the interior of Ω, from the equality case of Theorem 3.3 it follows that Ω^* is isometric to Ω. □

Proof of Proposition 1.4: As we already noticed in the introduction, the proof can be deduced immediately from Lemma 1.5. In fact, Definition 2.3 implies that (M, g, x_0) is a PIMS for (M, g) itself in the sense of the Definition 3.1; thus (M, g) and (M^*, g^*) coincide and the constant $\alpha(M, M^*)$ is equal to 1. □

4 Proofs of Theorems 1.9 and 1.10

In order to compare mean exit-times for two domains on two different compact manifolds lying in the same class, we have to compare the isoperimetric inequalities on these two manifolds.

Revisiting Paul Lévy's work [16] (initially applied to convex bodies in the Euclidean space), Gromov [11] proved the following celebrated isoperimetric inequality.

Theorem 4.1 *For every Riemannian manifold (M, g) whose Ricci curvature satisfies $\mathrm{Ric}_g \geq (n-1)g$, for every compact domain with smooth boundary Ω in M, let Ω^* be a geodesic ball of the canonical sphere $(\mathbb{S}^n, g_0)$ such that $\dfrac{\mathrm{Vol}(\Omega^*, g_0)}{\mathrm{Vol}(\mathbb{S}^n, g_0)} = \dfrac{\mathrm{Vol}(\Omega, g)}{\mathrm{Vol}(M, g)}$, then*

$$\frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}(M, g)} \geq \frac{\mathrm{Vol}_{n-1}(\partial\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n, g_0)}.$$

Moreover, this last inequality is an equality if and only if Ω *is isometric to* Ω^**. In other words, for any* $x_0 \in \mathbb{S}^n$*,* $(\mathbb{S}^n, g_0, x_0)$ *is a strict PIMS for all the Riemannian manifolds* (M, g) *which satisfy* $\mathrm{Ric}_g \geq (n-1)g$.

Remark 4.2 The isoperimetric inequality given by Theorem 4.1 is sharp: in fact, as the canonical sphere $(\mathbb{S}^n, g_0)$ satisfies $\mathrm{Ric}_{g_0} = (n-1)g_0$, we may apply Theorem 4.1 to the sphere, and then deduce an inequality which is an equality when Ω is a geodesic ball of $(\mathbb{S}^n, g_0)$.

Applying Lemma 1.5 and Theorem 4.1, we obtain:

Corollary 4.3 *For every Riemannian manifold* (M, g) *whose Ricci curvature satisfies* $\mathrm{Ric}_g \geq (n-1)g$*, for every compact domain with smooth boundary* Ω *in* M*, let* Ω^* *be a geodesic ball of the canonical sphere* $(\mathbb{S}^n, g_0)$ *such that* $\mathrm{Vol}(\Omega^*, g_0)/\mathrm{Vol}(\mathbb{S}^n, g_0) = \mathrm{Vol}(\Omega, g)/\mathrm{Vol}(M, g)$*, then* $\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*)$*. Moreover, the equality* $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ *is realized if and only if* Ω *is isometric to* Ω^**.*

Remark 4.4 For every $K > 0$, it is easy to extend Corollary 4.3 to every Riemannian manifold (M, g) which satisfies $\mathrm{Ric}_g \geq K(n-1)g$: just replace the canonical sphere by the sphere of constant sectional curvature K in the statement of Corollary 4.3, apply Corollary 4.3 to the Riemannian manifold (M, Kg) and then use the homogeneity formula (3).

Remark 4.5 Corollary 4.3 is sharp and answers Questions 1.6 when the class $\mathcal{M}$ under consideration is the class of Riemannian manifolds whose Ricci curvature is bounded from below by the Ricci curvature of the canonical sphere: in fact, for every fixed $\beta \in]0, 1[$, let $\mathcal{W}_\beta$ be the set of all domains Ω, in all the Riemannian manifolds $(M, g) \in \mathcal{M}$, such that $\mathrm{Vol}(\Omega, g)/\mathrm{Vol}(M, g) = \beta$. It is clear that the the geodesic ball Ω^* of the canonical sphere $(\mathbb{S}^n, g_0)$ such that $\mathrm{Vol}(\Omega^*, g_0) = \beta \mathrm{Vol}(\mathbb{S}^n, g_0)$ is an element of $\mathcal{W}_\beta$, and Corollary 4.3 proves that the functional $\Omega \mapsto \mathcal{E}(\Omega)$, when restricted to the set $\mathcal{W}_\beta$, attains its absolute maximum when $\Omega = \Omega^*$ and that this maximum is strict. Moreover, Theorem 1.9 proves that, if $\mathcal{E}(\Omega)$ is not far from this maximal value, then M is diffeomorphic to $\mathbb{S}^n$.

Theorem 4.1 was improved and generalized to the case where the Ricci curvature has any sign by Bérard, Besson and Gallot ([3, Theorem 2] and [10, Theorem 6.16] for a quantitatively improved version):

Theorem 4.6 *For any* $K \in \mathbb{R}$*, a PIMS for all the* n*-dimensional Riemannian manifolds* (M, g) *which satisfy* $\mathrm{Ric}_g \geq (n-1)Kg$ *and* $\mathrm{diameter}(M, g) \leq D$ *is given by the Euclidean sphere of radius* $R(K, D)$ *(PIMS at any point) where* $R(K, D)$ *is defined by*

$$R(K,D)=\begin{cases}\frac{1}{\sqrt{K}}\left(\frac{\int_0^{\frac{D\sqrt{K}}{2}}(\cos t)^{n-1}dt}{\int_0^{\frac{\pi}{2}}(\cos t)^{n-1}dt}\right)^{\frac{1}{n}} & \text{if } K>0\\ \frac{n}{2}\left(\int_0^{\frac{\pi}{2}}(\cos t)^{n-1}dt\right)^{-\frac{1}{n}}D & \text{if } K=0\\ \frac{1}{\sqrt{|K|}}\mathrm{Max}\left(\frac{\int_0^{D\sqrt{|K|}}(\cosh 2t)^{\frac{n-1}{2}}dt}{\int_0^{\pi}(\sin t)^{n-1}dt},\left(\frac{\int_0^{D\sqrt{|K|}}(\cosh 2t)^{\frac{n-1}{2}}dt}{\int_0^{\pi}(\sin t)^{n-1}dt}\right)^{\frac{1}{n}}\right) & \text{if } K<0\end{cases}$$

In other terms, for every compact domain with smooth boundary Ω in M, if Ω^* is a geodesic ball on the Euclidean sphere $\mathbb{S}^n(R(K,D))$ of radius $R(K,D)$ and if Ω^{**} is a geodesic ball of the canonical sphere $\mathbb{S}^n(1)=(\mathbb{S}^n,g_0)$ such that

$$\frac{\mathrm{Vol}(\Omega^{**})}{\mathrm{Vol}(\mathbb{S}^n,g_0)}=\frac{\mathrm{Vol}(\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n(R(K,D)))}=\frac{\mathrm{Vol}(\Omega,g)}{\mathrm{Vol}(M,g)}\ ,$$

then

$$\frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}(M,g)}\geq\frac{\mathrm{Vol}_{n-1}(\partial\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n(R(K,D)))}=\frac{1}{R(K,D)}\frac{\mathrm{Vol}_{n-1}(\partial\Omega^{**})}{\mathrm{Vol}(\mathbb{S}^n,g_0)}. \tag{5}$$

Remark 4.7 Let us first remark that the smaller $R(K,D)$ is, the stronger is the isoperimetric inequality (5). When $K>0$, Theorem 4.6 is sharp, because (if we choose $D=\frac{\pi}{\sqrt{K}}$) the sphere $\mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$ satisfies its assumptions and the conclusion of Theorem 4.6 in this case is that the isoperimetric inequality (5) is verified when $(M,g)=\mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$. Notice that, in this case, this inequality is indeed an equality because $R(K,D)=R\left(K,\frac{\pi}{\sqrt{K}}\right)=\frac{1}{\sqrt{K}}$. Moreover, under the assumptions "$\mathrm{Ric}_g\geq(n-1)Kg$" and "$(M,g)$ not isometric to $\mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$", Myers' theorem (and its equality case) implies that $\mathrm{diameter}(M,g)<\frac{\pi}{\sqrt{K}}$, and thus we can apply Theorem 4.6 with the values $K=1$ and $D<\frac{\pi}{\sqrt{K}}$ of the constants, which implies that, under these assumptions, $R(K,D)<\frac{1}{\sqrt{K}}$. The isoperimetric inequality (5) is then strictly stronger than the one of the sphere $\mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$.

On the contrary, when $K\leq 0$, Theorem 4.6 is not sharp because one always has diameter $(\mathbb{S}^n(R(K,D)))>D$ in this case, and thus the sphere $\mathbb{S}^n(R(K,D))$ does not satisfy the assumptions of Theorem 4.6.

Corollary 4.8 *Let K be an arbitrary real number (of any sign), for any n-dimensional Riemannian manifold (M,g) which satisfies* $\mathrm{Ric}_g\geq(n-1)Kg$ *and* $\mathrm{diameter}(M,g)\leq D$*, for every compact domain with smooth boundary Ω in M, if Ω^* is a geodesic ball on the Euclidean sphere $\mathbb{S}^n(R(K,D))$ and if Ω^{**} is a geodesic ball of the canonical sphere $\mathbb{S}^n(1)=(\mathbb{S}^n,g_0)$ such that*

$$\frac{\mathrm{Vol}(\Omega^{**})}{\mathrm{Vol}(\mathbb{S}^n, g_0)} = \frac{\mathrm{Vol}(\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n(R(K, D)))} = \frac{\mathrm{Vol}(\Omega, g)}{\mathrm{Vol}(M, g)} ,$$

then

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*) = R(K, D)^2 \, \mathcal{E}(\Omega^{**}). \tag{6}$$

Proof Theorem 4.6 shows that the Euclidean sphere $\mathbb{S}^n(R(K, D))$ of radius $R(K, D)$ is a PIMS for the Riemannian manifold (M, g). Let $\Omega \subset M$ be a compact domain with smooth boundary, Ω^* a geodesic ball on the Euclidean sphere $\mathbb{S}^n(R(K, D))$ of radius $R(K, D)$ and Ω^{**} a geodesic ball of the canonical sphere $\mathbb{S}^n(1) = (\mathbb{S}^n, g_0)$ such that

$$\frac{\mathrm{Vol}(\Omega^{**})}{\mathrm{Vol}(\mathbb{S}^n, g_0)} = \frac{\mathrm{Vol}(\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n(R(K, D)))} = \frac{\mathrm{Vol}(\Omega, g)}{\mathrm{Vol}(M, g)}.$$

Then Lemma 1.5 gives

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*) = R(K, D)^2 \mathcal{E}(\Omega^{**}),$$

where the last equality is deduced from the fact that the sphere of radius $R(K, D)$ is isometric to $(\mathbb{S}^n, R(K, D)^2 g_0)$ and from formula (3). □

We now recall an inequality due to G. Perelman [20] (which is an improvement of a previous result of S. Ilias [12]).

Theorem 4.9 *Let (M, g) be a n-dimensional compact Riemannian manifold. Assume that M is not diffeomorphic to $\mathbb{S}^n$, that $\mathrm{Ric}_g \geq (n-1)g$ and that the sectional curvature of (M, g) is $\geq -\kappa^2$. Then there exists a constant $\varepsilon(n, \kappa) > 0$ such that* $\mathrm{diameter}(M, g) \leq \pi - \varepsilon(n, \kappa)$.

Remark 4.10 By applying Theorem 4.6 with the values $K = 1$ and $D = \pi - \varepsilon(n, \kappa)$ of the constants, which implies that, under these assumptions,

$$R(K, D) = R(1, \pi - \varepsilon(n, \kappa)) = \left(1 - \frac{\int_0^{\frac{\varepsilon(n,\kappa)}{2}} (\sin t)^{n-1} dt}{\int_0^{\frac{\pi}{2}} (\sin t)^{n-1} dt}\right)^{\frac{1}{n}} , \tag{7}$$

we observe that, with respect to the isoperimetric inequality of the canonical sphere, the isoperimetric inequality on (M, g) induced by (5) is improved by some factor which is bounded away from 1.

Proof of Theorem. 1.9: Applying Theorem 4.6 (with the values $K = 1$ and $D = \mathrm{diameter}(M, g)$ of the constants) we prove that the Euclidean sphere $\mathbb{S}^n(R(1, D))$ of radius $R(1, D)$ is a PIMS (at any point) for the Riemannian manifold (M, g). For every compact domain with smooth boundary Ω in M, if Ω^0 is a geodesic ball on the

Euclidean sphere $\mathbb{S}^n(R(1, D))$ of radius $R(1, D)$ and if Ω^* is a geodesic ball of the canonical sphere $\mathbb{S}^n(1) = (\mathbb{S}^n, g_0)$ such that

$$\frac{\text{Vol}(\Omega^*)}{\text{Vol}(\mathbb{S}^n, g_0)} = \frac{\text{Vol}(\Omega^0)}{\text{Vol}(\mathbb{S}^n(R(1, D)))} = \frac{\text{Vol}(\Omega, g)}{\text{Vol}(M, g)},$$

then Lemma 1.5 yields

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^0) = R(1, D)^2 \mathcal{E}(\Omega^*), \tag{8}$$

where the last equality is deduced from the fact that $\mathbb{S}^n(R(1, D))$ is isometric to $(\mathbb{S}^n, R(1, D)^2 g_0)$ and from formula (3). Let us first suppose that (M, g) is not isometric to $(\mathbb{S}^n, g_0)$: then Myers' theorem (and its equality case) implies that diameter$(M, g) < \pi$, and thus that $R(1, D) < 1$ (if $D =$ diameter(M, g)) by the definition of $R(K, D)$. Using the fact that $R(1, D) < 1$ in the inequality (8), we conclude that, if (M, g) is not isometric to $(\mathbb{S}^n, g_0)$, then $\mathcal{E}(\Omega) < \mathcal{E}(\Omega^*)$ for every compact domain with smooth boundary Ω in M, which proves the part (i) of Theorem 1.9.

If we now suppose that M is not diffeomorphic to $\mathbb{S}^n$, Theorem 4.9 implies that the value $D = \pi - \varepsilon(n, \kappa)$ is a upper bound of the diameter of (M, g). Using the inequality (8) and formula (7) we get

$$\mathcal{E}(\Omega) \leq \left(1 - \frac{\int_0^{\frac{\varepsilon(n,\kappa)}{2}} (\sin t)^{n-1} dt}{\int_0^{\frac{\pi}{2}} (\sin t)^{n-1} dt}\right)^{\frac{2}{n}} \mathcal{E}(\Omega^*)$$

for every compact domain with smooth boundary Ω in M, which proves part (ii) of Theorem 1.9. □

4.1 Proof of Theorem 1.10:

In the sequel, as in Sect. 3, for any continuous function f, we shall denote by Ω_t the set of points $x \in \Omega$ such that $f(x) > t$ and by $A(t)$ the volume of Ω_t. We first need the following:

Lemma 4.11 *Let (M, g) be a Riemannian manifold. Then, for any compact domain $\Omega \subset M$ and for any nonnegative continuous function f on Ω which vanishes on $\partial\Omega$, one has:*

$$\int_\Omega f^p dv_g = p \int_0^{\sup f} t^{p-1} A(t) dt, \quad \forall p \in [1, +\infty[.$$

Proof Let $t_i = \frac{i}{N} \sup f$ (for every $i \in \{0, \ldots N\}$). The function $t \mapsto A(t) = \text{Vol}(\Omega_t)$ being strictly decreasing, we have

$$\sum_{i=0}^{N-1} t_i (A(t_i) - A(t_{i+1})) \leq \int_\Omega f dv_g \leq \sum_{i=0}^{N-1} t_{i+1} (A(t_i) - A(t_{i+1})). \tag{9}$$

Let S_N^+ (resp. S_N^-) denote the right (resp. the left) hand side of (9). This is an approximation from above (resp. from below) of the integral $\int_0^{\sup f} A(t)dt$. As $0 \le S_N^+ - S_N^- \le \frac{\sup f}{N} A(0)$, when $N \to \infty$, $S_N^+ - S_N^- \to 0_+$ and S_N^+, S_N^- both go to $\int_0^{\sup f} A(t)dt$ (and to $\int_\Omega f dv_g$ by (9)). This proves that $\int_\Omega f dv_g = \int_0^{\sup f} A(t)dt$. Applying this equality to the function f^p and noticing that $\mathrm{Vol}\left(\{f^p > t\}\right) = \mathrm{Vol}\left(\{f > t^{\frac{1}{p}}\}\right) = A\left(t^{\frac{1}{p}}\right)$, we obtain

$$\int_\Omega f^p dv_g = \int_0^{(\sup f)^p} A\left(t^{\frac{1}{p}}\right) dt = p \int_0^{\sup f} t^{p-1} A(t)dt.$$

□

Proof of Theorem.1.10: For any compact domain $\Omega \subset M$ with smooth boundary, let $\mathcal{C}(f_\Omega)$ be the set of critical values of f_Ω and $\mathcal{S}(f_\Omega) := f_\Omega\big(\mathcal{C}(f_\Omega)\big)$ be the set of its singular values; using the definition of $\mathcal{E}(\Omega)$, Lemma 4.11 and the fact that $\mathcal{S}(f_\Omega)$ has measure zero by Sard's Theorem, we obtain:

$$\mathrm{Vol}(\Omega)\ \mathcal{E}(\Omega) = \int_\Omega f_\Omega dv_g = \int_{[0, \sup f_\Omega] \setminus \mathcal{S}(f_\Omega)} A(t)dt. \tag{10}$$

For every regular value t of f_Ω one has $A(t) \le \mathrm{Vol}(\Omega) \le \mathrm{Vol}(M, g)/2$ and thus, by the definition of Cheeger's isoperimetric constant, $\mathrm{Vol}_{n-1}(\partial\Omega_t) \ge H(M, g)A(t)$. From this and from (10) we deduce:

$$\mathrm{Vol}(\Omega)\ \mathcal{E}(\Omega) \le \frac{1}{H(M, g)} \int_{[0, \sup f_\Omega] \setminus S(f_\Omega)} \mathrm{Vol}_{n-1}(\partial\Omega_t)dt = \frac{1}{H(M, g)} \int_\Omega |\nabla f_\Omega| dv_g,$$

where, in the last equality, we have used the coarea formula (11). Thus, by Cauchy–Schwarz inequality

$$\mathrm{Vol}(\Omega)\ \mathcal{E}(\Omega) \le \frac{1}{H(M, g)} \left(\mathrm{Vol}(\Omega)\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla f_\Omega|^2 dv_g\right)^{\frac{1}{2}}$$

and hence, since $\mathcal{E}(\Omega) = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega |\nabla f_\Omega|^2 dv_g$, one gets $(\mathcal{E}(\Omega))^{\frac{1}{2}} \le \frac{1}{H(M,g)}$ which ends the proof of the theorem. □

5 Appendix: proof of Theorem 3.3

The main tool for the proof of Theorem 3.3 is the coarea formula (see for instance [6, pp. 104–107]), which writes, for every measurable function φ on a Riemannian manifold (M, g):

$$\int_M \varphi(x)\ \|\nabla f(x)\| dv_g(x) = \int_{\inf f}^{\sup f} \left(\int_{f^{-1}(\{t\})} \varphi(x) da_t(x)\right) dt\ . \tag{11}$$

Since Federer's proof (see [9] or [6]), this equation is known to be valid when the function $f : M \to \mathbb{R}$ is Lipschitz; however we shall essentially use it when f is a smooth function[12] on a given compact domain with smooth boundary $\Omega \subset M$ vanishing on the boundary of Ω. As before we denote by $\Omega_t := \{f > t\}$ (resp. by $\{f = t\}$) the set of points $x \in \Omega$ such that $f(x) > t$ (resp. such that $f(x) = t$), by $\mathcal{S}(f)$ the (compact) set of critical values (i.e., the image of the compact set of critical points) of f and by I^0_{reg} the open set $]0, \sup f[\setminus \mathcal{S}(f)$ of regular values of f. By Sard's theorem almost every $t \in [0, \sup f]$ belongs to I^0_{reg}. Thus in formula (11) the integral $\int_{\inf f}^{\sup f}$ must thus be interpreted as the integral on I^0_{reg}, the set $\{f = t\}$ is then a smooth (generally not connected) hypersurface of M, equal to $\partial\Omega_t$ and da_t is the induced $(n-1)$-dimensional measure of this hypersurface.

Let (M^*, g^*, x^*) be a PIMS for (M, g) (in the sense of Definition 3.1). In the sequel, we will denote by α the constant $\alpha(M, M^*)$. Let $\Omega \subset M$ be any compact domain with smooth boundary (such that $\text{Vol}(\Omega) < \text{Vol}(M, g) \leq +\infty$) and let $\Omega^* = B_{M^*}(x^*, R_0) \subset M^*$ be its "symmetrized domain" in the sense of Definition 3.1. Given any smooth nonnegative function f on Ω which vanishes on the boundary of Ω, for every $t \in [0, \sup f]$, we have defined in Sect. 3 the functions $A(t) := \text{Vol}\{f > t\} = \text{Vol}(\Omega_t)$ and $R(t)$ (where $R(t)$ is the solution of the equation $\text{Vol}[B_{M^*}(x^*, R(t))] = \alpha^{-1} A(t))$ and the geodesic ball $\Omega_t^* := B_{M^*}(x^*, R(t))$ of (M^*, g^*), which is the symmetrized domain associated to Ω_t. We have also defined the functions $\bar{f} : [0, R_0] \to \mathbb{R}$ and $f^* : \Omega^* \to \mathbb{R}^+$ by the equalities $\bar{f}(r) := \inf\left(R^{-1}([0, r])\right)$ and $f^* := \bar{f} \circ \rho$, where $\rho(\cdot) = d_{M^*}(x^*, \cdot)$.

Proposition 5.1 *The basic properties of the functions $\bar{f}$ and R are:*

(i) *R is strictly decreasing and $\bar{f}$ is decreasing (not strictly in general);*
(ii) *$\bar{f}(R(t)) = t$ for every $t \in [0, \sup f]$;*
(iii) *$\bar{f}(0) = \sup \bar{f} = \sup f$, $\bar{f} = 0$ on the interval $[R(0), R_0]$;*
(iv) *$\bar{f}(r) = \sup\{t : R(t) \geq r\}$ for every $r \in [0, R(0)]$;*
(v) *$t \mapsto R(t)$ is continuous on the right at every point $t \in [0, \sup f]$;*
(vi) *$\{f^* > t\} = \Omega_t^* = B_{M^*}(x^*, R(t))$ for every $t \in [0, \sup f]$.*

Proof The proofs of (i), (ii) and (iii) are straightforward and left to the reader. Notice that, for any decreasing sequence $(t_n)_{n\in\mathbb{N}}$ converging to t, $\{f > t\}$ is the increasing union of the $\{f > t_n\}$'s and thus $A(t_n) \to A(t)$ and $R(t_n) \to R(t)$; this proves (v). By (i) and (ii) one gets (iv):

$$r < R(t) \implies (\exists n \text{ s. t. } r < R(t_n) < R(t)) \implies \bar{f}(r) \geq \bar{f}\left(R(t_n)\right)) = t_n > t.$$

Again by properties (i) and (ii) one obtains $\bar{f}(r) > t \implies r < R(t)$ and we deduce that $\{\bar{f} > t\} = [0, R(t)[$. Thus $\{f^* > t\} = B_{M^*}(x^*, R(t)) = \Omega_t^*$ and also (vi) is proved. □

A less trivial property is expressed by the following:

[12] Except at the end of this section, where the coarea formula is applied to a decreasing Lipschitz function $[0, R_0] \to \mathbb{R}$.

Lemma 5.2 *$\bar{f}$ and f^* are Lipschitz with Lipschitz constant $\|\nabla f\|_{L^\infty}$.*

Proof As $f^* := \bar{f} \circ \rho$, it is sufficient to prove the lemma for $\bar{f}$. By a classical result of measure theory, as $R : [0, \sup f] \to \mathbb{R}$ is a decreasing function which is continuous on the right, there exists a positive Borel measure μ_R such that $\mu_R(]t, t']) = R(t) - R(t')$ for every $t, t' \in [0, \sup f]$ such that $t \leq t'$. This measure is the derivative of $-R$ in the sense of distributions.

As I^0_{reg} is an open set, for every $t \in I^0_{\mathrm{reg}}$ and every sufficiently small h, the interval $[t, t+h]$ (this notations meaning $[t+h, t]$ when $h < 0$) is entirely contained in I^0_{reg}. By applying formula (11) to the function $\varphi = \frac{1}{\|\nabla f\|}$, which is smooth on $f^{-1}([t, t+h])$, one gets:

$$\alpha\Big(\mathrm{Vol}\left[B_{M^*}(x^*, R(t))\right] - \mathrm{Vol}\left[B_{M^*}(x^*, R(t+h))\right]\Big) = A(t) - A(t+h)$$
$$= \int_{f^{-1}([t,t+h])} \frac{1}{\|\nabla f(x)\|} \, \|\nabla f(x)\| \, dv_g(x) = \int_t^{t+h} \left(\int_{\{f=s\}} \frac{1}{\|\nabla f\|} \, da_s \right) ds.$$

By dividing both sides by h and taking the limit when $h \to 0$, the mean value theorem implies that $R'(t)$ exists at every $t \in I^0_{\mathrm{reg}}$ and satisfies:

$$\alpha \; R'(t) \; \mathrm{Vol}_{n-1}\left[\partial B_{M^*}(x^*, R(t))\right] = -\int_{\{f=t\}} \frac{1}{\|\nabla f(x)\|} \, da_t(x). \tag{12}$$

Let t, t' be any points of $[0, \sup f]$ such that $t < t'$ and let us denote by $I^{t,t'}_{\mathrm{reg}}$ the open set $]t, t'[\setminus \mathcal{S}(f)$. By (12) and by the positivity of μ_R, we get

$$\|\nabla f\|_{L^\infty}\big(R(t) - R(t')\big) \geq \|\nabla f\|_{L^\infty} \mu_R\left(I^{t,t'}_{\mathrm{reg}}\right) = -\|\nabla f\|_{L^\infty} \int_{I^{t,t'}_{\mathrm{reg}}} R'(s) ds$$
$$\geq \int_{I^{t,t'}_{\mathrm{reg}}} \frac{1}{\alpha \mathrm{Vol}_{n-1}\left(\partial \Omega^*_s\right)} \left(\int_{f^{-1}(\{s\})} da_s \right) ds \geq \int_{I^{t,t'}_{\mathrm{reg}}} \frac{\mathrm{Vol}_{n-1}\left(\partial \Omega_s\right)}{\alpha \mathrm{Vol}_{n-1}\left(\partial \Omega^*_s\right)} ds \geq t' - t.$$

For every $r', r \in [0, R_0]$ with $r' < r$ and $\bar{f}(r') = \bar{f}(r)$ we are done. Thus (i) of Proposition 5.1 allows to suppose that $\bar{f}(r) < \bar{f}(r')$. Therefore, for every ε such that $0 < \varepsilon < \frac{1}{2}(\bar{f}(r') - \bar{f}(r))$, by the definition of $\bar{f}$ and by (iv) of Proposition 5.1, there exist $t, t' \in [0, \sup f]$ such that $R(t) \leq r, t < \bar{f}(r) + \varepsilon$, $R(t') \geq r'$ and $t' > \bar{f}(r') - \varepsilon$. Using these last inequalities and the previous one, we get

$$0 < \bar{f}(r') - \bar{f}(r) - 2\varepsilon < t' - t \leq \|\nabla f\|_{L^\infty}(R(t) - R(t')) \leq \|\nabla f\|_{L^\infty}(r - r').$$

We conclude by making $\varepsilon \to 0_+$. □

Lemma 5.2 proves (i) of Theorem 3.3. As f^* is Lipschitz, we may apply Lemma 4.11 to both functions f and f^* and, for every $p \in [1, +\infty[$, we get:

$$\int_\Omega f^p dv_g = p \int_0^{\sup f} t^{p-1} A(t) dt = \alpha p \int_0^{\sup f} t^{p-1} A^*(t) dt = \alpha \int_{\Omega^*} \left(f^*\right)^p dv_{g^*} \,,$$

where $A^*(t) := \text{Vol}(\{f^* > t\}) = \text{Vol}(\Omega_t^*) = \alpha^{-1}\text{Vol}(\Omega_t) = \alpha^{-1}A(t)$. This proves (ii) of Theorem 3.3.

In order to prove (iii) of Theorem 3.3 we apply formula (11) to the function $-\bar{f}$ (because $-\bar{f}$ is Lipschitz by Lemma 5.2). This gives[13], for any measurable function $\varphi : [0, R_0] \to \mathbb{R}$,

$$\int_{[0,R_0]} \varphi(r)|\bar{f}'(r)|dr = \int_{[0,\sup f]} \left(\sum_{r \in \bar{f}^{-1}(\{t\})} \varphi(r) \right) dt = \int_{I^0_{\text{reg}}} \varphi(R(t))dt,$$

where the last equality is a consequence of (ii) of Proposition 5.1 and of the fact that $\bar{f}$ is decreasing everywhere and strictly decreasing when restricted to the open set $R\big(I^0_{\text{reg}}\big)$ [because the restriction of R to I^0_{reg} is a derivable, open, strictly decreasing map by (i) of Proposition 5.1 and equality (12)]. Denoting $\text{Vol}_{n-1}\left[\partial B_{M^*}(x^*, r)\right]$ by $V_{n-1}(r)$, integrating with respect to the normal coordinates centered at x^* on (M^*, g^*) and then applying the previous equality (where we replace $\varphi(r)$ by $-\bar{f}'(r)V_{n-1}(r)$), we obtain:

$$\begin{aligned}\int_{\Omega^*} \|\nabla f^*(x)\|^2 dv_{g^*}(x) &= \int_0^{R_0} |\bar{f}'(r)|^2 V_{n-1}(r)dr = \int_{I^0_{\text{reg}}} |\bar{f}'(R(t))|V_{n-1}(R(t))dt \\ &= \int_{I^0_{\text{reg}}} \frac{\text{Vol}_{n-1}(\partial\Omega_t^*)}{|R'(t)|}dt = \int_{I^0_{\text{reg}}} \frac{\alpha\text{Vol}_{n-1}(\partial\Omega_t^*)^2}{\int_{\{f=t\}} \frac{1}{\|\nabla f\|}da_t}dt, \qquad (13)\end{aligned}$$

where we made use of (ii) of Proposition 5.1 to prove the third equality and of equality (12) to prove the last equality.

On the other hand, applying the coarea formula (11) to the function f, replacing in this formula the integrand $\varphi(x)$ by $\|\nabla f(x)\|$ and using Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}\int_M \|\nabla f(x)\|^2\, dv_g(x) &= \int_{I^0_{\text{reg}}} \left(\int_{\partial\Omega_t} \|\nabla f\|da_t \right) dt \geq \int_{I^0_{\text{reg}}} \frac{\text{Vol}_{n-1}(\partial\Omega_t)^2}{\int_{\{f=t\}} \frac{1}{\|\nabla f\|}da_t}dt \\ &\geq \int_{I^0_{\text{reg}}} \frac{\alpha^2\text{Vol}_{n-1}(\partial\Omega_t^*)^2}{\int_{\{f=t\}} \frac{1}{\|\nabla f\|}da_t}dt \geq \alpha \int_{\Omega^*} \|\nabla f^*(x)\|^2 dv_{g^*}(x), \qquad (14)\end{aligned}$$

where the last inequality can be deduced from (13) and where the second inequality is a consequence of the assumption that (M^*, g^*, x^*) is a PIMS for (M, g). This proves the inequality (iii) of Theorem 3.3.

[13] In the present 1−dimensional case, the coarea formula is much more simple to prove: in fact, by the Rademacher theorem $-\bar{f}$, being Lipschitz by the Lemma 5.2, is a. e. differentiable; moreover its derivative (in the sense of distributions) is a nonnegative measure $\mu_{\bar{f}} = \bar{f}'(r)\, dr$ (of bounded density with respect to the Lebesgue measure dr) which is the pulled back by $-\bar{f}$ of the Lebesgue measure of $[0, \sup f]$, because $\mu_{\bar{f}}(]r, r'[) = \bar{f}(r) - \bar{f}(r')$.

Finally, let us assume that this inequality is an equality then, in (14), all inequalities are equalities, in particular $\mathrm{Vol}_{n-1}(\partial\Omega_t) = \alpha\mathrm{Vol}_{n-1}(\partial\Omega_t^*)$ for every $t \in I_{\mathrm{reg}}^0$ and, if (M^*, g^*, x^*) is a strict PIMS for (M, g), this implies that Ω_t is isometric to Ω_t^*. As $\{f > 0\}$ is the increasing union of the sets $\{f > t_n\}$ when $t_n \in I_{\mathrm{reg}}^0$ and $t_n \to 0_+$, we have proved that $\Omega_0 = \{f > 0\}$ is isometric to its symmetrized domain $\Omega_0^* = \{f^* > 0\}$. This concludes the proof of Theorem 3.3.

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