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Abel summability in topological spaces

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Abstract The classical summability theory can not be used in the topological spaces as it needs addition operator. Recently some authors have studied the summability theory in the topological spaces by assuming the topological space to have a group structure or a linear structure or introducing some summability methods those do not need a linear structure in the topological space as statistical convergence and distributional convergence. In the present paper we introduce a new concept of density and we study the summability theory in an arbitrary Hausdorff space by introducing a new type of statistical convergence and distirbutional convergence via Abel method that is a sequence-to-function transformation. Moreover we give a Bochner integral representation of $Abel$-summability in the Banach spaces.

Keywords $Abel$-distributional convergence · $Abel$-density · Hausdorff spaces · $Abel$ summability

Mathematics Subject Classification 54A20 · 40J05 · 40A05 · 40G15 · 11B05

1 Preliminaries

The classical summability theory can not be used in the topological spaces as it needs addition operator. Recently some authors have studied summability theory in the topological spaces by assuming the topological space to have a group structure or a linear

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structure or introducing some summability methods those do not need a linear structure in the topological space as statistical convergence and distributional convergence [4–6,9,16–19,22]. One of the main aims of the present paper is to study the summability theory in an arbitrary Hausdorff topological space by introducing a new type of statistical convergence and distributional convergence via Abel method that is a sequence-to-function transformation.

A-statistical convergence, one of the main concepts of the summability theory, has been studied by several authors for scalar sequences [2,3,8,11–13,20]. Recently some authors have dealt with this concept in the Hausdorff topological spaces [4,22]. Moreover some topological definitions and results for the space of all real numbers have been given via Abel convergence in [7] and some Tauberian theorems in the distributional sense for power series where Cesàro summability follows from Abel summability can be found in [10]. In the present paper by defining *Abel*-statistical convergence not only do we study summability theory in the Hausdorff topological spaces but also we give a new type of convergence method that is stronger than statistical convergence.

Now we recall the A-summability and A-density concepts those are closely linked to A-statistical convergence.

Let $A = (a_{nk})$ be a summability matrix and let $z = (z_k)$ be a real valued sequence. If the sequence $(Az)_n = \sum_{k=0}^{\infty} a_{nk} z_k$ exists, i.e., the series $\sum_{k=0}^{\infty} a_{nk} z_k$ is convergent for each $n \in \mathbb{N}_0$ then we say that Az is the A-transformation of z where $\mathbb{N}_0$ is the set of all nonnegative integers. If the sequence Az converges to a number L then we say that z is A-summable to L. A summability matrix A is said to be regular if $\lim_{n\to\infty}(Az)_n = L$ whenever $\lim_{k\to\infty} z_k = L$. The following theorem characterizes the regular matrices:

Theorem 1 *A summability matrix $A = (a_{nk})$ is regular if and only if:*

(i) $\sup_n \sum_k |a_{nk}| < \infty$,
(ii) $\lim_n \sum_k a_{nk} = 1$,
(iii) $a_k := \lim_{n\to\infty} a_{nk} = 0$ *for all* $k \in \mathbb{N}_0$ [1].

Let $A = (a_{nk})$ be a nonnegative regular summability matrix. Then A-*density* of $E \subseteq \mathbb{N}_0$ is given by

$$\delta_A(E) := \lim_{n\to\infty} \sum_{k\in E} a_{nk}$$

whenever the limit exists [2,3,12]. If $A = C_1$ then we say E has natural density $\delta(E)$ instead of A-density where $C_1 = (c_{nk})$ is a summability matrix defined as

$$c_{nk} = \begin{cases} \frac{1}{n+1}, & 0 \le k \le n \\ 0, & otherwise \end{cases}.$$

Let (X, τ) be a Hausdorff topological space and let $A = (a_{nk})$ be a nonnegative summability matrix such that each row adds up to one. Then a sequence $x = (x_k)$ in

X is said to be A-statistically convergent to $\alpha \in X$ if for any open set U that contains α,

$$\lim_{n\to\infty} \sum_{x_k \notin U} a_{nk} = 0$$

i.e., the A-density of the set $E(U) := \{k : x_k \notin U\}$ is zero or equivalently $\chi_{E(U)}$ that is the characteristic sequence of $E(U)$ is A-summable to zero. If $A = C_1$ then x is said to be statistically convergent to α.

Now we recall the Abel method and then we define the *Abel*-density concept:

Let $z = (z_k)$ be a real valued sequence. If the series

$$\sum_{k=0}^{\infty} z_k y^k$$

converges for all $y \in (0, 1)$ and

$$\lim_{y\to 1^-} (1 - y) \sum_{k=0}^{\infty} z_k y^k = L \tag{1.1}$$

then we say that z is Abel convergent to L [15]. The *Abel*-density of $E \subseteq \mathbb{N}_0$ is given by

$$\mu(E) := \lim_{y\to 1^-} (1 - y) \sum_{k\in E} y^k$$

whenever the limit exists. Since

$$\frac{1}{1 - y} = \sum_{k=0}^{\infty} y^k$$

for $|y| < 1$ it is obvious that $\mu(\mathbb{N}_0) = 1$. In the present paper we also deal with the existence of *Abel*-density of sets by using spliced sequences.

Let (X, τ) be a Hausdorff topological space. Then a sequence $x = (x_k)$ in X is said to be *Abel*-statistically convergent to $\alpha \in X$ if for any open set U that contains α,

$$\lim_{y\to 1^-} (1 - y) \sum_{x_k \notin U} y^k = 0$$

i.e., the *Abel*-density of the set $E(U)$ is zero or equivalently $\chi_{E(U)}$ is *Abel*-convergent to zero. Now we recall the following theorem of Powell and Shah [15]:

Theorem 2 *If a real valued sequence is C_1-summable then it is Abel-convergent to the same value (but not conversely).*

Remark 1 As Abel summability is stronger than C_1-summability the following results hold immidately:

(i) If the natural density of a set $E \subset \mathbb{N}_0$ exists then the *Abel*-density of it exists as well. More precisely if A-density of a set $E \subset \mathbb{N}_0$ exists then Abel density of it exists where A is a nonnegative summability matrix such that *Abel*-summability is stronger than A-summability.

(ii) A sequence in a Hausdorff topological space is *Abel*-statistically convergent whenever it is statistically convergent. Indeed a sequence in a Hausdorff space is *Abel*-statistically convergent whenever it is A-statistically convergent where A is a nonnegative summability matrix such that each row adds up to one and *Abel*-summability is stronger than A-summability.

Example 1 As the set $E = \left\{k = n^2 : n \in \mathbb{N}_0\right\}$ has zero natural density it has zero *Abel*-density as well.

Example 2 Let $E = \{2k : k \in \mathbb{N}_0\}$. Since $\dfrac{1}{1-y} = \sum_{k=0}^{\infty} y^k$ for $|y| < 1$, we get

$$\begin{aligned}
\lim_{y\to 1^-}(1-y)\sum_{k\in E} y^k &= \lim_{y\to 1^-}(1-y)\sum_{k=0}^{\infty} y^{2k} \\
&= \lim_{y\to 1^-}(1-y)\frac{1}{1-y^2} \\
&= \lim_{y\to 1^-}\frac{1}{1+y} \\
&= \frac{1}{2}
\end{aligned}$$

which implies $\mu(E) = 1/2$.

The following example shows that there exist some sets those do not have *Abel*-density.

Example 3 Let $\mathbb{D}$ be the set of all real valued sequences $z = (z_n)$ such that for all $n \in \mathbb{N}_0$ $0 < z_n < 1$ and $\lim_{n\to\infty} z_n = 1$ and let $\mathbb{B} = \left\{B(z) = (b_{nk}) : b_{nk} = (1 - z_n)z_n^k, \quad z \in \mathbb{D}\right\}$. Using Theorem 1 it is not difficult to see that each matrix in $\mathbb{B}$ is regular whose each row adds up to one. Now take a fixed $z \in \mathbb{D}$. By a theorem of Steinhaus [21], there exists a sequence $t = (t_k)$ consisting of 0's and 1's that is not $B(z)$-summable. Thus the limit

$$\lim_{n\to\infty}(1-z_n)\sum_{k=0}^{\infty} t_k z_n^k$$

does not exist which implies the limit

$$\lim_{y\to 1^-}(1-y)\sum_{k=0}^{\infty} t_k y^k$$

does not exist. Now if we consider the set $E \subset \mathbb{N}_0$ defined by

$$E = \{k \in \mathbb{N}_0 : t_k = 1\}$$

we get

$$\lim_{y \to 1^-} (1-y) \sum_{k \in E} y^k = \lim_{y \to 1^-} (1-y) \sum_{k=0}^{\infty} t_k y^k$$

which implies $\mu(E)$ does not exist.

2 Abel distributional convergence and Abel density

Osikiewicz [14] has studied the existence of the A-densities of sets by using spliced real sequences. Recently Unver et al. [22] have given a characterization of A-statistical convergence in the Hausdorff topological spaces via A-distributional convergence and they have studied the existence of the densities of sets of a partition as well. Let (X, τ) be a Hausdorff topological space and let $\sigma(\tau)$ be the Borel sigma field of subsets of (X, τ). Consider a set function $F : \sigma(\tau) \to [0, 1]$ such that $F(X) = 1$ and if G_0, $G_1, \ldots$ are disjoint sets in $\sigma(\tau)$ then

$$F\left(\bigcup_{j=0}^{\infty} G_j\right) = \sum_{j=0}^{\infty} F(G_j).$$

Such a function is called a distribution on $\sigma(\tau)$. Recall the following definition:

Definition 1 Let F be a distribution on $\sigma(\tau)$ and let $A = (a_{nk})$ be a nonnegative summability matrix whose each row adds up to one and let $x = (x_k)$ be a sequence in X. Then the sequence x is said to be A-distributionally convergent to F if for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$ we have

$$\lim_{n \to \infty} \sum_{x_k \in G} a_{nk} = F(G)$$

where ∂G is the boundary of G [22].

The following definition is the Abel type analog of Definition 1:

Definition 2 Let F be a distribution on $\sigma(\tau)$ and let $x = (x_k)$ be a sequence in X. Then the sequence x is said to be *Abel*-distributionally convergent to F if for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$ we have

$$\lim_{y \to 1^-} (1-y) \sum_{x_k \in G} y^k = F(G).$$

Now we give a characterization of *Abel*-statistical convergence and then we study the existence of *Abel*-densities of sets.

Theorem 3 *Let X be a Hausdorff topological space and let $x = (x_k)$ be a sequence in X. Then x is Abel-statistically convergent to $\alpha \in X$ if and only if it is Abel-distributionally convergent to $F : \sigma(\tau) \to [0, 1]$ defined by*

$$F(G) = \begin{cases} 0, & \text{if } \alpha \notin G \\ 1, & \text{if } \alpha \in G \end{cases}.$$

Proof Let x be an *Abel*-statistically convergent sequence to α. Then for any open set U that contains α we have

$$\lim_{y \to 1^-} (1 - y) \sum_{x_k \notin U} y^k = 0$$

Thus for each $z \in \mathbb{D}$

$$\lim_{n \to \infty} (1 - z_n) \sum_{k=0}^{\infty} z_n^k = 0$$

which implies x is $B(z)$-statistically convergent to α for each $B(z) \in \mathbb{B}$. Therefore from Proposition 1 of [22] for each $B(z) \in \mathbb{B}$ it is $B(z)$-distributionally convergent to F which implies x is *Abel*-distributionally convergent to F. Conversely let x be an *Abel*-distributionally convergent sequence to F. Then for all $B(z) \in \mathbb{B}$, x is $B(z)$-distributionally convergent to F. Thus from Proposition 1 of [22] it is $B(z)$-statistically convergent to α which implies x is *Abel*-statistically convergent to α. □

Now we recall some definitions (see [14]):

Definition 3 Let M be a fixed positive integer. An M-partition of $\mathbb{N}_0$ consists of infinite sets $K_i = \{\vartheta_i(j)\}$ for $i = 0, 1, \ldots, M-1$ such that $\bigcup_{i=0}^{M-1} K_i = \mathbb{N}_0$ and $K_i \cap K_j = \varnothing$ for all $i \neq j$. An ∞-partition on $\mathbb{N}_0$ consists of a countably infinite number of infinite sets $K_i = \{\vartheta_i(j)\}$ for $i \in \mathbb{N}_0$ such that $\bigcup_{i=0}^{\infty} K_i = \mathbb{N}_0$ and $K_i \cap K_j = \varnothing$ for all $i \neq j$.

Definition 4 Let $\{K_i : i = 0, 1, \ldots, M-1\}$ be a fixed M-partition of $\mathbb{N}_0$, let $x^{(i)} = \left(x_j^{(i)}\right)$ be a sequence in X with $\lim_{j \to \infty} x_j^{(i)} = \alpha_i$, $i = 0, 1, \ldots, M-1$. If $k \in K_i$, then $k = \vartheta_i(j)$ for some j. Define $x = (x_k)$ by $x_k = x_{\vartheta_i(j)} = x_j^{(i)}$. Then x is called an M-splice over $\{K_i : i = 0, 1, \ldots, M-1\}$ with limit points $\alpha_0, \alpha_1, \ldots, \alpha_{M-1}$.

Definition 5 Let $\{K_i : i \in \mathbb{N}_0\}$ be a fixed ∞-partition of $\mathbb{N}_0$, let $x^{(i)} = \left(x_j^{(i)}\right)$ be a sequence in X with $\lim_{j \to \infty} x_j^{(i)} = \alpha_i$, $i \in \mathbb{N}_0$. If $k \in K_i$, then $k = \vartheta_i(j)$ for some j. Define $x = (x_k)$ by $x_k = x_{\vartheta_i(j)} = x_j^{(i)}$. Then x is called an ∞-splice over $\{K_i : i \in \mathbb{N}_0\}$ with limit points $\alpha_0, \alpha_1, \ldots, \alpha_M, \ldots$.

Theorem 4 *Let X be a Hausdorff topological space and let $\{K_i : i = 0, 1, \ldots, M-1\}$ be an M-partition of $\mathbb{N}_0$. Then the following statements are equivalent:*

(i) *$\mu(K_i)$ exists for all $i = 0, 1, \ldots, M-1$.*
(ii) *There exist $p_0, p_1, \ldots, p_{M-1} \in [0, 1]$ such that $\sum_{i=0}^{M-1} p_i = 1$ and any M-spliced sequence over $\{K_i : i = 0, 1, \ldots, M-1\}$ with limit points $\alpha_0, \alpha_1, \ldots, \alpha_{M-1}$ is Abel-distributionally convergent to the distribution F: $\sigma(\tau) \to [0, 1]$ where*

$$F(G) = \sum_{\substack{0 \le i \le M-1 \\ \alpha_i \in G}} p_i, \text{ for all } G \in \sigma(\tau).$$

(iii) *There exist $p_0, p_1, \ldots, p_{M-1} \in [0, 1]$ such that $\sum_{i=0}^{M-1} p_i = 1$ and the M-splice of $x^{(0)}, x^{(1)}, \ldots, x^{(M-1)}$ over $\{K_i : i = 0, 1, \ldots, M-1\}$ where $x^{(i)} = (\alpha_i, \alpha_i, \ldots)$ being a constant sequence, is Abel-distributionally convergent to the distribution $F : \sigma(\tau) \to [0, 1]$ where*

$$F(G) = \sum_{\substack{0 \le i \le M-1 \\ \alpha_i \in G}} p_i, \; for \quad all \quad G \in \sigma(\tau).$$

Proof *$i \Longrightarrow ii$:* Asume that $\mu(K_i)$ exists for each $i = 0, 1, \ldots, M-1$. Let $p_i = \mu(K_i)$ for $i = 0, 1, \ldots, M-1$. Since $\{K_i : i = 0, 1, \ldots, M-1\}$ is an M-partition of $\mathbb{N}_0$ it is obvious that

$$1 = \sum_{i=0}^{M-1} \mu(K_i) = \sum_{i=0}^{M-1} p_i.$$

Let $F : \sigma(\tau) \to [0, 1]$ be a set function defined by

$$F(G) = \sum_{\substack{0 \le i \le M-1 \\ \alpha_i \in G}} p_i, \quad \text{for all } G \in \sigma(\tau).$$

It is not difficult to see that F is a distribution on $\sigma(\tau)$. Now it is enough to show that x is *Abel*-distributionally convergent to the distribution F. For each $i = 0, 1, \ldots, M-1$ the existence of $\mu(K_i)$ implies that for each $B(z) \in \mathbb{B}$ and each $i = 0, 1, \ldots, M-1$, $\delta_{B(z)}(K_i)$ exists and equals to $\mu(K_i)$. Since for each $i = 0, 1, \ldots, M-1$, $p_i = \mu(K_i) = \delta_{B(z)}(K_i)$ from Theorem 1 of [22] for each $B(z) \in \mathbb{B}$, x is $B(z)$-distributionally convergent to F i.e for each $z \in \mathbb{D}$ and for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$

$$\lim_{n\to\infty} (1 - z_n) \sum_{x_k \in G} z_n^k = F(G)$$

which implies

$$\lim_{y\to 1^-}(1-y)\sum_{x_k\in G} y^k = F(G).$$

Thus x is *Abel*-distributionally convergent to F.

ii $\Longrightarrow$ *iii: Since* $x^{(i)} = (\alpha_i, \alpha_i, \ldots)$ is convergent for all $i = 0, 1, \ldots, M-1$ the proof follows immediately.

iii $\Longrightarrow$ *i: Assume that* there exist $p_0, p_1, \ldots, p_{M-1} \in [0,1]$ such that $\sum_{i=0}^{M-1} p_i = 1$ and the sequence x that is the M-splice of $x^{(0)}, x^{(1)}, \ldots, x^{(M-1)}$ over $\{K_i : i = 0, 1, \ldots, M-1\}$ where $x^{(i)} = (\alpha_i, \alpha_i, \ldots)$ being a constant sequence, is *Abel*-distributionally convergent to the distribution F. Then for each $B(z) \in \mathbb{B}$, x is $B(z)$-distributionally convergent to F. Then from Theorem 1 of [22] for each $i = 0, 1, \ldots, M-1$ and for each $B(z) \in \mathbb{B}$ $\delta_{B(z)}(K_i)$ exists and equals to p_i which implies

$$\lim_{n\to\infty}(1-z_n)\sum_{k\in K_i} z_n^k = p_i.$$

Thus for each $i = 0, 1, \ldots, M-1$

$$\mu(K_i) = \lim_{y\to 1^-}(1-y)\sum_{k\in K_i} y^k = p_i$$

which concludes the proof. □

The following theorem characterizes the sigma additivity of *Abel*-densities of an infinite partition.

Theorem 5 *Let X be a Hausdorff topological space and let $\{K_i = \{\vartheta_i(j)\} : i \in \mathbb{N}_0\}$ be an ∞-partition of $\mathbb{N}_0$. Then $\mu(K_i)$ exists for all $i \in \mathbb{N}_0$ and $\sum_{i=0}^{\infty} \mu(K_i) = 1$ if and only if there exist $p_i \in [0,1]$ for $i \in \mathbb{N}_0$ such that $\sum_{i=0}^{\infty} p_i = 1$ and any ∞-splice sequence over $\{K_i : i \in \mathbb{N}_0\}$ with limit points $\alpha_1, \alpha_2, \ldots$ is Abel-distributionally convergent to the distribution $F : \sigma(\tau) \to [0,1]$ where*

$$F(G) = \sum_{\alpha_i\in G} p_i, \quad \textit{for all } G \in \sigma(\tau).$$

Proof Assume that $\mu(K_i)$ exists for each $i \in \mathbb{N}_0$. Then for each $i \in \mathbb{N}_0$ and for each $B(z) \in \mathbb{B}$, $\delta_{B(z)}(K_i)$ exists and equals to $\mu(K_i)$. Thus we have for each $i \in \mathbb{N}_0$ and $B(z) \in \mathbb{B}$ that

$$\sum_{i=0}^{\infty} \delta_{B(z)}(K_i) = 1.$$

From Theorem 2 of [22] for each $B(z) \in \mathbb{B}$ we get that any ∞-splice sequence $x = (x_k)$ over $\{K_i : i \in \mathbb{N}_0\}$ with limit points $\alpha_1, \alpha_2, \ldots$ is $B(z)$-distributionally convergent to the distribution $F : \sigma(\tau) \to [0, 1]$ where

$$F(G) = \sum_{\alpha_i \in G} \delta_{B(z)}(K_i) = \sum_{\alpha_i \in G} \mu(K_i), \quad \textit{for all } G \in \sigma(\tau)$$

i.e., for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$

$$\lim_{n\to\infty} (1 - z_n) \sum_{x_k \in G} z_n^k = F(G)$$

for each $z \in \mathbb{D}$. Hence for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$

$$\lim_{y\to 1^-} (1 - y) \sum_{x_k \in G} y^k = F(G)$$

which implies x is *Abel*-distributionally convergent to F.

Conversely assume that there exist $p_i \in [0, 1]$ for $i \in \mathbb{N}_0$ such that $\sum_{i=0}^{\infty} p_i = 1$ and any ∞-splice sequence over $\{K_i : i \in \mathbb{N}_0\}$ with limit points $\alpha_1, \alpha_2, \ldots$ is *Abel*-distributionally convergent to F. Then for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$

$$\lim_{y\to 1^-} (1 - y) \sum_{x_k \in G} y^k = F(G).$$

Therefore for all $G \in \sigma(\tau)$ with $F(\partial G) = 0$ we have

$$\lim_{n\to\infty} (1 - z_n) \sum_{x_k \in G} z_n^k = F(G) \tag{2.1}$$

for each $z \in \mathbb{D}$. Then from Theorem 2 of [22] and equality (2.1) imply for each $B(z) \in \mathbb{B}$ and for all $i \in \mathbb{N}_0$, $\delta_{B(z)}(K_i)$ exists and equals to p_i with

$$\sum_{i=0}^{\infty} p_i = 1.$$

Thus for each $z \in \mathbb{D}$ and for all $i \in \mathbb{N}_0$

$$\lim_{n\to\infty} (1 - z_n) \sum_{k \in K_i} z_n^k = p_i$$

which implies

$$\mu(K_i) = \lim_{y\to 1^-} (1 - y) \sum_{k \in K_i} y^k = p_i$$

for all $i \in \mathbb{N}_0$. Hence $\mu(K_i)$ exists for all $i \in \mathbb{N}_0$ and $\sum_{i=0}^{\infty} \mu(K_i) = 1$. $\square$

In [22] Unver et al. have given a Bochner integral representation of A-limits of the spliced sequences in the Banach spaces. Our final theorem gives a similar result for the *Abel*-limits of the spliced sequences. Of course we use the result of [22].

Theorem 6 *Let* $(X, \|\cdot\|)$ *be a Banach space,and let* $\{K_i = \{\vartheta_i(j)\} : i \in \mathbb{N}_0\}$ *be an* ∞*-partition of* $\mathbb{N}_0$. *If* $\mu(K_i)$ *exists for all* $i \in \mathbb{N}_0$ *and* $\sum_{i=0}^{\infty} \mu(K_i) = 1$ *then for any bounded* ∞*-spliced sequence* $x = (x_k)$ *over* $\{K_i : i \in \mathbb{N}_0\}$

$$\lim_{y \to 1^-} \sum_{k=0}^{\infty} x_k y^k = \int_X t\,dF \tag{2.2}$$

where F *is a distribution defined by*

$$F(G) = \sum_{\alpha_i \in G} \mu(K_i), \quad \text{for all } G \in \sigma(\tau).$$

and the integral in (2.2) *is the Bochner integral.*

Proof Assume that $\mu(K_i)$ exists for all $i \in \mathbb{N}_0$ and $\sum_{i=0}^{\infty} \mu(K_i) = 1$. Then for each $B(z) \in \mathbb{B}$ and for all $i \in \mathbb{N}_0$, $\delta_{B(z)}(K_i)$ exists and equals to $\mu(K_i)$ with

$$\sum_{i=0}^{\infty} \delta_{B(z)}(K_i) = 1.$$

Then from Proposition 2 of [22] we have for any bounded ∞-spliced sequence $x = (x_k)$ over $\{K_i : i \in \mathbb{N}_0\}$

$$\lim_{n \to \infty} (1 - z_n) \sum_{k=0}^{\infty} x_k z_n^k = \int_X t\,dF \tag{2.3}$$

for each $B(z) \in \mathbb{B}$ where F is a distribution defined by

$$F(G) = \sum_{\alpha_i \in G} \delta_{B(z)}(K_i) = \sum_{\alpha_i \in G} \mu(K_i). \tag{2.4}$$

Hence from equalities (2.3) and (2.4) we have

$$\lim_{y \to 1^-} \sum_{k=0}^{\infty} x_k y^k = \int_X t\,dF$$

which concludes the proof. □

References

1. Boos, J.: Classical and Modern Methods in Summability. Oxford University Press, Oxford (2000)
2. Buck, R.C.: Generalized asymptotic density. Am. J. Math. **75**, 335–346 (1953)
3. Buck, R.C.: The measure theoretic approach to density. Am. J. Math. **68**, 560–580 (1946)
4. Cakalli, H., Khan, M.K.: Summability in topological spaces. Appl. Math. Lett. **24**(3), 348–352 (2011)
5. Cakalli, H.: Lacunary statistical convergence in topological groups. Indian J. Pure Appl. Math. **26**(2), 113–119 (1995)
6. Cakalli, H.: On statistical convergence in topological groups. Pure Appl. Math. Sci. **43**(1–2), 27–31 (1996)
7. Cakalli, H., Albayrak, M.: New type continuities via Abel convergence. Sci. World J. 2014 (2014)
8. Connor, J.S.: The statistical and strong p-Cesàro convergence of sequences. Analysis **8**(1–2), 47–63 (1988)
9. Di Maio, G., Kočinac, L.D.R.: Statistical convergence in topology. Topol. Appl. **156**(1), 28–45 (2008)
10. Estrada, R., Vindas, J.: Distributional versions of Littlewood's Tauberian theorem. Czech. Math. J. **63**(2), 403–420 (2013)
11. Fast, H.: Sur la convergence statistique. Colloq. Math. **2**, 241–244 (1951)
12. Fridy, J.A.: On statistical convergence. Analysis **5**(4), 301–313 (1985)
13. Fridy, J.A., Miller, H.I.: A matrix characterization of statistical convergence. Analysis **11**(1), 59–66 (1991)
14. Osikiewicz, J.A.: Summability of spliced sequences. Rocky Mt. J. Math. **35**(3), 977–996 (2005)
15. Powell, R.E., Shah, S.M.: Summability Theory and Its Applications. Prentice-Hall of India, New Delhi (1988)
16. Prullage, D.L.: Summability in topological groups. Math. Z. **96**, 259–278 (1967)
17. Prullage, D.L.: Summability in topological groups. II. Math. Z. **103**, 129–138 (1968)
18. Prullage, D.L.: Summability in topological groups. IV. Convergence fields. Tôhoku Math. J. **2**(21), 159–169 (1969)
19. Prullage, D.L.: Summability in topological groups. III. Metric properties. J. Anal. Math. **22**, 221–231 (1969)
20. Šalát, T.: On statistically convergent sequences of real numbers. Math. Slovaca **30**(2), 139–150 (1980)
21. Steinhaus, H.: Some remarks on the generalizations of the notion of limit. Prace Mat. Fiz. 22, 121–134 (1911) (in Polish)
22. Unver, M., Khan, M.K., Orhan, C.: A-distributional summability in topological spaces. Positivity **18**, 131–145 (2014)