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An explicit construction of automorphic representations of the symplectic group with a given quadratic unipotent Arthur parameter

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Abstract To a large class of unipotent quadratic Arthur parameters for symplectic groups, we attach (explicitly realized in the space of square-integrable automorphic forms) an irreducible automorphic representation using degenerate Eisenstein series.

Keywords Unipotent Arthur parameter · Symplectic group · Automorphic representation

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1 Introduction and preliminaries

1.1 Introduction

In this paper we explicitly realize, using degenerate Eisenstein series, an automorphic representation of a (global) symplectic group with a prescribed quadratic unipotent Arthur parameter. We manage to obtain this result for a large class of quadratic unipotent Arthur parameters, namely for those ones satisfying a technical condition (Δ) which is easy to check. In [1] is given a detailed study of parametrization of automorphic (square-integrable) representations of global symplectic group by these parameters, but without separation into cuspidal and residual spectrum. On the other hand,

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even with the detailed knowledge about the constituents of the local Arthur packets, when a global representation is formed (by choosing members of local packets), it is still difficult to decide whether it is automorphic; a very non-trivial multiplicity formula is involved. In this respect, our method is very direct and gives an explicit realization of an automorphic representation, with lots of information on the local components.

We want to stress that our construction of automorphic representations is unconditional. It explicitly constructs a residual representation attached to an Arthur parameter. This parameter is of special kind (conforming to conditions (Δ) we describe in the third section), but for such parameter the construction is not making any additional assumptions. It is not applying the trace formula, thus is independent of the still unproven stabilization of the twisted trace formula for GL_n, and makes no additional assumptions, such as regular infinitesimal character or cohomological requirements.

By our (inductive) construction, all these representations we obtain (except maybe at the basis of our construction) are residual. A result on classification of residual representations is given in [12], and in [11] are given, by standard procedure of Langlands spectral decomposition, the quadratic unipotent Arthur parameters which parametrize automorphic representations of symplectic group with a non-zero constant term along Borel subgroup.

The idea of using degenerate Eisenstein series to insure the automorphicity of the constructed representation was suggested by Goran Muić in relation with his earlier work [18]. We construct automorphic representations for quite general quadratic unipotent Arthur parameters, and these representations need not to have a non-zero constant term along Borel subgroups (unlike to those in [18]); the inductive construction starts from the unipotent automorphic representation with the general support (i.e., a minimal parabolic subgroup with non-vanishing constant term along that parabolic subgroup). In our inductive construction, to prove the non-vanishing of degenerate Eisenstein series applied to certain global representation (and thus proving that this representation is automorphic) we calculate the constant term of the Eisenstein series in question. In that calculation, a sum of intertwining operators occurs. We use an argument which is essentially the one in [15] (the second chapter, the first section), but since the assumptions are somewhat different than in loc.cit (where representation of which Eisenstein series is being taken is cuspidal, and in our situation, this might not be the case) for the benefit of the reader, we make an explicit calculation (Theorem 5.3).

In the second section, we prove some claims about the structure of local components of the automorphic representations of symplectic and general linear groups related to quadratic unipotent parameters. These claims are not consequences of our constructions; they are quite general and apply to every automorphic square-integrable representation with quadratic unipotent Arthur parameter. Let Π be (any) irreducible automorphic representation appearing (discretely) in the space of square-integrable automorphic forms on $Sp_{2n}(\mathbb{A})$. Then, there exists a standard parabolic subgroup $P(\mathbb{A})$ of $Sp_{2n}(\mathbb{A})$ with the standard Levi subgroup $M(\mathbb{A})$ and a cuspidal automorphic representation π_0 of M such that, as abstract representations, π is embedded in $\mathrm{ind}_{P(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}\pi_0$ (the precise statement of the result and realization in the space of automorphic forms which we need in the fourth chapter are given in Theorem 5.1). We know that a Levi subgroup M is a product of general linear factors with the symplectic group of smaller rank. Now, using some simple consequences of the classification of the unitary unramified representations of classical groups [19] we get that the factors of π_0 corresponding to the general linear factors are of Ramanujan type [22], and the factor corresponding to the representation of the smaller symplectic group is a cuspidal automorphic representation with almost all local components which are negative representations (the notion is defined further on in the preliminaries section and is essential in understanding of the local components of the automorphic representations attached to quadratic unipotent parameters). The results of the second section can be obtained even without the embedding of Theorem 5.1, i.e., of its abstract-representation counterpart mentioned above-these results would also follow from the global Langlands' subquotient theorem [6], but we need Theorem 5.1 for the calculations in the fourth section.

In the third section we explain in detail the inductive nature of finding automorphic representations with the prescribed quadratic unipotent Arthur parameter. We explain how, for a given parameter, we find "a basic parameter", and then how we start with the inductive procedure. This procedure is divided in two steps, Step 1 and Step 2, which we explain. We state the form of the Arthur parameter (Conditions (Δ)) for which this procedure is carried out.

In the fourth section we cover the case in which, in each step, all the intermediate (and final) automorphic representation have a non-zero constant term along the standard Borel subgroup. At the end, to show that the final automorphic square-integrable representation of $Sp_{2n}(\mathbb{A})$ is really corresponding to the given Arthur parameter ϕ, we use local results from the Preliminaries (sub)section (Proposition 1.2).

In the fifth section, using a result mentioned above, about embedding of a non-cuspidal automorphic representation in the representation induced from a cuspidal automorphic representation, we obtain the same results as in the fourth section about construction of the representation with the prescribed quadratic unipotent parameter, but now we do not assume that "a basic representation" has a non-zero constant term along the standard Borel subgroup. In this case, we also have to use some extra ingredients, due to a number of different authors (Arthur, Mœglin, Ban), concerning some properties of local Arthur packets, as is briefly explained at the beginning of the third section.

1.2 Preliminaries

Let k be an algebraic number field and k_v its completion at place v. The symplectic group of rank n over k is defined in the following way:

$$Sp_{2n} = \left\{ g \in GL_{2n} : g \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} g^t = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} \right\}$$

where J_n is $n \times n$ matrix defined by $J_n = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$. In general, if G is an algebraic group over k, B is an k-algebra, then $G(B)$ denotes the group of B-points of G. The adele ring of k we denote by $\mathbb{A} = \mathbb{A}_k$, and for a finite place v of k, we denote by O_v the ring of integers of k_v, and by $\overline{\omega}_v$ a generator of it's maximal ideal (we fix it). We let $|\cdot|_v$ (or ν_v) denote the normalized absolute value on k_v. We choose a non-trivial additive character $\psi : k\backslash\mathbb{A} \to \mathbb{C}^*$; we have $\psi = \prod_v' \psi_v$ (a restricted product; ψ_v is unramified almost everywhere). We fix a Haar measure $\mathrm{d}x = \prod_v' \mathrm{d}x_v$ such that $\mathrm{d}x_v$ self-dual with respect to character ψ_v (meaning self-duality with respect to the Fourier transform). A unitary Grossencharacter $\mu : k^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ can be written as a restricted direct product of local characters $\mu_v : k_v \to \mathbb{C}^*$; in this set up, Tate defined local L-function $L(s, \mu_v)$ and ε-factor $\varepsilon(s, \mu_v, \psi_v)$, here $\varepsilon(s, \mu) = \prod \varepsilon(s, \mu_v, \psi_v)$ is independent of ψ. For all places where μ_v and ψ_v are unramified, $\varepsilon(s, \mu_v, \psi_v) = 1$. Meromorphic continuation of $L(s, \mu) = \prod L(s, \mu_v)$ (the product initially converges for $Re(s) > 1$) satisfies the following functional equation

$$L(s, \mu) = \varepsilon(s, \mu)L(1 - s, \mu^{-1}).$$

If $\mu \neq |\cdot|^{it},\ t \in \mathbb{R},\ L(s, \mu)$ is entire. Otherwise, it has simple poles for $s = -it$ and $s = 1 - it$. Since we are interested in the cases where $\mu^2 = 1$, to detect the poles, it will be enough only to look at $s \in \mathbb{R}$. Thus only the case of $\mu = 1$ produces a pole of $L(s, \mu)$.

For Sp_{2n} defined above, we fix a Borel subgroup consisting of upper-triangular matrices, and maximal (k-split) torus T_n consisting of diagonal matrices in B_n. In this setting, we denote by U_n the unipotent radical of B_n. With this choice of maximal torus and positive roots (Σ^+) determined by B_n, the set of simple roots $\Delta = \Delta(Sp_{2n})$ is given by $\alpha_i = e_i - e_{i+1},\ i = 1, \ldots, n-1$ and $\alpha_n = 2e_n$ (here $e_i(t) = t_i$, where $t = diag(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$ is a diagonal matrix belonging to the maximal torus T_n). Corresponding coroots are given by $\check{\alpha}_i = \check{e}_i - \check{e}_{i+1},\ \check{\alpha}_n = \check{e}_n$, where cocharacters $\check{e}_i$ have obvious meaning. For any $\alpha \in \Sigma^+$, the corresponding root subgroup $U(\mathbb{A})$ is isomorphic to $\mathbb{A}$; similarly, for every place $v,\ U(k_v) \cong k_v$. Using this isomorphisms, we define Haar measures on these roots subgroups of unipotent radicals, moreover, in this way we fix Haar measures on unipotent radicals of standard parabolic subgroups, since they are spanned by root subgroups (these are isomorphisms of k-varieties). The normalization of measures on unipotent radicals is important because of the definition of intertwining operators.

For every $v < \infty$ we fix $K_v = Sp_{2n}(O_{k_v})$ as a maximal good compact subgroup of $Sp_{2n}(k_v)$, and for archimedean places we fix some maximal compact subgroup and denote $K = \prod_v K_v \subset Sp_{2n}(\mathbb{A})$. Also we denote $K_\infty = \prod_{v|\infty} K_v$. Let $\mathfrak{g}_\infty$ be a (real) Lie algebra of $\prod_{v|\infty} Sp_{2n}(k_v)$. We say that an irreducible admissible representation of $Sp_{2n}(k_v)$ $(v < \infty)$ is unramified if it has a non-zero K_v-fixed vector (also called K_v-spherical). Then, necessarily, this vector is unique, up to a scalar. We have the following important result:

Lemma 1.1 *Let $G = Sp_{2n}(k_v)$ and K_v as above. Assume that σ is K_v-spherical smooth representation of G, and σ is a subquotient of* $\mathrm{Ind}_{MN}(\sigma' \otimes 1_N)$, *for some smooth representation σ' of M (where M is a standard Levi subgroup of a standard parabolic subgroup MN). Then σ' is $M \cap K_v$-spherical.*

Proof This is Lemma 1.1 (ii) of [16]. □

The main aim of this paper is attaching (fairly explicitly) an automorphic representation to a quadratic unipotent Arthur parameter (cf. [1]). So we recall the form of Arthur parameters we need (without invoking the conjectural Langlands group L_k). Let W_k denote the global Weil group attached to an algebraic number field k.

The general quadratic unipotent Arthur parameter is an admissible homomorphism

$$\phi : W_k \times SL(2, \mathbb{C}) \to SO(2n + 1; \mathbb{C})$$

such that the following holds

1. $\phi_{|W_k}$ is continuous, semisimple, and it's image is bounded,
2. $\phi_{|SL(2,\mathbb{C})}$ is a morphism of algebraic groups,
3. $\phi(W_k \times SL(2, \mathbb{C}))$ is discrete, that is, the image is not contained in a proper Levi subgroup of $SO(2n + 1; \mathbb{C})$,
4. $\phi_{|W_k}$ factors through W_k^{ab}, where W_k^{ab} is the maximal abelian Hausdorff quotient of W_k.

We can thus decompose homomorphism ϕ into the sum of irreducible representations of $W_k^{ab} \times SL(2, \mathbb{C})$ so that

$$\phi \cong \oplus_{(\mu,a)\in \mathrm{Jord}(\phi)} \mu \otimes V_a,$$

where V_a is the unique irreducible algebraic representation of $SL(2, \mathbb{C})$ of dimension a and $\mu : k^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ is a quadratic Grossencharacter of k, obtained by class field theory from a character of W_k^{ab}. This decomposition defines $\mathrm{Jord}(\phi)$. It can be shown (cf. Theorem 4.1 of [18]) that the set of parameters like this is in the bijective correspondence with the collection of finite sets Jord (called Jordan blocks) consisting of pairs (μ, a), where $\mu : k^*\backslash\mathbb{A} \to \mathbb{C}^*$ is a quadratic Grossencharacter of k, a is an odd positive rational integer such that $\sum_{(\mu,a)\in \mathrm{Jord}} a = 2n + 1$ and $\prod_{(\mu,a)\in \mathrm{Jord}} \mu = 1$. So the bijection is $\mathrm{Jord} \Longleftrightarrow \mathrm{Jord}(\phi)$.

To this global Arthur parameter we attach a local parameter:

$$\phi_v \cong \oplus_{(\mu,a)\in \mathrm{Jord}(\phi)} \mu_v \otimes V_a.$$

Note that, in this (local) case, $\mathrm{Jord}(\phi_v)$ (defined analogously to the global case) does not have to be a set (i.e., for different global μ and λ we can have $(\mu_v, a) = (\lambda_v, a)$).

We say that an irreducible subrepresentation $X \cong \otimes_v X_v$ of the space $A^2(Sp_{2n}(k)\backslash Sp_{2n}(\mathbb{A}))$ of square-integrable automorphic forms on $Sp_{2n}(\mathbb{A})$ is attached to a global unipotent parameter ϕ, if for all but a finite number of places v, X_v is unramified representation attached to parameter ϕ_v (so necessarily negative representation; the definition of a (strongly) negative representation of $Sp_{2n}(k_v)$, $v < \infty$ and its Arthur parameter (i.e., the Jordan blocks) is recalled below).

In this paper we construct inductively an automorphic representation with the prescribed quadratic unipotent Arthur parameter like this, in addition, we assume the condition (Δ) explained in the beginning of the third section.

For an irreducible representation π of $Sp_{2n}(k_v)$ $(v < \infty)$, by $r_{1,\ldots,1;0}(\pi)$ we denote the Jacquet module of that representation with respect to the Borel subgroup B_n. An irreducible admissible unramified representation of $Sp_{2n}(k_v)$ is strictly (or strongly) negative if for every irreducible subquotient $\chi_1\nu^{s_1} \otimes \cdots \otimes \chi_n\nu^{s_n}$ of $r_{1,\ldots,1;0}(\pi)$ (here χ_i is unitary character of $GL(1, k_v)$, $s_i \in \mathbb{R}$, $i = 1, \ldots, n$) the following holds

$$\begin{aligned} s_1 &< 0,\\ s_1 + s_2 &< 0,\\ &\vdots\\ s_1 + s_2 + \cdots + s_n &< 0. \end{aligned} \tag{1}$$

In an analogous circumstances, an unramified representation is negative if the inequalities above are not necessarily strict. Note that this criterion is reverse of Casselman's criterion for square-integrability for the representations of symplectic groups; indeed the Aubert duals of strictly negative (resp. negative) representations are square-integrable (resp. tempered) representations.

We use Zelevinsky notation for the normalized parabolic induction for the general linear groups and the symplectic group. Let $M \cong GL_{n_1}(k_v) \times \cdots \times GL_{n_k}(k_v)$ be a (k_v-points of) Levi subgroup of a standard parabolic subgroup P of $GL_n(k_v)$ so that $n_1 + \cdots + n_k = n$. Let $\pi_1, \ldots, \pi_k$ be admissible representations of $GL_{n_i}(k_v)$, $i = 1, \ldots, k$. Then, we denote $\mathrm{Ind}_P^{GL_n(k_v)}(\pi_1 \otimes \cdots \otimes \pi_k)$ by $\pi_1 \times \cdots \times \pi_k$. Similarly, let $M \cong GL_{n_1}(k_v) \times \cdots \times GL_{n_k}(k_v) \times Sp_{2n'}(k_v)$ be a (k_v-points of) Levi subgroup of a standard parabolic subgroup P of $Sp_{2n}(k_v)$ so that $n_1 + \cdots + n_k + n' = n$. Let $\pi_1, \ldots, \pi_k$ be admissible representations of $GL_{n_i}(k_v)$, $i = 1, \ldots, k$ and let σ be an admissible representation of $Sp_{2n'}(k_v)$. Then, we denote $\mathrm{Ind}_P^{Sp_{2n}(k_v)}(\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma)$ by $\pi_1 \times \cdots \times \pi_k \rtimes \sigma$. If $n' = 0$, i.e., P is a subgroup of the Siegel parabolic subgroup, then we denote this induced representation by $\pi_1 \times \cdots \times \pi_k \rtimes 1$. Here 1 thus denotes the trivial representation of the trivial group.

For $v < \infty$, a unitary character χ of $GL_1(k_v)$, and $\alpha, \beta \in \mathbb{R}$ such that $\beta + \alpha + 1 \in \mathbb{Z}_{\geq 1}$, by $\zeta(-\alpha, \beta; \chi)$ we denote the unique irreducible subrepresentation of $\nu^{-\alpha}\chi \times \nu^{-\alpha+1}\chi \times \cdots \times \nu^{\beta}\chi$ (which is a character $\nu^{\frac{\beta-\alpha}{2}}\chi \circ \det$ of $GL_{\beta+\alpha+1}(k_v)$). For $v < \infty$, let χ_0 be the unique quadratic non-trivial unramified character of k_v^*.

The **Jordan blocks** are defined for an unramified strongly negative and negative representations of a symplectic group $Sp_{2n}(k_v)$ $(v < \infty)$ as follows. For an unramified strictly negative representation σ of $Sp_{2n}(k_v)$ there exists (a unique) set of positive odd rational integers $2m_1+1 < 2m_2+1 < \cdots < 2m_l+1$ and $2n_1+1 < 2n_2+1 < \cdots < 2n_k+1$ such that k is even and $2m_1+1+\cdots+2m_l+1+2n_1+1\cdots+2n_k+1 = 2n+1$ (so l is odd) such that

$$\sigma \hookrightarrow \zeta(-n_k, n_{k-1}; \chi_0) \times \cdots \times \zeta(-n_2, n_1; \chi_0) \\ \times \zeta(-m_l, m_{l-1}; 1) \times \cdots \times \zeta(-m_3, m_2; 1) \times \zeta(-m_1, -1; 1) \rtimes 1, \qquad (2)$$

where, if $m_1 = 0$, there is no factor $\zeta(-m_1, -1; 1)$ (cf. [16, Lemma 5.5]). Then, we define, for χ_0 (the quadratic unramified character defined above) and $1 = 1_{GL_1(k_v)}$, the trivial character of $GL_1(k_v)$, the following set, called the Jordan block of σ :

$$\mathrm{Jord}(\sigma) = \{(\chi_0, 2n_1 + 1), \ldots, (\chi_0, 2n_k + 1), (1, 2m_1 + 1), \ldots, (1, 2m_l + 1)\}.$$

For a negative representation σ_n there exists a unique strongly negative representation σ_{sn} and pairs $(\chi_1, l_1), \ldots, (\chi_j, l_j)$ ($l_i \in \mathbb{Z}_{\geq 1}$, χ_i unramified unitary characters) unique up to a permutation and taking inverses of characters, such that (cf. [19, Theorem 0-3])

$$\sigma_n \hookrightarrow \times_{i=1}^{j} \zeta\left(-\frac{l_i - 1}{2}, \frac{l_i - 1}{2}; \chi_i\right) \rtimes \sigma_{sn}.$$

Then we define a multiset $\mathrm{Jord}(\sigma_n) = \mathrm{Jord}(\sigma_{sn}) + \sum_{i=1}^{k}\{(\chi_i, l_i), (\chi_i^{-1}, l_i)\}$.

Assume ϕ is an Arthur parameter as above. Then, for a global quadratic Grossencharacter χ, we denote $\mathrm{Jord}_\chi(\phi) = \{a : (\chi, a) \in \mathrm{Jord}(\phi)\}$. In the same way, for a local component ϕ_v of an Arthur parameter and local character χ_v, we define $\mathrm{Jord}_{\chi_v}(\phi_v)$. Analogously, for a (local) negative representation σ and a local character χ, we define $\mathrm{Jord}_\chi(\sigma) = \{a : (\chi, a) \in \mathrm{Jord}(\sigma)\}$.

What we need in future from this information about negative representation is the following result.

Proposition 1.2 *Let $\beta > \alpha > 0$ be integers, and let χ be an unramified quadratic character of k_v. Let π be a negative representation. Then, the irreducible unramified subquotient of $\zeta(-\beta, \alpha; \chi) \rtimes \pi$ is a subrepresentation; it is also a negative representation.*

Proof This is the main result in [9]. ◻

2 Restriction on the local components

In this section we comment on some information about local components of automorphic representations of symplectic group attached to unipotent Arthur parameter and some global consequences that we can get relatively straightforward from the classification of spherical representations for general linear and symplectic groups. These consequences we relate to generalized Ramanujan conjecture (cf. [22]).

Firstly, for the convenience of reader, we recall the classification of unitary unramified representations for p-adic general linear groups (cf. [19, Theorem 4-1]). By $\mathrm{Irr}^{unr}(GL)$ we denote a set of the equivalence classes of irreducible unramified representations for the general linear groups $GL_n(k_v)$ for all $n \in \mathbb{Z}_{\geq 1}$. Here $\mathrm{Irr}^{u,unr}(GL)$ denotes a subset of $\mathrm{Irr}^{unr}(GL)$ consisting of all the classes of unitary unramified representations.

Theorem 2.1 (i) *Let* $\phi_1, \ldots, \phi_a, \psi_1, \ldots, \psi_b \in \mathrm{Irr}^{unr}(GL)$ *be a sequence of unramified unitary characters (one-dimensional unramified representations). Let* $\alpha_1, \ldots, \alpha_b \in \langle 0, \frac{1}{2}\rangle$ *be a sequence of real numbers (the possibility of* $a = 0$ *or* $b = 0$ *is not excluded here). Then*

$$\phi_1 \times \cdots \times \phi_a \times (\nu^{\alpha_1}\psi_1 \times \nu^{-\alpha_1}\psi_1) \times \cdots \times (\nu^{\alpha_b}\psi_b \times \nu^{-\alpha_b}\psi_b) \in \mathrm{Irr}^{u,unr}(GL).$$

(ii) *Let* $\pi \in \mathrm{Irr}^{u,unr}(GL)$. *Then there exist* $\phi_1, \ldots, \phi_a, \psi_1, \ldots, \psi_b$ *and* $\alpha_1, \ldots, \alpha_b$ *as in (i) such that* π *is isomorphic to the induced representation given above. Each sequence* $\phi_1, \ldots, \phi_a$ *and* $(\psi_1, \alpha_1), \ldots, (\psi_b, \alpha_b)$ *is uniquely determined by* π *up to a permutation.*

Now we return to automorphic representation Π of $Sp_{2n}(\mathbb{A})$ (more precisely, of $\prod_{v<\infty} Sp_{2n}(k_v) \times (\mathfrak{g}_\infty, K_\infty))$ in square-integrable discrete spectrum of $Sp_{2n}(\mathbb{A})$ attached to an unipotent Arthur parameter. In [10] we prove that there is an embedding (as abstract representations)

$$\Pi \hookrightarrow \pi_1 \times \cdots \pi_k \rtimes \sigma, \tag{3}$$

where $\pi_1, \ldots, \pi_k$ are (essentially) cuspidal automorphic representations of appropriate general linear groups, σ is an automorphic cuspidal representation of $Sp_{2m}(\mathbb{A})$.

We know that, at almost all places v, Π_v is an unramified negative representation (with the cuspidal support expressible in terms of quadratic characters entering the expression for the local Arthur parameter). At these places we look at the local embedding

$$\Pi_v \hookrightarrow \pi_{1,v} \times \cdots \pi_{k,v} \rtimes \sigma_v. \tag{4}$$

By Lemma 1.1, we know that $\pi_{1,v}, \ldots, \pi_{k,v}, \sigma_v$ are spherical representations. By the main result of [19, (Theorem 0.8)] σ_v, as a unitary unramified representation of $Sp_{2m}(k_v)$, is irreducibly induced

$$\sigma_v \cong \times_{(l,\chi,\alpha)} \left(\zeta\left(-\frac{l-1}{2}, \frac{l-1}{2}; 1_v\right)\nu^\alpha\chi\right) \rtimes \sigma_{v,neg}. \tag{5}$$

Here (l, χ, α) varies through members of certain finite set, χ is unramified unitary character (but we know that cuspidal support of σ_v consists of quadratic characters, so $\chi \in \{\chi_{0,v}, 1_v\}$); l is positive rational integer, and $\alpha \in \langle 0, 1\rangle$, satisfying some technical conditions. Here $\chi_{0,v}$ denotes the unique unramified non-trivial quadratic character of k_v^* and 1_v is the trivial character of k_v^*. The representation $\sigma_{v,neg}$ is negative. Since the exponents in the cuspidal support of σ_v must be rational integers, in the above expression we must have $\frac{l-1}{2} + \alpha \in \mathbb{Z}$, and $-\frac{l-1}{2} + \alpha \in \mathbb{Z}$, so $\alpha = \frac{1}{2}$ and l is even. Having this in mind, we examine some of these technical conditions mentioned above (cf. Definition 0–7 of [19]); namely we must apply Definition 0–7 (3) of [19], where, for fixed l and χ, exponents in $e(l, \chi)$ are collected. Here $e(l, \chi)$ denotes all the α's which occur in (5) for fixed l and χ. We thus have only one exponent, $\alpha = \frac{1}{2}$. Then, using notation from loc.cit, we have that $v = 0$. On the other hand, from (3)(b) it follows that $u = 1$. So, $u + v$ is odd, meaning, by 3(a), that $(\chi, l) \in \mathrm{Jord}(\sigma_{v,neg})$, for every (χ, l) appearing in (5). We know ([16], [19, Theorem 0-3]) that, for $\sigma_{v,neg}$ there exists $\sigma_{v,sn}$, a strongly negative representation of some $Sp_{2s_0}(k_v)$, and unitary unramified characters $\chi_1, \ldots, \chi_r$ and $l_1, \ldots, l_r \in \mathbb{Z}_{\geq 1}$ such that

$$\sigma_{v,neg} \hookrightarrow \times_{j=1}^{r} \zeta\left(-\frac{l_j - 1}{2}, \frac{l_j - 1}{2}; \chi_j\right) \rtimes \sigma_{v,sn}, \tag{6}$$

where, in our situation, again $\chi_j \in \{\chi_{0,v}, 1_v\}$ and $\mathrm{Jord}(\sigma_{v,neg}) = \mathrm{Jord}(\sigma_{v,sn}) \cup \sum_{j=1}^{r}\{(\chi_j, l_j), (\chi_j^{-1}, l_j^{-1})\}$. Since we are in the symplectic group case, every number l in $(l, \chi) \in \mathrm{Jord}(\sigma_{v,sn})$ is odd. This means that every l in $(l, \frac{1}{2})$ from (5) must be one of (χ_j, l_j) in (6). But, since such l is even, this would, by (6), mean that there is an exponent in the cuspidal support of $\sigma_{v,neg}$, so in the cuspidal support of Π_v, which is not an integer; a contradiction. This means that (5) reduces to $\sigma_v = \sigma_{v,neg}$ is a negative representation.

Now we analyze spherical representations of general linear group in (4). We note that, for every $j = 1, \ldots, k$, there exists a real number β_j such that $\nu^{-\beta_j}\pi_j$ is a cuspidal (unitary) representation of some global general linear group (cf. 3). For the simplicity of notation, assume $j = 1$. Then,

$$\nu^{-\beta_1}\pi_{1,v} \cong \phi_1 \times \cdots \times \phi_a \times (\nu^{\alpha_1}\psi_1 \times \nu^{-\alpha_1}\psi_1) \times \cdots \times (\nu^{\alpha_b}\psi_b \times \nu^{-\alpha_b}\psi_b)$$

(Theorem 2.1). Here $\phi_j = \zeta(-\frac{t-1}{2}, \frac{t-1}{2}; \phi_j)$, i.e., we look at ϕ_j as a character of $GL_t(k_v)$; analogously for ψ_j. Note that $\nu^{-\beta_1}\pi_{1,v}$ has to be a generic representation, so every factor in the previous relation must be generic, forcing ϕ's and ψ's to be characters of $GL_1(k_v)$. Note that then $\beta_1 + \alpha_j$ and $\beta_1 - \alpha_j$ have to be integers, for $j = 1, \ldots, b$. By subtracting these expressions, we get that $2\alpha_j \in \mathbb{Z}$, which is impossible (by assumptions in Theorem 2.1). This means that $\nu^{-\beta_1}\pi_{1,v} \cong \phi_1 \times \cdots \times \phi_a$, where ϕ_j's are unramified (quadratic) characters of $GL_1(k_v)$.

Keeping the notation from (3) and (4) we have proved the following theorem:

Theorem 2.2 *Let Π be an irreducible automorphic representation in the discrete (square-integrable) spectrum of $Sp_{2n}(\mathbb{A})$ attached to an unipotent Arthur parameter, so that, at almost all places, Π_v is an unramified negative representation.*

For those places v, the local components σ_v of an automorphic cuspidal representation σ such that (3) holds, are negative (unramified) representations. Also, for those places, the local components of a cuspidal automorphic (unitary) representation of $\nu^{-\beta_j}\pi_j$ of a general linear group are fully induced from quadratic characters. This moreover means, that, at those places, the local components are tempered, i.e., every representation $\nu^{-\beta_j}\pi_j$, $j = 1, \ldots, k$ is of Ramanujan type (cf. [22] and conjectures in the sixth section there).

Remark Note that for this Theorem we do not actually need the global embedding of (3); it is enough to just use Langlands' "subquotient version" [6]; we though do need (3) in the fifth section.

We now study a certain specific case of (3). Assume that all the representations $\pi_1, \ldots, \pi_k$ are global characters of $GL_1(\mathbb{A})$, denote them $\nu^{\beta_1}\chi_1, \ldots, \nu^{\beta_k}\chi_k$, where

χ's a unitary, and β's are real numbers. From the fact that almost everywhere the characters $\chi_{i,v}$, $i = 1, \dots k$, are quadratic, it follows that they are quadratic (globally).

Definition 2.1 For a strongly negative unramified representation σ or for a negative unramified σ whose cuspidal support consists only of quadratic characters, by supp(Jord(σ)) we denote the multiset

$$\mathrm{supp}(\mathrm{Jord}(\sigma)) = \sum_{(\chi,a)\in \mathrm{Jord}(\sigma)} [\nu^{-\frac{a-1}{2}}\chi, \nu^{\frac{a-1}{2}}\chi].$$

Here $[\nu^{-\frac{a-1}{2}}\chi, \nu^{\frac{a-1}{2}}\chi]$ denotes the set $\{\nu^{-\frac{a-1}{2}}\chi, \nu^{-\frac{a-1}{2}+1}\chi, \dots, \nu^{\frac{a-1}{2}}\chi\}$.

In the fourth section we need the following:

Corollary 2.3 *In the above situation, we have*

$$\mathrm{supp}(\mathrm{Jord}(\Pi_v)) = \mathrm{supp}(\mathrm{Jord}(\sigma_v)) + \sum_{i=1}^{k}\{\chi_{i,v}\nu^{\beta_i}, \chi_{i,v}\nu^{-\beta_i}\}.$$

Proof Since Π_v and σ_v are unramified negative representations and $\chi_{i,v}$ quadratic, this is obvious. □

3 Description of the inductive construction

The method presented in this paper is, as we have explained in the Introduction, of the inductive nature, the important question is how to construct the first automorphic representation explicitly realized in the space of square-integrable automorphic forms, which corresponds to the part of the given Arthur parameter (i.e., which corresponds to the analogous Arthur parameter for the symplectic group of smaller rank). We now discuss how to see what the "basis" Arthur parameter should look like.

Let $\mathrm{Jord}_\mu(\phi) = \{a : (\mu, a) \in \mathrm{Jord}(\phi)\}$. We denote different grossencharacters appearing in the parameter ϕ by $\mu_1, \dots, \mu_k$ and we denote $|\mathrm{Jord}_{\mu_i}(\phi)| = r_i$. From the conditions in the definition of Arthur parameter it follows that there is an odd number of different μ_i's such that r_i is odd. We let $\mathrm{Jord}_{\mu_i}(\phi) = \{a_{i1}, \dots, a_{ir_i}\}$, where $a_{i1} < a_{i_2} < \dots < a_{ir_i}$.

We let exactly the first l characters (l is odd) be such that their r_i is odd. It follows that $\prod_i^l \mu_i = 1$. Now, we denote by $\phi_0 = \mu_1 \otimes V_{a_{11}} \oplus \mu_2 \otimes V_{a_{21}} \oplus \dots \oplus \mu_l \otimes V_{a_{l1}}$ an Arthur parameter of an automorphic representation $\sigma(\phi_0)$ (realized directly in the space of the square-integrable automorphic forms). This representation will be the basis of our inductive procedure.

If $l = 1$, we get $\mu_1 = 1$. In that case we know that the trivial representation of $Sp_{a_{11}-1}(\mathbb{A})$ is attached to the Arthur parameter $\phi_0 = \mathbf{1} \otimes V_{a_{11}}$; and we know that this representation is automorphic (realized on the space of constant functions on $Sp_{a_{11}-1}(\mathbb{A})$); moreover it belongs to the residual part of the spectrum. Moreover, Kim and Shahidi [11] proved that, among the automorphic square-integrable representations with the unipotent Arthur parameters, exactly these (i.e., the ones with $l = 1$)

are the ones for which some constant term along the Borel subgroup does not vanish; this means (as can be seen, for example, in [18]) that these representations can be realized (as abstract representations) as subrepresentations of some (global) principal series representations.

We assume that we can find $\sigma(\phi_0)$, even when $l > 1$. There are some instances for which we know, more or less explicitly, representations attached to a parameter of this form; e.g., a cuspidal representation of $SL(2, \mathbb{A})$ attached to the Gelbart–Jacquet lift of a cuspidal representation of $GL(2, \mathbb{A})$ to $GL(3, \mathbb{A})$. The precise statement can be found in [8, the third section]. The case of special interest in this context is a cuspidal representation of $SL(2, \mathbb{A})$ with the parameter $\chi \otimes 1 \oplus \mu \otimes 1 \oplus \chi\mu \otimes 1$, where χ, μ are non-trivial global quadratic characters (related in a way described in [8]. Some of the examples can be obtained by functoriality (from symplectic to general linear group).

To get Arthur parameter ϕ from the Arthur parameter ϕ_0 we have to add an even number of elements from $\mathrm{Jord}_{\mu_i}(\phi)$, $i = 1, \ldots, l$ and whole sets of elements of $\mathrm{Jord}_{\mu_i}(\phi)$, $i = l + 1, \ldots, k$ (every set from this collection has even number of elements).

So assume that we have already constructed some automorphic representation $\sigma(\phi')$ attached to the parameter ϕ' such that ϕ' is obtained by adjoining a $\mu_t \otimes V_{a_{t,j}} \oplus \mu_t \otimes V_{a_{t,j+1}}$ in each step (we of course, have a finite number of steps). Here, if $t \in \{1, \ldots, l\}$ we start with $\mu_t \otimes V_{a_{t2}} \oplus \mu_t \otimes V_{a_{t3}}$, and when we have $t \in \{l + 1, \ldots, k\}$ we start with $\mu_t \otimes V_{a_{t1}} \oplus \mu_t \otimes V_{a_{t2}}$. So, we assume that we have an explicit realization of $\sigma(\phi')$, (one of) the automorphic representations with the Arthur parameter ϕ', inside the space of square integrable automorphic forms $A_2(Sp_{2n'}(k)\backslash Sp_{2n'}(\mathbb{A}))$.

We want to construct explicitly (i.e., realize explicitly in the space of the square-integrable automorphic forms) an automorphic representation with the Arthur parameter $\phi = \phi' \oplus \mu_t \otimes V_{a_{t,j}} \oplus \mu_t \otimes V_{a_{t,j+1}}$ (so that μ_t may or may not appear in ϕ'; if $\mu_t \otimes V_b$ appears in ϕ' then $b < a_{t,j}$).

To be able to perform our inductive procedure, we assume that our quadratic unipotent parameter is of such a form that the addition of new elements in the Jordan block can be arranged to satisfy the following

Conditions (Δ)

1. Each of the numbers $a_{11}, \ldots, a_{l1}$ entering the definition of ϕ_0 is strictly smaller than any element $a_{t,j}$ we subsequently add in the Jordan block
2. We add elements $a_{t,j}$ and $a_{t,j+1}$ in the Jordan block in such an order that $a_{t,j}$ is equal or greater to any other element added previously in the Jordan block (this means that we do not necessarily add all the elements of Jord_{μ_1} and then all the elements of Jord_{μ_2} and so on).

The method we use (for both cases, $l = 1$ and $l > 1$) is the global analog of the construction of discrete series of classical p-adic groups [14]; this (global) method is already present in [18, (cf. Theorem 5.2)]. We briefly describe it. It consists of two steps:

Step 1

We first form a global representation of $Sp_{2n'+2a_{t,j}}(\mathbb{A})$ induced from a representation $|\det|^s\mu_t \otimes \sigma(\phi')$ of $GL(a_{t,j}, \mathbb{A}) \otimes Sp_{2n'}(\mathbb{A})$. In our realization of this representation in a certain space of automorphic forms, we can get Eisenstein series (as an intertwining operator) acting on it. This Eisenstein series is holomorphic at $s = 0$. We will then describe a certain subrepresentation π of this space (for $s = 0$) on which this Eisenstein series will be non-zero, giving our subrepresentation a different realization $E_0(\pi)$ (we use a constant term along the appropriate parabolic subgroup to detect a suitable subrepresentation).

Step 2

Now, we again form a global representation of $Sp_{2n}(\mathbb{A})$ (where $n = n' + \frac{a_{t,j}+a_{t,j+1}}{2}$) induced from a representation $|\det|^s \mu_t \otimes E_0(\pi)$ of $GL(\frac{a_{t,j+1}-a_{t,j}}{2}, \mathbb{A}) \otimes Sp_{2n'+2a_{t,j}}(\mathbb{A})$. Again, in the appropriate realization, we can take an Eisenstein series acting on this representation. We prove that this Eisenstein series has a pole of order two for $s = \frac{a_{t,j}+a_{t,j+1}}{4}$. Then, when we normalize the Eisenstein series with $(s - \frac{a_{t,j}+a_{t,j+1}}{4})^2$, we get an image representation in the space of square-integrable automorphic forms, with the local structure which relatively easy to analyze for almost every finite place. Now, using our local results from the second section, we see that almost everywhere, the local image is an irreducible negative representation, and we obtain an automorphic representation with the Arthur parameter ϕ.

We see that this construction is the global analogue of the local construction of the discrete series representations [14], so that the first step corresponds to the formation of the tempered representations, and the second step gives us (the new) discrete series representations.

In the next section we prove all the intermediate steps described above for the situation when $l = 1$; in the fifth section we prove the analogous claims for the parameters with $l > 1$ (but sometimes we slightly change our approach, e.g., in the proof of Proposition 5.7). To do so, we need some additional arguments (the embedding of the global automorphic representation in the representation induced from the automorphic cuspidal representation of a Levi subgroup [10], explicit description of the calculation of the constant term of Eisenstein series in a general situation (Theorem 5.3), analogous to a similar result in [15]; we also need some results of Arthur on the structure (with respect to Langlands parameters) of the representations in Arthur packets [1] and of Mœglin about the action of the (Aubert)-duality operator on the representations in the Arthur's packets [1,13].

4 Explicit construction I: representations supported in the Borel subgroup

If the representation $\sigma(\phi_0)$ (realized on the $\prod_{v<\infty} Sp_{2n_0}(k_v) \times (\mathfrak{g}_\infty, K_\infty)$-invariant irreducible subspace of $A(Sp_{2n_0}(k)\backslash Sp_{2n_0}(\mathbb{A}))$ is concentrated on the Borel subgroup (i.e., we are in $l = 1$ case, as explained in the beginning of the third section), then from the proof of Lemma I.3.2 of [15] (as is explained in the third section of [18]), by taking the constant term of $\sigma(\phi_0)$ along the Borel subgroup, we get intertwining which realizes our representation inside the globally induced representation (normalized induction, K-finite vectors) of $Sp_{2n_0}(\mathbb{A})$, induced from a character of the (standard) Borel subgroup (cf. also section 2 of [18]). Because we want an automorphic discrete series representation $\sigma(\phi_0)$ to be attached to the parameter ϕ_0, by looking at the corresponding local Arthur parameters we get that it is embedded (i.e., can be realized), as explained above, into the global representation induced from the character $\lambda(\phi_0)$, where

$$\lambda(\phi_0) = \nu^{-\frac{a_{1,1}-1}{2}} \otimes \cdots \otimes \nu^{-1}$$

is a character of a maximal split torus of $Sp_{2n_0}(\mathbb{A})$ (so $2n_0 = a_{1,1} - 1$). This means that $\sigma(\phi')$ is embedded in the global representation induced from the character $\lambda(\phi')$

$$\lambda(\phi') = (|\cdot|^{-\frac{a_{i,l}-1}{2}}\mu_i \otimes |\cdot|^{-\frac{a_{i,l}-1}{2}+1}\mu_i \otimes \cdots \otimes |\cdot|^{\frac{a_{i,l-1}-1}{2}}\mu_i) \otimes \cdots \\ \otimes |\cdot|^{-\frac{a_{1,1}-1}{2}} \otimes \cdots \otimes |\cdot|^{-1}. \tag{7}$$

Here $\mu_i \in \{\mu_1, \ldots, \mu_k\}$ and $l \in \{2, \ldots, r_i\}$. Here, we assume that, inductively, we have proved, that in each adding of two elements (χ, α) and (χ, β) (where $\alpha < \beta$ are odd positive integers; and we add elements in Jord_χ by increasing order) we get that our two-step procedure, in a case of descripting this character of a maximal torus, boils down to adding $|\cdot|^{-\frac{\beta-1}{2}}\chi \otimes \cdots \otimes |\cdot|^{\frac{\alpha-1}{2}}\chi$ in front of the previous character (of a smaller torus). Only the character of a basic torus $(|\cdot|^{-\frac{a_{1,1}-1}{2}} \otimes \cdots \otimes |\cdot|^{-1})$ is different, corresponding to the embedding of the trivial representation (global).

4.1 Step 1

We return to the (first step of the) proof. We consider the global induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{Sp_{2n'+2a_{t,j}}(\mathbb{A})}(|\det|^s \mu_{t,GL(a_{t,j},\mathbb{A})} \otimes \sigma(\phi')), \tag{8}$$

where P is a standard parabolic subgroup with Levi subgroup isomorphic to $GL(a_{t,j}) \otimes Sp_{2n'}$ (this representation is realized in the analogous way as in [2, p. 32]). Then, the degenerate Eisenstein series, for a section f_s belonging to (8), is given by

$$E(s, f_s)(g) = \sum_{P(k)\backslash Sp_{2n'+2a_{t,j}}(k)} f_s(\gamma g)$$

is acting on this representation. It is holomorphic at $s = 0$ (Langlands, c.f. [2, Theorem 7.2]) and defines an intertwining operator

$$\mathrm{Ind}_{P(\mathbb{A})}^{Sp_{2n'+2a_{t,j}}(\mathbb{A})}(\mu_{t,GL(a_{t,j},\mathbb{A})} \otimes \sigma(\phi')) \to A(Sp_{2n'+2a_{t,j}}(k)\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A})). \tag{9}$$

Here we assume that $\sigma(\phi')$ is realized in the space of square-integrable automorphic forms in $A(Sp_{2n'}(k)\backslash Sp_{2n'}(\mathbb{A}))$.

Let Y be a subrepresentation of (8) such that the Eisenstein series does not send it to zero. We want to detect what is a form of such representation. To do so, we calculate the constant term along the Borel subgroup of $E(f_s, \cdot)|_{s=0}$, $f_s \in Y$, and examine when it is non-zero. Since $E(f_s, \cdot)|_{s=0}Y \subset A(Sp_{2n'+2a_{t,j}}(k))\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A}))$, taking the constant term along the appropriate Borel subgroup will give us automatically realization of $E(f_s, \cdot)|_{s=0}Y$ (i.e., of Y since it is irreducible) as a subrepresentation of a representation globally induced from the character of the appropriate maximal torus. So, if we put this together, we calculate $E_0(s, f_s)(g)$. We denote

$$\lambda_{\phi',s,a_{t,j}} = \mu_t|\cdot|^{s-\frac{a_{t,j}-1}{2}} \otimes \cdots \otimes \mu_t|\cdot|^{s+\frac{a_{t,j}-1}{2}} \otimes \lambda(\phi'),$$

and let α be a simple root such that the parabolic subgroup appearing in (8) corresponds to the set of roots $\Delta\backslash\{\alpha\}$, i.e., $\alpha = \alpha_{a_{t,j}}$. We have

$$\begin{aligned} E_0(s, f_s)(g) &= \int_{U_{n'+a_{t,j}}(k)\backslash U_{n'+a_{t,j}}(\mathbb{A})} E(f_s, ug)\mathrm{d}u \\ &= \sum_{w\in W; w(\Delta\backslash\{\alpha\})>0} M(\lambda_{\phi',s,a_{t,j}}, w)(f_s)_0(g), \qquad (10) \\ (f_s)_0(g) &= \int_{U_{n'}(k)\backslash U_{n'}(\mathbb{A})} f_s(u'g)\mathrm{d}u'; \qquad (11) \end{aligned}$$

the last integral is actually a canonical map (for $g = k \in K$ it does not depend on s) from the automorphic realization of $\sigma(\phi')$ to the space of it's constant terms along $B_{n'}$, i.e., embedding into the global representation induced from the character $\lambda(\phi')$ of the maximal torus $T_{n'}(k)\backslash T_{n'}(\mathbb{A})$.

This formula for the calculation of the constant term of the Eisenstein series of the representation (8) is proved in Lemma 2.1 of [17] (c.f. Lemma 2.2 there).

We further have

$$\begin{aligned} \sum_{w\in W; w(\Delta\backslash\{\alpha\})>0} M(\lambda_{\phi',s,a_{t,j}}, w)(f_s)_0 &= (f_s)_0 + M(\lambda_{\phi',s,a_{t,j}}, w_0)(f_s)_0 \\ &+ \sum_{w\in W; w(\Delta\backslash\{\alpha\})>0; w\neq w_0,1} M(\lambda_{\phi',s,a_{t,j}}, w)(f_s)_0, \qquad (12) \end{aligned}$$

where $w_0 = w_{l,\Delta} w_{l,\Delta\backslash\{\alpha\}}^{-1}$ is the longest element in the Weyl group for Σ^+ modulo the longest one in $\Sigma^+_{\Delta\backslash\{\alpha\}}$.

Now we examine more thoroughly the decomposition of the global intertwining operator $M(\lambda_{\phi',s,a_{t,j}}, w_0)$ into the local ones.

We decompose the representation (8) into the local components

$$\begin{aligned} &\mathrm{Ind}_{P(\mathbb{A})}^{Sp_{2n'+2a_{t,j}}(\mathbb{A})}(|\det|^s \mu_{t,GL(a_{t,j},\mathbb{A})} \otimes \sigma(\phi')) \\ &\cong \otimes_v \mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)}(|\det|_v^s \mu_{t,GL(a_{t,j},k_v)} \otimes \sigma_v(\phi')). \qquad (13) \end{aligned}$$

We are interested how the local components of (8) decompose for $s = 0$ (so we get finite length, unitary representations for $s = 0$). To study this decomposition, we need local intertwining operators

$$A(s, |\det|_v^s \mu_{t,GL(a_{t,j},k_v)} \otimes \sigma_v(\phi'), w_0) : \mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)} (|\det|_v^s \mu_{t,GL(a_{t,j},k_v)} \otimes \sigma_v(\phi'))$$
$$\to \mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)} (|\det|_v^{-s} \mu_{t,GL(a_{t,j},k_v)} \otimes \sigma_v(\phi'))$$

defined, standardly, by

$$A(s, |\det|_v^s \mu_{t,GL(a_{t,j},k_v)} \otimes \sigma_v(\phi'), w_0) f_v(g_v) = \int_{N(k_v)} f_v(w_0^{-1} n_v g_v) \mathrm{d}n_v.$$

w_0 in the defining integral is the fixed representative of the Weyl group element w_0 (fixed like in [21]). These operators converge for $Re(s) >> 0$, and admit meromorphic continuation on the whole complex plane. Since (11) gives (global) embedding into the representation induced from the character of the maximal torus, this also happens locally, so that we have embedding

$$\sigma_v(\phi') \hookrightarrow \lambda_v(\phi'),$$

and

$$\mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{tj}}(k_v)} (|\det|_v^s \mu_{t,GL(a_{tj},v)} \otimes \sigma_v(\phi')) \hookrightarrow \mathrm{Ind}_{B_{n'+a_{tj}}(k_v)}^{Sp_{2n'+2a_{tj}}(k_v)} (\lambda_{\phi',s,a_{t,j},v}).$$

Then, the above intertwining operator $A(s, |\det|_v^s \mu_{t,GL(a_{t,j},k_v)} \otimes \sigma_v(\phi'), w_0)$ is just the restriction of the intertwining operator $A(\lambda_{\phi',s,a_{t,j},v})$ acting on

$$\mathrm{Ind}_{B_{n'+a_{t,j}}(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)} (\lambda_{\phi',s,a_{t,j},v}) \to \mathrm{Ind}_{B_{n'+a_{t,j}}(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)} (w_0(\lambda_{\phi',s,a_{t,j},v})).$$

To normalize intertwining operators $A(s, |\det|_v^s \mu_{t,GL(a_{t,j},k_v)} \otimes \sigma_v(\phi'), w_0)$ we use the same normalization as we use (standardly) for the intertwining operators $A(\lambda_{\phi',s,a_{t,j},v}, w_0)$ which act on the principal series representations, so the normalization is by well understood L-functions:

$$N(|\det|_v^s \mu_{t,GL(a_{t,j},k_v)} \otimes \sigma_v(\phi'), w_0)$$
$$= r(\lambda_{\phi',s,a_{t,j},v}, w_0) A(s, |\det|_v^s \mu_{t,GL(a_{t,j},k_v)} \otimes \sigma_v(\phi'), w_0),$$

where, for $\underline{s} = (s_1, \ldots, s_k) \in \mathbb{C}^k$ and a general local character $\lambda(\underline{s})_v = |\cdot|_v^{s_1} \lambda_{1,v} \otimes \cdots \otimes |\cdot|_v^{s_k} \lambda_{k,v}$ of the maximal torus of the appropriate size, and w from the Weyl group attached to this torus, we have:

$$r(\lambda(\underline{s})_v, w) = \prod_{\alpha \in \sum^+, w(\alpha)<0} \frac{L(1, \lambda(\underline{s})_v \circ \check{\alpha}) \varepsilon(0, \lambda(\underline{s})_v \circ \check{\alpha}, \psi_v)}{L(0, \lambda(\underline{s})_v \circ \check{\alpha})}.$$

This might not be a standard normalization, but it satisfies usual properties with respect to taking hermitian contragredient, etc. (as is discussed in [18, relations (5-17), (5-18), (5-19)]); moreover, for $s=0$, $N(\mu_{t,GL(a_{t,j},k_v)}\otimes\sigma_v(\phi'),w_0)$ is holomorphic and unitary (because the analogous claim holds for $N(\lambda_{\phi',s,a_{t,j},v},w_0)$, cf. Theorem 2.5 of [18]) and $N(0,\mu_{t,GL(a_{t,j},k_v)}\otimes\sigma_v(\phi'),w_0)^2=Id$. We use this normalized intertwining operators to define subspaces $Y_v^{\pm}\hookrightarrow \mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)}(\mu_{t,GL(a_{t,j},k_v)}\otimes\sigma_v(\phi'))$ as follows:

$$
\begin{aligned}
Y_v(\sigma(\phi'))^{\pm}=\{&f\in \mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)}(\mu_{t,GL(a_{t,j},k_v)}\otimes\sigma_v(\phi')):\\
&\times N(\mu_{t,GL(a_{t,j},k_v)}\otimes\sigma_v(\phi'),w_0)f=\pm f.\}
\end{aligned}
$$

Moreover

$$
\mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)}(\mu_{t,GL(a_{t,j},k_v)}\otimes\sigma_v(\phi'))=Y_v(\sigma(\phi'))^{+}\oplus Y_v(\sigma(\phi'))^{-}.
$$

Note that if $\sigma(\phi')_v$ is spherical and $\mu_{t,GL(a_{t,j},k_v)}$ unramified quadratic character, then $Y_v(\sigma(\phi'))^{+}\neq\{0\}$, since it necessarily contains a K_v-fixed vector, by the property (iv) of Theorem 2.5 in [18]. If, additionally, v is non-archimedean, then (spherical) representation $Y_v(\sigma(\phi'))^{+}$ is irreducible (and so is $Y_v(\sigma(\phi'))^{-}$) (we can see by working little bit with [16, Corollary 5.1] or directly, when we apply Aubert involution).

So we have

$$
\mathrm{Ind}_{P(\mathbb{A})}^{Sp_{2n'+2a_{t,j}}(\mathbb{A})}(\mu_{t,GL(a_{t,j},\mathbb{A})}\otimes\sigma(\phi'))\cong\oplus_S[\otimes_{v\notin S}Y_v(\sigma(\phi'))^{+}\otimes_{v\in S}Y_v(\sigma(\phi'))^{-}],
$$

where S ranges over finite subsets of the set of places of k.

Claim *If $|S|$ is even and π an irreducible subrepresentation of*

$$
\otimes_{v\notin S}Y_v(\sigma(\phi'))^{+}\otimes_{v\in S}Y_v(\sigma(\phi'))^{-},
$$

then (9) is non-trivial on π.

We now proceed to prove this claim.

We return to (12). Let f_s be a function in (8), so that $(f_s)_0$ is factorizable function in $\mathrm{Ind}_{B_{n'+a_{t,j}}(\mathbb{A})}^{Sp_{2n'+2a_{t,j}}(\mathbb{A})}(\lambda_{\phi',s,a_{t,j}})$. We assume that $(f_s)_0=\otimes_v f_{s,v}$ and for $v\notin S$ $f_{s,v}$ is spherical vector normalized so that $f_{s,v}(e)=1$. To simplify the notation, we denote the action of the Weyl group on $\lambda_{\phi',s,a_{t,j}}$ by $w(s)$ (instead of $w(\lambda_{\phi',s,a_{t,j}})$). Then, for the choice of π as in the claim, (12) becomes

$$
\begin{aligned}
E_0(s,f_s)(g)=&(f_s)_0+r(\lambda_{\phi',s,a_{t,j},v},w_0)^{-1}\otimes_{v\notin S}f_{w_0(s),v}\\
&\otimes_{v\in S}N(|det|^s\mu_{t,GL(a_{t,j},k_v)}\otimes\sigma_v(\phi'),w_0)f_{s,v}\\
&+\sum_{w\in W;w(\Delta\backslash\{\alpha\})>0;w\neq w_0,1}M(\lambda_{\phi',s,a_{t,j}},w)(f_s)_0. \qquad (14)
\end{aligned}
$$

Lemma 4.1 *In the above setting, we have $w(\lambda_{\phi',0,a_{t,j}}) = \lambda_{\phi',0,a_{t,j}}$ for $w \in W$ such that $w(\Delta\backslash\{\alpha\}) > 0$ if and only if $w = w_0$ or $w = 1$.*

Proof Indeed, by (Lemma 4.4 of [23]), (we let $i = a_{t,j}$) the set $\{w \in W : w(\Delta\backslash\{\alpha_i\}) > 0\}$ can be described as $\cup_{0\le j\le i} W_j$ where each W_j consists of elements of the form $p\varepsilon$, where $p \in S_n$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n) \in \{\pm 1\}^n$, such that the action of p and ε can be described as (the last row is a description of the ε-action)

$$\begin{bmatrix} 1\ 2 & \overbrace{\qquad\ldots\qquad}^{p \text{ is increasing}} & j|\ j+1 & \overbrace{\qquad\ldots\qquad}^{p \text{ is decreasing}} & i|\ \ i+1 & \overbrace{\qquad\ldots\qquad}^{p \text{ is increasing}} & n \\ 1\ 1 & \ldots & 1|\ \ -1 & \ldots & -1|\ \ \ 1 & \ldots & 1 \end{bmatrix}.$$

We write down $\lambda_{\phi',0,a_{t,j}}$ as follows:

$$\lambda_{\phi',0,a_{t,j}} = \mu_t \nu^{-\frac{i-1}{2}} \otimes \cdots \otimes \mu_t \nu^{\frac{i-1}{2}} \otimes \lambda_{i+1}\nu^{s_{i+1}} \otimes \cdots \otimes \lambda_n \nu^{s_n}.$$

Assume that $w(\lambda_{\phi',0,a_{t,j}}) = \lambda_{\phi',0,a_{t,j}}$ and $w = p\varepsilon \in W_j$ describe above. Because of our construction, if for some λ_k, $k \in \{i+1, \ldots, n\}$ we have $\lambda_k = \mu_t$, then $|s_k| < \frac{i-1}{2}$. We observe how to get $\mu_t \nu^{\frac{i-1}{2}}$, i.e., the factor on the ith place. Since it cannot come from the factors on the places from $i+1, \ldots n$ (as we just observed), it must come from either factor on the first place ($\mu_t \nu^{-\frac{i-1}{2}}$) either from the factor on the ith place ($\mu_t \nu^{\frac{i-1}{2}}$). Assume it comes from the factor on the first place. This means $p(1) = i$ and $\varepsilon_1 = -1$. This means $j = 0$, $p(1) = i$, and p is decreasing on $\{1, 2, \ldots, i\}$. This means $p(2) = i-1, \ldots, p(i) = 1$, and $p|_{\{i+1,\ldots,n\}} = id$, so that $w = p\varepsilon = w_0$. On the other hand, if it comes from the factor on the ith place ($\mu_t \nu^{\frac{i-1}{2}}$), this means $p(i) = i$ and $\varepsilon_i = 1$. This means $j = i$ and $p(1) = 1, \ldots, p(i) = i$, also $p|_{\{i+1,\ldots,n\}} = id$, so that $w = p\varepsilon = 1$. □

From this Lemma follows that in (14) the second line cannot cancel the expression in the first line for $s = 0$.

We have the following important ingredient.

Lemma 4.2 *The following holds:*

$$\lim_{s\to 0} r(\lambda_{\phi',s,a_{t,j}}, w_0)^{-1} = 1.$$

Note here that if $\mu_t = 1$, then $a_{t,j} > 1$ because of the description of the basis of our inductive procedure.

Proof (of the Lemma) Note that, according to the notation in Lemma 4.1, the element w_0 belongs to the set W_0. Now, according to [23] and Corollary 1-3 of [18], we have explicit description of roots $\alpha \in \Sigma^+$ such that $w_0(\alpha) < 0$, and these roots play a role in the definition of the normalizing factor $r(\lambda_{\phi',s,a_{t,j},v}, w_0)$. Let $i = a_{t,j}$. Then the set of those $\alpha \in \Sigma^+$ such that $w_0(\alpha) < 0$ can be divided in several sets:

(i) $\{2e_k : 1 \le k \le i\}$,

(ii) $\{e_k + e_l : 1 \leq k < l \leq i\}$,
(iii) $\{e_k - e_l : 1 \leq k \leq i, i+1 \leq l \leq n\}$,
(iv) $\{e_k + e_l : 1 \leq k \leq i, i+1 \leq l \leq n,\ p(l) > p(k)\}$.

Now we study the expression

$$\prod_{\alpha \in \sum^{+}, w_0(\alpha)<0} \frac{L(1, \lambda_{\phi', s, a_{t,j}} \circ \check{\alpha}) \varepsilon(0, \lambda(\underline{s}) \circ \check{\alpha})}{L(0, \lambda(\underline{s}) \circ \check{\alpha})}$$

for α from the each of the above four groups of roots. The contribution from the first group of roots is given by

$$\prod_{k=1}^{i} \frac{L\left(-\frac{i-1}{2} + s + k, \mu_t\right) \epsilon\left(-\frac{i-1}{2} + s + k - 1, \mu_t\right)}{L\left(-\frac{i-1}{2} + s + k - 1, \mu_t\right)}. \tag{15}$$

The contribution from the second group of roots is given by

$$\prod_{1 \leq k < l \leq i} \frac{L(2s - i + k + l, 1) \epsilon(2s - i + k + l - 1, 1)}{L(2s - i + k + l - 1, 1)}. \tag{16}$$

Note that in the fourth set of roots, the condition $p(l) > p(k)$ is automatically satisfied, due to the description of the permutation p in $w_0 = p\epsilon$ given in the proof of 4.1. So the contribution of the third and the fourth group of roots is given by

$$\prod_{1 \leq k \leq i, i+1 \leq l \leq n} \frac{L\left(s - \frac{i-1}{2} + k - l', \mu_t \chi_l\right) L\left(s - \frac{i-1}{2} + k + l', \mu_t \chi_l\right)}{L\left(s - \frac{i-1}{2} + k - l' - 1, \mu_t \chi_l\right) L\left(s - \frac{i-1}{2} + k + l' - 1, \mu_t \chi_l\right)}$$
$$\cdot \epsilon\left(s - \frac{i-1}{2} + k - l' - 1, \mu_t \chi_l\right) \epsilon\left(s - \frac{i-1}{2} + k + l' - 1, \mu_t \chi_l\right) \tag{17}$$

Here $\chi_l \nu^{l'}$ denotes a character appearing as a factor in $\lambda_{\phi', s, a_{t,j}}$ at the lth place (χ_l is quadratic). We now use a functional equation satisfied by the L-functions: for a quadratic Grossencharacter χ we have

$$L(s, \chi) = \epsilon(s, \chi) L(1 - s, \chi).$$

Now we apply this on the denominators in the expression (15) and then introduce the a change of variables $k \mapsto i + 1 - k$ to obtain

$$\prod_{k=1}^{i} \frac{L\left(s - \frac{i-1}{2} + k, \mu_t\right)}{L\left(-s - \frac{i-1}{2} + k, \mu_t\right)}.$$

If $\mu_t \neq 1$ $L(s, \mu_t)$ is holomorphic in s, so that each of the above factors tends to 1 when $s \to 0$. If $\mu_t = 1$, we have the following simple fact

$$\lim_{s\to 0} \frac{L(-s+t,1)}{L(s+t,1)} = \begin{cases} 1; & \text{if } t \notin \{0,1\} \\ -1; & \text{if } t \in \{0,1\}. \end{cases}$$

This means that exactly for $k = \frac{i-1}{2}$ and for $k = \frac{i+1}{2}$ the factors are equal to -1 when $s \to 0$ so that overall product consisting from the contributions of the first set of roots equals 1 (unless $i = a_{t,j} = 1$ and $\mu_t = 1$ so that whole product consists only of one factor, but this cannot happen).

After applying the functional equation on the denominators of the contribution (16) and then applying change of variables $(k,l) \mapsto (i+1-l, i+1-k)$ we obtain

$$\prod_{1\leq k<l\leq i} \frac{L(2s-i+k+l,1)}{L(-2s-i+k+l,1)}.$$

This means that in the cases when $k+l = i$ and $k+l = i+1$ these factors tend to -1 when $s \to 0$. There are exactly $2\frac{i-1}{2} = i-1$ (even number!) of these instances, so the overall product tends to 1. If $i = 1$ there is no second product.

After applying the functional equation on the denominators in the expression (17) and applying the change of variables $k \mapsto i+1-k$, and then grouping together numerators and denominators in a nice way, we obtain

$$\begin{aligned} &\prod_{i+1\leq l\leq n} \prod_{1\leq k\leq i} \frac{L\left(s+2+\frac{i-1}{2}-k-l', \mu_t\chi_l\right)}{L\left(-s+2+\frac{i-1}{2}-k-l', \mu_t\chi_l\right)} \cdot \\ &\prod_{i+1\leq l\leq n} \prod_{1\leq k\leq i} \frac{L\left(s+2+\frac{i-1}{2}-k+l', \mu_t\chi_l\right)}{L\left(-s+2+\frac{i-1}{2}-k+l', \mu_t\chi_l\right)}. \end{aligned} \tag{18}$$

We have to check what happens if $\chi_l = \mu_t$. The number of times when l' is such that $2+\frac{i-1}{2}-k-l' = 1$ for any $k \in \{1,2,\ldots,i\}$ is equal to the number of l' such that $2+\frac{i-1}{2}-k+l' = 1$ any $k \in \{1,2,\ldots,i\}$. Analogously we cancel of the -1's in the numerators, keeping in mind that $|l'| < \frac{i-1}{2}$ (since we study the exponents corresponding to μ_t). □

Now we have that (so $|S|$ is even)

$$\begin{aligned} &\lim_{s\to 0} (f_s)_0 + r(\lambda_{\phi',s,a_{t,j}}, w_0)^{-1} \otimes_{v\notin S} f_{w_0(s),v} \otimes_{v\in S} N(\mu_{t,GL(a_{t,j},k_v)} \\ &\quad \otimes\sigma_v(\phi'), w_0) f_{s,v} = 2(f_0)_0 \neq 0, \end{aligned}$$

since, $|S|$ is even and for $v \in S$ we have $N(\mu_{t,GL(a_{t,j},k_v)} \otimes \sigma_v(\phi'), w_0) f_{0,v} = -f_{0,v}$, so for such choice of π we have that intertwining (9) is non-trivial on π (since after calculation of the constant term we still get a non-zero contribution) and the **Claim** is proved.

4.2 Step 2

Proposition 4.3 *Assume that π is an irreducible subrepresentation of (8) such that the intertwining (9) does not send it to zero (e.g., one for which we take $|S|$ to be even). Let $E(\pi)$ be the image of π under this intertwining (so that $E(\pi) \subset A(Sp_{2n'+2a_{t,j}}(k))\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A}))$). Let $E_0(\pi)$ be a constant term along $B_{n'+a_{t,j}}$. Now we construct degenerate Eisenstein series $f_s \mapsto E(f_s, g) = \sum_{\gamma \in P(k)\backslash Sp_{2n}(k)} f_s(\gamma g)$ attached to the global induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\mu_t|det|^s \otimes E_0(\pi))$, where P is the standard parabolic subgroup with Levi factor isomorphic to $GL(\frac{a_{t,j+1}-a_{t,j}}{2}) \times Sp_{2n'+2a_{t,j}}$. Then, map*

$$\mathrm{Ind}_{P(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\mu_t|det|^{\frac{a_{t,j}+a_{t,j+1}}{4}} \otimes E_0(\pi)) \to A(Sp_{2n}(k)\backslash Sp_{2n}(\mathbb{A}))$$

given by

$$f_{\frac{a_{t,j}+a_{t,j+1}}{4}} \to \left(s - \frac{a_{t,j}+a_{t,j+1}}{4}\right)^2 E(f_s, \cdot)|_{s=\frac{a_{t,j}+a_{t,j+1}}{4}}$$

is well-defined and non-trivial, and its image $E(\pi, \mu_t, a_{t,j+1})$ is contained in the space of square-integrable automorphic forms. Every irreducible subrepresentation of $E(\pi, \mu_t, a_{t,j+1})$ (which is semi-simple representation) is thus automorphic square-integrable, and has a global Arthur parameter equal to ϕ.

Proof We denote

$$\lambda_s = \mu_t|\cdot|^{s-\frac{\frac{a_{t,j+1}-a_{t,j}}{2}-1}{2}} \otimes \cdots \otimes \mu_t|\cdot|^{s+\frac{\frac{a_{t,j+1}-a_{t,j}}{2}-1}{2}} \otimes \lambda_{\phi',0,a_{t,j}},$$

and α denote a simple root such that the standard parabolic subgroup P of Sp_{2n} from the statement of Proposition corresponds to set of roots $\Delta\backslash\{\alpha\}$. Let S' be a finite set of places containing the archimedean ones. Let $f_s = \otimes_v (f_s)_v$ be a factorizable function in the space of $\mathrm{Ind}_{P(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\mu_t|det|^s \otimes E_0(\pi))$ such that for $v \notin S'$, the function f_s is K_v-invariant. We calculate the constant term of the degenerate Eisenstein series $f_s \mapsto E(f_s, \cdot)$; similarly as in the first step of the construction, but now, since we immediately realized π in the space of it's constant terms (cf. [17, section 2]) and embed $\mu_t|det|^s \hookrightarrow \mu_t|\cdot|^{s-\frac{\frac{a_{t,j+1}-a_{t,j}}{2}-1}{2}} \otimes \cdots \otimes \mu_t|\cdot|^{s+\frac{\frac{a_{t,j+1}-a_{t,j}}{2}-1}{2}}$ (also by taking an appropriate constant term), we get

$$\begin{aligned} E_0(s, f_s)(g) &= \sum_{w \in W_n, w(\Delta\backslash\{\alpha\})>0} M(\lambda_s, w) f_s \\ &= M(\lambda_s, w_0) f_s + \sum_{w \in W_n, w(\Delta\backslash\{\alpha\})>0, w \neq w_0} M(\lambda_s, w) f_s \\ &= r(\lambda_s, w_0)^{-1} (\otimes_{v \notin S'} f_{w_0(\lambda_s, v)}) \otimes_{v \in S'} N(s, \mu_{t,v} \otimes (E_0(\pi))_v, w_0) f_{s,v}) \end{aligned}$$

$$+ \sum_{w \in W_n, w(\Delta\backslash\{\alpha\})>0, w \neq w_0} r(\lambda_s, w)^{-1} (\otimes_{v \notin S'} f_{w(\lambda_s, v)})$$
$$\otimes_{v \in S'} N(s, \mu_{t,v} \otimes E_0(\pi))_v, w) f_{s,v}).$$

Here we use the analog of the normalization for the operators $N(s, \mu_{t,v} \otimes (E_0(\pi))_v, w_0)$ described in the previous proposition, and use the fact that (un)normalized operators $M(\lambda_s, w)$ and $M(s, \mu_t \otimes E_0(\pi))$ (and, consequently, $N(\lambda_s, w)$ and $N(s, \mu_t \otimes (E_0(\pi)), w))$ agree on $\mathrm{Ind}_{P(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\mu_t|det|^s \otimes E_0(\pi))$ which is, by taking a constant term, realized as a subspace of $\mathrm{Ind}_{B_n(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\lambda_s)$ (cf. section 3 of [18]).

We are interested in the situation for $s = s_0 = \frac{a_{t,j}+a_{t,j+1}}{4}$. We have an important lemma.

Lemma 4.4 *In the above situation, the order of the pole at* $s = \frac{a_{t,j}+a_{t,j+1}}{4}$ *of* $r(\lambda_s, w_0)^{-1}$ *is two, and of* $r(\lambda_s, w)^{-1}$ *for* $w \neq w_0$ *is at most one. Note that either* $a_{t,j} > 1$ *or* $\mu_t \neq 1$.

Proof We first calculate the order of pole for $s = s_0$ of $r(\lambda_s, w_0)^{-1}$. Again,we use the similar notation as in the proof of Lemma 4.2. Let now $i = \frac{a_{t,j+1}-a_{t,j}}{2}$. We return to the contributions (15)–(17). We now want to examine possible poles and zeros of these expressions for $s_0 = \frac{a_{t,j}+a_{t,j+1}}{4}$. It is easy to see that the enumerator of the expression (15) does not have any poles or zeros, and the denominator has no zeros and a simple pole if $a_{t,j} = 1$ and $\mu_t = 1$, but this is impossible here. As for the expression (16), we see that, since both numerator and denominator express values for L-function $L(z, 1)$ for (real) $z > 2$ there are no zeros or poles. In (17) there are no poles in the numerator if $\chi_l = \mu_t$, and there is a pole of second order in the denominator (if $\chi_l = \mu_t$, and that happens exactly for the factors with $l' = \nu^{-\frac{a_{t,j}-1}{2}}$ and $l' = \nu^{\frac{a_{t,j}-1}{2}}$, and these factors do appear in $\lambda_{\phi',0,a_{t,j}}$, as we have seen in the previous step of the construction).

If l is such that $\chi_l \neq \mu_t$, the corresponding L-functions are entire, but there are also no zeros appearing since $L(*, \mu_t \chi_l)$ is evaluated at the rational integers. This means that $r(\lambda_s, w_0)$ has a zero of order 2 at $s = s_0$ if $a_{t,j} \neq 1$ and of order 3 if $a_{t,j} = 1$ and $\mu_t = 1$ and the claim about poles of $r(\lambda_s, w_0)^{-1}$ follows.

Now we deal with $w \neq w_0$. Let $w = p\epsilon \in W_j$ (what this means is described in the proof of Lemma 4.1). We need to describe all $\alpha \in \Sigma^+$ such that $w(\alpha) < 0$ to calculate the normalizing factor $r(\lambda_s, w)$. The set of these α's consists of six sets

1. $S_1 = \{2e_k : j+1 \leq k \leq i\}$,
2. $S_2 = \{e_k + e_l : j+1 \leq k < l \leq i\}$,
3. $S_3 = \{e_k - e_l : j+1 \leq k \leq i,\ i+1 \leq l \leq n\}$,
4. $S_4 = \{e_k + e_l : 1 \leq k \leq j, j+1 \leq l \leq i,\ p(l) < p(k)\}$,
5. $S_5 = \{e_k - e_l : 1 \leq k \leq j, i+1 \leq l \leq n,\ p(l) < p(k)\}$,
6. $S_6 = \{e_k + e_l : j+1 \leq k \leq i, i+1 \leq l \leq n, p(l) > p(k)\}$.

We thus have

$$r(\lambda_s, w) = \prod_{t=1}^{6} \prod_{\alpha \in S_t} \frac{L(1, \lambda_s \circ \check{\alpha})\varepsilon(0, \lambda_s \circ \check{\alpha})}{L(0, \lambda_s \circ \check{\alpha})}.$$

Now, reasoning analogously as in the case of w_0, we see that the contribution (in L-functions) to $r(\lambda_{s_0}, w)$ is holomorphic (and non-zero) for $\alpha \in S_1 \cup S_2 \cup S_4$. There is a pole of the first order of the denominator for $\alpha \in S_3$ if $k = 1$, so that $j = 0$, and for $\alpha \in S_5$ if $k = 1$ so that $j \geq 1$. Analogously, $\alpha \in S_6$ contributes with the pole of the first order in the denominator if $k = 1$, i.e., $j = 0$. (Again, we have a separate case if $a_{t,j} = 1$ and $\mu_t = 1$, in that case, $\alpha \in S_1$ also contributes if $k = 1$, so $j = 0$). Assume now that $a_{t,j} > 1$ or $\mu_t \neq 1$. Then if $r(\lambda_s, w)^{-1}$ has a pole of second order for $s = s_0$, we have $j = 0$, and $p(i + 1) > p(1)$ (this comes from the condition on $\alpha \in S_6$). Now, from the scheme in the proof of Lemma 4.1, we see that $w = w_0$. □

We denote $r(w_0) = \lim_{s \to s_0}(s - s_0)^2 r(\lambda_s, w_0)^{-1} \neq 0$. Lemma 4.4 insures that

$$\begin{aligned} &\lim_{s \to s_0}(s - \frac{a_{t,j} + a_{t,j+1}}{4})^2 E_0(s, f_s)(g) \\ &\quad = r(w_0)(\otimes_{v \notin S'} f_{w_0(\lambda_{s_0}, v)}) \otimes_{v \in S'} N(s_0, \mu_{t,v} \otimes (E_0(\pi))_v, w_0) f_{s_0,v}). \end{aligned}$$

Indeed, the operator $N(s_0, \mu_{t,v} \otimes (E_0(\pi))_v, w)$ is holomorphic at $s = s_0 = \frac{a_{t,j}+a_{t,j+1}}{4}$ for all $w \in W_n$, $w(\Delta \backslash \{\alpha\}) > 0$ (Lemma 3.5(i) of [17]) so we get the limit above (we just calculate $r(\lambda_s, w)^{-1}$ as $s \to s_0$). Also, $M(s_0, \mu_{t,v} \otimes (E_0(\pi))_v, w_0)$ is nonzero for $s = s_0$, so we get that $\lim_{s \to s_0}(s - \frac{a_{t,j}+a_{t,j+1}}{4})^2 E_0(s, f_s)(g)$ is non-zero, specifically $\lim_{s \to s_0}(s - \frac{a_{t,j}+a_{t,j+1}}{4})^2 E(f_s, g) = E(\pi, \mu_t, a_{t,j+1})$ is non-zero, and $E(\pi, \mu_t, a_{t,j+1})$ belongs to the space of the square-integrable automorphic forms on $Sp_{2n}(k) \backslash Sp_{2n}(\mathbb{A})$ (as we can see from the calculation of it's constant term). We see that the constant term of $E(\pi, \mu_t, a_{t,j+1})$ (which is isomorphic to this representation) is generated, for all $v \notin S'$, by the spherical vector $f_{w_0(\lambda_{s_0}, v)}$ inside $\mathrm{Ind}_{P(k_v)}^{Sp_{2n}(k_v)}(\mu_{t,v}|det|_v^s \otimes E_0(\pi)_v) \hookrightarrow \mathrm{Ind}_{B_n(k_v)}^{Sp_{2n}(k_v)}(w_0(\lambda_{s_0,v}))$.

Note that $w_0(\lambda_{s_0,v}) = \lambda(\phi)_v$, the (local component) of a character of the maximal torus, attached to the Arthur parameter ϕ.

Lemma 4.5 *For all $v \notin S'$ the subspace in $\mathrm{Ind}_{B_n(k_v)}^{Sp_{2n}(k_v)}(w_0(\lambda_{s_0,v}))$ generated by the spherical vector is an irreducible (spherical) subrepresentation σ_v of $\mathrm{Ind}_{B_n(k_v)}^{Sp_{2n}(k_v)}(w_0(\lambda_{s_0,v}))$.*

This lemma follows from Proposition 1.2 and it guarantees that every irreducible subrepresentation of $E(\pi, \mu_t, a_{t,j+1})$ is attached to the Arthur parameter ϕ. □

5 Explicit construction II: representations with general support

To be able to execute the same steps for general case of automorphic unipotent representations (as we have done in the previous section in the case of such representation with some constant term non-vanishing along the Borel subgroup) we need some additional results. One of these results actually states that any automorphic representation, occurring as a subrepresentation in the space of automorphic forms (so especially, a representation occurring in the discrete part of square-integrable spectrum) can be embedded (as an abstract representation) in the representation induced from a (essentially) cuspidal representation of a Levi subgroup. This result is proved in [10]. For our purposes, we need an explicit realization of the space of the induced representation, and we actually have that same realization in [10]. This result is straightforward application of Langlands' method in his Corvallis lecture combined with various decompositions of the spaces of automorphic forms which can be found in the second and the fourth chapter of [15], and we briefly explain it.

Assume that we have an irreducible representation realized in the space of automorphic forms $Sp_{2n'}(k)\backslash Sp_{2n'}(\mathbb{A})$ (in this setting, it is important to be aware of the explicit realizations of the global representations in certain spaces of automorphic forms). Assume that this representation, say (Π, V), is concentrated on the (standard) parabolic subgroup P_{θ_0}, (parabolic subgroup which corresponds to the subset θ_0 of the set of simple roots of $Sp_{2n'}$ with respect to the maximal diagonal torus (and upper triangular Borel subgroup)), and assume that θ_0 is minimal with this property. This means that the constant term of (Π, V) along P_{θ_0} does not vanish. We show in [10] that there is a character ξ of $Z_{M_{\theta_0}}(\mathbb{A})$ such that (maybe after left translation by an element of $Z_{M_{\theta_0}}(\mathbb{A})$), the space of constant terms of (Π, V), denoted $V_0 \neq 0$, belongs to the space of automorphic forms $A_0(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'}(\mathbb{A}))_\xi$, (for the notation and the definitions of these spaces of automorphic forms we refer to [15], I.2.17, and for the constant term I.2.6). Then we show that there is an automorphic representation π_0 of $M_{\theta_0}(\mathbb{A})$ (cuspidal, because of the minimality of the set θ_0) such that V_0 belongs to $A(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'}(\mathbb{A}))_{\pi_0}$ [15, II.1.1.]. We recall that $A(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'}(\mathbb{A}))_{\pi_0}$ is defined as

$$A(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'}(\mathbb{A}))_{\pi_0} = \{\phi \in A(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'}(\mathbb{A})) : \\ \forall k \in K, \phi_k \in A(M_{\theta_0}(k)\backslash M_{\theta_0}(\mathbb{A}))_{\pi_0}\}$$

where $\phi_k(m) = \delta_{P_{\theta_0}}^{-\frac{1}{2}}\phi(mk)$; this space can canonically be identified with

$$\mathrm{ind}_{M_{\theta_0}(\mathbb{A})\cap K}^{K} A(M_{\theta_0}(k)\backslash M_{\theta_0}(\mathbb{A}))_{\pi_0},$$

hence the statement about embedding in the induced representation (as abstract representations).

So, to conclude, the following holds [10].

Theorem 5.1 *Let (Π, V) be a $((\mathfrak{g}_\infty, K_\infty) \times \prod_{v<\infty} Sp_{2n'}(k_v))$-irreducible subspace of the space of automorphic forms inside $A(Sp_{2n'}(k)\backslash Sp_{2n'}(\mathbb{A}))$ such that some constant term of the functions from V does not vanish along a parabolic subgroup P_{θ_0} of $Sp_{2n'}$; assume that θ_0 is minimal (set of simple roots) with this property. Then, there exists an irreducible automorphic representation π_0 of $M_{\theta_0}(\mathbb{A})$ (appearing in $A_0(M_{\theta_0}(k)\backslash M(\mathbb{A}))$ such that the space of constant terms of V along P_{θ_0}, denoted by V_0, belongs (up to a left translation by an element from $Z_{M_{\theta_0}}$)) to the space $A_0(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'}(\mathbb{A}))_{\pi_0}$ of cuspidal automorphic forms.*

Remark The embedding of the representation Π inside the space $A_0(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'}(\mathbb{A}))_{\pi_0}$ given in this theorem is referred to as "given by the constant term."

We show (Theorem 5.3) that the constant term of certain (degenerate Eisenstein series) related to the representation π as above decomposes as a sum of global intertwining operators, similarly to the case of (ordinary) Eisenstein series.

Calculation of this sort were executed also in [18], but only in the case $\theta_0 = \emptyset$, i.e., when the representation Π is supported on the Borel subgroup. To treat this (more general) case, we need some additional results from [15].

So, recall that in the first step of our inductive construction we use (a degenerate) Eisenstein series as an intertwining operator from the space of automorphic forms related to the space from the previous theorem, equivalent to a certain induced representation, to the space of automorphic forms on $Sp_{2n''}(k)\backslash Sp_{2n''}(\mathbb{A})$, for certain n'' to be defined later. The local components of this induced (unitary) representation will chosen so that the resulting representation has a non-zero Eisenstein series; this would be guaranteed by the fact that the constant term of this Eisenstein series (along the parabolic P_{θ_0}) is non-zero.

5.1 Step 1

Let ϕ' be the Arthur parameter build up in the previous step of our inductive procedure, and let $\sigma = \otimes_v \sigma_v$ be an irreducible automorphic representation of $Sp_{2n'}(\mathbb{A})$ with the Arthur parameter ϕ' realized in the space of the automorphic forms $A(Sp_{2n'}(k)\backslash Sp_{2n'}(\mathbb{A}))$. We assume that the basis of our inductive procedure is Arthur parameter ϕ_0, and that we have an automorphic representation σ_0 of $Sp_{2n_0}(\mathbb{A})$ with that Arthur parameter, and that σ_0 is concentrated on the parabolic subgroup P_{θ_0}. Now, as we add new elements in the Jordan block, and the rank of the symplectic group in question raises, we always make this convention about the notation of a subset θ_0 of the set of simple roots: let say that θ_0 is a subset of the set of simple roots of Sp_{2n_0} such that $M_{\theta_0} \cong GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_k} \times Sp_{2m_1}$ (of course, depending on θ_0, the last Sp_{m_1} does not have to be there). Now, we enlarge the rank of the symplectic group; say that we have $Sp_{2n'}$ now. Now, the simple roots in $Sp_{2n'}$ are rearranged in a such away that M'_{θ_0} (Levi subgroup in $Sp_{2n'}$ attached to θ_0) is isomorphic to $GL_1 \times \cdots \times GL_1 \times GL_{n_1} \times \cdots \times GL_{n_k} \times Sp_{m_1}$, where the number of GL_1's at the beginning is exactly $n' - n_0$. With this convention, we have the following lemma and theorem.

We need next observation about the action of the elements of the Weyl group W of Sp_{2n}.

Lemma 5.2 *Let $\alpha_i = e_i - e_{i+1}$ be a simple root with $i \leq n - n_0$ and $\emptyset \neq \theta_0 \subset \Delta(Sp_{2n_0})$. Let $w \in W$ be such that $w(\Delta\backslash\{\alpha_i\}) > 0$, and $w(\theta_0) = \theta_0$. Then, w acts as the identity on θ_0, moreover it acts as a identity on every simple root of $\Delta(Sp_{2n_0})$ "between" the simple roots in θ_0. As a consequence, if $e_{n-n_0+1} - e_{n-n_0+2} \in \theta_0$ (the first simple root of Sp_{2n_0} in the standard ordering with our convention above about the numeration of roots), then w acts as identity on $\Delta(Sp_{2n_0})$.*

Proof We use the description of those elements of Weyl group for which $w(\Delta\backslash\{\alpha_i\}) > 0$ given in Lemma 4.1. Since $w = p$ (in the notation of this Lemma) is increasing on $\{e_{n-n_0+1}, \ldots, e_n\}$ and $w(\theta_0) = \theta_0$, it is obvious that w acts on θ_0 as an identity, and also it then must act as a identity on the simple roots in $\Delta(Sp_{2n_0})$ between the roots in θ_0. □

For this theorem we also define $i = a_{t,j}$; W denotes the Weyl group of $Sp_{2n'+2a_{t,j}}$ attached to our standard choices of maximal torus and Borel subgroup. The standard parabolic subgroups (and their unipotent radicals) of the group $Sp_{2n'}$ are denoted by prime. Calculation in the next theorem is essentially the one from [15, II.1.7], but we had to adapt it, since the inducing data in our case is not cuspidal so we had to use a bit different realizations of global representations.

Theorem 5.3 *Let $\sigma(\phi')$ be an automorphic representation of $Sp_{2n'}(\mathbb{A})$ and ϕ' Arthur parameter of $\sigma(\phi')$, as described above. We assume that $\sigma(\phi')$ is concentrated on P'_{θ_0}, and let π'_0 be an automorphic cuspidal representation of $M_{\theta_0}(\mathbb{A})'$ such that $\sigma(\phi')$ is realized as a subspace in $A(N'_{\theta_0}(\mathbb{A})M'_{\theta_0}(k)\backslash Sp_{2n'}(\mathbb{A}))_{\pi'_0}$ (cf. Theorem 5.1). We consider the global induced representation*

$$\pi_s = \mathrm{Ind}_{P_{\Delta\backslash\{\alpha_i\}}(\mathbb{A})}^{Sp_{2n'+2a_{t,j}}(\mathbb{A})}(|\det|^s \mu_{t,GL(a_{t,j},\mathbb{A})} \otimes \sigma(\phi')). \tag{19}$$

realized in the space $A(U_{\Delta\backslash\alpha_i}(\mathbb{A})M_{\Delta\backslash\alpha_i}(k)\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A}))_{|\det|^s\mu_{t,GL(a_{t,j},\mathbb{A})}\otimes\sigma(\phi')}$ (keeping in mind our realization of $\sigma(\phi')$). For f_s from this space, we define Eisenstein series

$$E(s, f_s)(g) = \sum_{\gamma\in P_{\Delta\backslash\{\alpha_i\}}(k)\backslash Sp_{2n'+2a_{t,j}}(k)} f_s(\gamma g),$$

cf. [15, II.1.5]. *The constant term of $E(s, f_s)$ along P_{θ_0} (a standard parabolic of $Sp_{2n'+2a_{t,j}}(\mathbb{A})$) is given by*

$$E_{P_{\theta_0}}(s, f_s)(g) = \sum_{w\in W, w(\Delta\backslash\alpha_i)>0, w(\theta_0)=\theta_0} M(w, \pi, s)(f_s)(g)$$

where

$$M(w,\pi,s) : A(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A}))_{\mu_t\nu^{s-\frac{i-1}{2}}\otimes\cdots\otimes\mu_t\nu^{s+\frac{i-1}{2}}\otimes\pi'_0}$$
$$\to A(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A}))_{w(\mu_t\nu^{s-\frac{i-1}{2}}\otimes\cdots\otimes\mu_t\nu^{s+\frac{i-1}{2}}\otimes\pi'_0)}$$

is an intertwining operator.

Proof Let $f_s \in A(U_{\Delta\backslash\alpha_i}(\mathbb{A})M_{\Delta\backslash\alpha_i}(k)\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A}))_{|\det|^s\mu_{t,GL(a_{t,j},\mathbb{A})}\otimes\sigma(\phi')}$. We have the following disjoint decomposition [7] (since $Sp_{2n'+2a_{t,j}}$ is split over k, this decomposition works over k and over $\mathbb{A}$):

$$Sp_{2n'+2a_{t,j}} = \bigcup_{\substack{w\in W, w^{-1}(\theta_0)>0,\\ w(\Delta\backslash\{\alpha_i\})>0}} P_{\Delta\backslash\{\alpha_i\}} w^{-1} P_{\theta_0},$$

where P_* and N_* denote a standard parabolic subgroup and unipotent radical of a standard parabolic subgroup attached to a subset $*$ of the set of simple roots. For such $w \in W$, easily follows that

$$P_{\Delta\backslash\{\alpha_i\}}w^{-1}P_{\theta_0} = P_{\Delta\backslash\{\alpha_i\}}w^{-1}(N_{\theta_0}\cap w\overline{N_{\emptyset}}w^{-1})(M_{\theta_0}\cap w\overline{N_{\Delta\backslash\{\alpha_i\}}}w^{-1}),$$

where

$$\begin{aligned}&P_{\Delta\backslash\{\alpha_i\}}\times(N_{\theta_0}\cap w\overline{N_{\emptyset}}w^{-1})\times(M_{\theta_0}\cap w\overline{N_{\Delta\backslash\{\alpha_i\}}}w^{-1})\\&\quad\to P_{\Delta\backslash\{\alpha_i\}}w^{-1}(N_{\theta_0}\cap w\overline{N_{\emptyset}}w^{-1})(M_{\theta_0}\cap w\overline{N_{\Delta\backslash\{\alpha_i\}}}w^{-1})\end{aligned}$$

given by $(x, y, z) \mapsto xw^{-1}yz$ is an isomorphism of varieties (cf, e.g., [5, 14.12]). Now we calculate the constant term of the Eisenstein series in question:

$$\begin{aligned}E_{P_{\theta_0}}(s, f_s)(g) &= \int_{N_{\theta_0}(k)\backslash N_{\theta_0}(\mathbb{A})}\sum_{\gamma\in P_{\Delta\backslash\{\alpha_i\}}(k)\backslash Sp_{2n'+2a_{t,j}}(k)} f_s(\gamma ug)\mathrm{d}u\\&= \int_{N_{\theta_0}(k)\backslash N_{\theta_0}(\mathbb{A})}\left(\sum_{\substack{w\in W, w^{-1}(\theta_0)>0,\\ w(\Delta\backslash\{\alpha_i\})>0}}\ \sum_{\substack{\gamma\in(N_{\theta_0}\cap w\overline{N_{\emptyset}}w^{-1})(k)\\(M_{\theta_0}\cap w\overline{N_{\Delta\backslash\{\alpha_i\}}}w^{-1})(k)}} f_s(w^{-1}\gamma ug)\right)\mathrm{d}u.\end{aligned}$$

We can exchange the sum and the integral because of the convergence properties of the series, $N_{\theta_0}(k)\backslash N_{\theta_0(\mathbb{A})}$ being compact [15, II.1.5.]. Since $N_{\theta_0}(k) = (N_{\theta_0}(k)\cap wN_{\emptyset}w^{-1})(N_{\theta_0}(k)\cap w\overline{N_{\emptyset}}w^{-1})$, we can write down the last line as

$$\begin{aligned}&\sum_{\substack{w\in W, w^{-1}(\theta_0)>0,\\ w(\Delta\backslash\{\alpha_i\})>0}}\ \sum_{m\in(M_{\theta_0}\cap w\overline{N_{\Delta\backslash\{\alpha_i\}}}w^{-1})(k)}\int_{N_{\theta_0}(k)\cap wN_{\emptyset}w^{-1}(k)\backslash N_{\theta_0}(\mathbb{A})} f_s(w^{-1}umg)\mathrm{d}u\\&= \sum_{\substack{w\in W, w^{-1}(\theta_0)>0,\\ w(\Delta\backslash\{\alpha_i\})>0}}\ \sum_{m\in(M_{\theta_0}\cap w\overline{N_{\Delta\backslash\{\alpha_i\}}}w^{-1})(k)}\int_{N_{\theta_0}(k)\cap wN_{\emptyset}w^{-1}(k)\backslash N_{\theta_0}(\mathbb{A})\cap wN_{\emptyset}w^{-1}(\mathbb{A})}\\&\quad\int_{N_{\theta_0}(\mathbb{A})\cap wN_{\emptyset}w^{-1}(\mathbb{A})\backslash N_{\theta_0}(\mathbb{A})} f_s(w^{-1}u_1umg)\mathrm{d}u_1\mathrm{d}u.\end{aligned}$$

By the change of the variable $u_1 \mapsto w^{-1}u_1w$ the first integral is transformed into integral over $(w^{-1}N_{\theta_0}w\cap N_{\emptyset})(k)\backslash(w^{-1}N_{\theta_0}w\cap N_{\emptyset})(\mathbb{A})$, and then we can interchange the order of integration. We then decompose the set of integration of the (now) second integral:

$$w^{-1}N_{\theta_0}w\cap N_{\emptyset} = (w^{-1}N_{\theta_0}w\cap M_{\Delta\backslash\{\alpha_i\}})(w^{-1}N_{\theta_0}w\cap N_{\Delta\backslash\{\alpha_i\}}).$$

Since $f_s \in A(N_{\Delta\setminus\{\alpha_i\}}(\mathbb{A})M_{\Delta\setminus\{\alpha_i\}}(k)\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A}))$, the integration over $(w^{-1}N_{\theta_0}w \cap N_{\Delta\setminus\{\alpha_i\}})$ is irrelevant, so that we have this expression for $E_{P_{\theta_0}}(f_s, g)$

$$\sum_{\substack{w\in W, w^{-1}(\theta_0)>0,\\ w(\Delta\setminus\{\alpha_i\})>0}} \sum_{m\in(M_{\theta_0}\cap w\overline{N_{\Delta\setminus\{\alpha_i\}}}w^{-1})(k)} \int\int f_s(u_2 w^{-1}umg)\mathrm{d}u_2\mathrm{d}u \tag{20}$$

where the first integral is over $N_{\theta_0}(\mathbb{A}) \cap wN_{\emptyset}w^{-1}(\mathbb{A})\backslash N_{\theta_0}(\mathbb{A})$ and the second over $w^{-1}N_{\theta_0}w \cap M_{\Delta\setminus\{\alpha_i\}}(k)\backslash w^{-1}N_{\theta_0}w \cap M_{\Delta\setminus\{\alpha_i\}}(\mathbb{A})$. Now, since $M_{\Delta\setminus\{\alpha_i\}}(\mathbb{A}) \cong GL_i(\mathbb{A}) \times Sp_{2n'}(\mathbb{A})$, we can see that for $u_2 \in GL_i \cap w^{-1}N_{\theta_0}w$ we have $f_s(u_2*) = f_s(*)$, because of the domain of the definition of f_s. This means that in the expression for $E_{P_{\theta_0}}(s, f_s)(g)$ (20) the second integral is over $(w^{-1}N_{\theta_0}w\cap Sp_{2n'})(k)\backslash(w^{-1}N_{\theta_0}w\cap Sp_{2n'})(\mathbb{A})$. Of course, we constantly use our normalization of Haar measures on the unipotent radicals.

It is easy to see that $w^{-1}N_{\theta_0}w \cap Sp_{2n'} = w^{-1}N_{\theta_0}w \cap N'_{\emptyset}$ is a unipotent radical of standard parabolic subgroup of $Sp_{2n'}$. It is not difficult to show that $(N'_{\emptyset} \cap w^{-1}N_{\theta_0}w)N'_{\theta_0}$ is a unipotent subgroup which is the unipotent radical of a standard parabolic subgroup of $Sp_{2n'}$ attached to the set of roots $w^{-1}\theta_0 \cap \theta_0 \subset \Delta'$. Because we normalized Haar measures on the unipotent radicals as we did, and in our realization f_s is left invariant under $N'_{\theta_0}(\mathbb{A})$, the inner integral in (20) can be taken to be over $(N'_{\emptyset} \cap w^{-1}N_{\theta_0}w)N'_{\theta_0}$. Now, if $w^{-1}\theta_0 \cap \theta_0 \subsetneq \theta_0$ because of the cuspidality of π_0, this integral is 0. So, in order for this integral to be non-zero, we have $w(\theta_0) = \theta_0$. This forces $M_{\theta_0} \cap w\overline{N_{\Delta\setminus\{\alpha_i\}}}w^{-1} = \{e\}$, and (20) becomes

$$\sum_{\substack{w\in W, w(\theta_0)=\theta_0,\\ w(\Delta\setminus\{\alpha_i\})>0}} \int_{N_{\theta_0}(\mathbb{A})\cap wN_{\theta_0}w^{-1}(\mathbb{A})\backslash N_{\theta_0}(\mathbb{A})} f_s(w^{-1}ug)\mathrm{d}u.$$

In the previous expression we recognize the intertwining operator

$$\begin{aligned} M(w, \pi, s) : A(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A}))_{\mu_t\nu^{s-\frac{i-1}{2}}\otimes\cdots\otimes\mu_t\nu^{s+\frac{i-1}{2}}\otimes\pi'_0} \\ \to A(U_{\theta_0}(\mathbb{A})M_{\theta_0}(k)\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A}))_{w(\mu_t\nu^{s-\frac{i-1}{2}}\otimes\cdots\otimes\mu_t\nu^{s+\frac{i-1}{2}}\otimes\pi'_0)}, \end{aligned}$$

as claimed. For the convergence issues and the meromorphic continuation of the (constant term of) the Eisenstein series and intertwining operators, we refer to [15], the second and the fourth chapter. By a result of Langlands (e.g., [2, Theorem 7.2]), this meromorphic continuation of the Eisenstein series is holomorphic for $s = 0$ (the realization of $\sigma(\phi')$ in our case is a bit different than in the cited theorem, but since this realization is given by taking the constant term, the holomorphy argument is the same). □

We continue with the assumptions from the previous theorem.

Proposition 5.4 *For $w \in W$, $w(\theta_0) = \theta_0$, $w(\Delta \backslash \{\alpha_i\}) > 0$*

$$w(\mu_t \nu^{s-\frac{i-1}{2}} \otimes \cdots \otimes \mu_t \nu^{s+\frac{i-1}{2}} \otimes \pi_0') = \mu_t \nu^{-s-\frac{i-1}{2}} \otimes \cdots \otimes \mu_t \nu^{-s+\frac{i-1}{2}} \otimes \pi_0'$$

for $s = 0$ if and only if $w = 1$ or $w = w_0$ (the longest element in W modulo the longest one in M_{θ_0}).

Proof Assume that

$$w(\mu_t \nu^{s-\frac{i-1}{2}} \otimes \cdots \otimes \mu_t \nu^{s+\frac{i-1}{2}} \otimes \pi_0') = \mu_t \nu^{-s-\frac{i-1}{2}} \otimes \cdots \otimes \mu_t \nu^{-s+\frac{i-1}{2}} \otimes \pi_0'.$$

We fix $s = 0$ and analyze how we can get factor $\mu_t \nu^{\frac{i-1}{2}}$. Our inductive procedure says that $\pi_0' \cong \lambda_1 \nu^{s_1} \otimes \cdots \otimes \lambda_k \nu^{s_k} \otimes \pi_0''$, where $\lambda_1 \nu^{s_1}, \ldots \lambda_k \nu^{s_k}$ are Grossencharacters obtained by adding elements in the Jordan blocks, and π_0'' is a cuspidal representation of the appropriate Levi subgroup of Sp_{2n_0}, associated to the representation $\sigma(\phi_0)$, as explained in Theorem 5.1. Now, similarly as in Lemma 4.1, we see that $\mu_t \nu^{\frac{i-1}{2}}$ (the ith) factor cannot be obtained from $\lambda_1 \nu^{s_1}, \ldots \lambda_k \nu^{s_k}$. Assume now that $w(\chi_j \nu^{s_j}) = \mu_t \nu^{\frac{i-1}{2}}$, where $\chi_j \nu^{s_j}$ is a factor of π_0'' ($s_j \in \mathbb{R}$). Now we have two cases. If $\theta_0 \subset \Delta(Sp_{2n_0})$ is attached to Levi subgroup which is a product of GL-factors (and maybe a symplectic group of smaller rank) such that there is a factor GL_s such that $s \geq 2$, then there is a set of roots θ_0', conjugated to θ_0, such that the factor GL_s is the first factor in the Levi subgroup attached to θ_0'. This means that $e_{n-n_0+1} - e_{n-n_0+2}$ belongs to θ_0'. But if we assume that $\sigma(\phi_0)$ (so also $\sigma(\phi')$) has a non-zero constant term along θ_0, it has a non-zero constant term along θ_0', so we can immediately make this assumption about the structure of θ_0. Now, we apply Lemma 5.2 to conclude that w acts as identity on $\Delta(Sp_{2n_0})$, so we cannot have $w(\chi_j \nu^{s_j}) = \mu_t \nu^{\frac{i-1}{2}}$. On the other hand, if θ_0 is attached to Levi subgroup isomorphic to a product of GL_1's and symplectic group of smaller rank, we have (according to Theorem 5.1)

$$\sigma(\phi_0) \hookrightarrow \mathrm{Ind}_{P_{\theta_0}(\mathbb{A})}^{Sp_{2n_0}(\mathbb{A})}(\pi_0'') = \mathrm{Ind}_{P_{\theta_0}(\mathbb{A})}^{Sp_{2n_0}(\mathbb{A})}(\chi_1 \nu^{s_1} \otimes \cdots \otimes \chi_{k_1} \nu^{s_k} \otimes \sigma_0), \tag{21}$$

where σ_0 is a cuspidal automorphic representation of some Sp_{2m_0} and $\chi_1, \ldots, \chi_{k_1}$ are Grossencharacters and $s_i \in \mathbb{R}$. Since $\sigma(\phi_0)$ is attached to an unipotent Arthur parameter ϕ_0, we conclude that

$$|s_r| \leq \max \left\{ \frac{a_{h,1} - 1}{2}; 1 \leq h \leq l \right\} < \frac{i-1}{2}, \quad \forall r = 1, \ldots, k,$$

by **Conditions** (Δ) 1. so we also cannot have $s_j = \frac{i-1}{2}$. This means, that, if $w = p\varepsilon$, then either $p(i) = i$ or $p(1) = i$. In the first case, this means $j = i$ and $w = 1$, in the second case $j = 0$ and $w = w_0$, as claimed. □

Using the previous proposition we prove that the (meromorphic continuation of) Eisenstein series $E(s, f_s)$, for f_s in a certain (irreducible) subspace of

$\pi_s = \mathrm{Ind}_{P_{\Delta\setminus\{\alpha_i\}}(\mathbb{A})}^{Sp_{2n'+2a_{t,j}}(\mathbb{A})}(|\det|^s \mu_{t,GL(a_{t,j},\mathbb{A})} \otimes \sigma(\phi'))$ (for $s = 0$) is non-zero, and as such, gives a realization of this representation in the space of automorphic forms $A(Sp_{2n'+2a_{t,j}}(k)\backslash Sp_{2n'+2a_{t,j}}(\mathbb{A}))$. Namely, using Theorem 5.1 we prove that the constant term of the image (of this irreducible subspace of π_s for $s = 0$) by Eisenstein series in non-zero.

Let $f_s = \otimes_v f_{s,v} \in \pi_s = \mathrm{Ind}_{P_{\Delta\setminus\{\alpha_i\}}(\mathbb{A})}^{Sp_{2n'+2a_{t,j}}(\mathbb{A})}(|\det|^s \mu_{t,GL(a_{t,j},\mathbb{A})} \otimes \sigma(\phi')) \cong \otimes_v \pi_{s,v}$ where for almost all v, $\pi_{s,v}$ is an unramified representation, and $f_{s,v}$ is K_v-invariant vector in $\pi_{s,v}$, normalized in such a way that $f_{s,v}(e_v) = 1$. By Proposition 5.4, f_s and $M(w_0, \pi, s) f_s$ belong to the same space for $s = 0$ (and by the same Proposition the other $M(w, \pi, s) f_s$ do not belong to the same space) so to prove that $E_{P_{\theta_0}}(s, f_s)(g)$ is non-zero, it is enough to prove $f_s + M(w_0, \pi, s) f_s$ is not zero. Let S be a finite set of places such that for $v \notin S$, $\pi_{s,v}$ is unramified. Then, analogously as in the third section, we normalize local intertwining operators. If $v \notin S$ then $\pi_{s,v}$ is a principal series representation, so we can use the normalization introduced in the third section; in this case

$$\pi_{s,v} \hookrightarrow \mathrm{Ind}(\mu_{t,v}\nu^{s-\frac{i-1}{2}} \otimes \cdots \otimes \mu_{t,v}\nu^{s+\frac{i-1}{2}} \otimes \lambda_{1,v}\nu^{s_1} \otimes \cdots \otimes \lambda_{k,v}\nu^{s_k} \rtimes \sigma(\phi_0)_v) \quad (22)$$

(we use the notation from the proof of Proposition 5.4). Now, $\sigma(\phi_0)_v$ is (negative) unramified representation of $Sp_{2n_0}(k_v)$ corresponding to the local unipotent Arthur parameter $\phi_{0,v}$. Although there does not exist (global) character $\lambda_{\phi',s,a_{t,j},w_0}$ such that $\sigma(\phi_0)_v$, or more precisely, $\pi_{s,v}$ is embedded in $\mathrm{Ind}(\lambda_{\phi',s,a_{t,j},w_0,v})$, we can still easily express the normalizing factor analogous to the normalizing factor $r(\lambda_{\phi',s,a_{t,j},w_0,v})$, as we did in the third section (where our representation π_s was a subrepresentation globally induced from character). Later, we check that these normalizations give normalized operators which still have good properties, even for $v \in S$ (where locally, we might not have subquotients of principal series representations).

Indeed, assume that $\pi_{s,v}$ is unramified representation, so that $\sigma(\phi_0)_v$ is also unramified and a subquotient of a principal series representation; let us say

$$\sigma(\phi_0)_v \hookrightarrow \chi_1\nu^{s'_1} \times \chi_2\nu^{s'_2} \times \cdots \times \chi_r\nu^{s'_r} \rtimes 1. \quad (23)$$

The (local) normalizing factor we use in this situation is (according to the third section)

$$r_v(s, w_0) = \prod_{\substack{\alpha\in\Sigma^+ \\ w_0(\alpha)<0}} \frac{L(1, \lambda_v(s) \circ \check{\alpha})\varepsilon(0, \lambda_v(s) \circ \check{\alpha}, \psi_v)}{L(0, \lambda_v(s) \circ \check{\alpha})}.$$

Here $\lambda_v(s)$ is obvious local character obtained from (22) and (23); we can take $\varepsilon(0, \lambda_v(s) \circ \check{\alpha}) = 1$, because we are in the unramified situation. The set of $\alpha \in \Sigma^+$, $w_0(\alpha) < 0$ is divided into four subsets (as in the proof of Lemma 4.2). The contribution (this time locally) from the first and the second set of roots is the same as in (15) and (16). As for the third and the fourth group of roots, we can divide them into two subsets-the first related to the (localization of) global characters $\lambda_1, \ldots, \lambda_k$ (in (22)) and the other subset-local characters $\chi_1, \ldots, \chi_r$ appearing in (23). While the first subset is obviously related to the global characters, we can also relate (global) Hecke L-functions with the second subset of roots; namely if exponents $s'_1, \dots, s'_r$ appear in (23), they appear in the local L function in the third group of roots, and $-s'_1, \dots, -s'_r$ appear in the fourth group of roots. This means, that, if we examine the third and the fourth group of roots together (and the corresponding local L-functions in the normalizations) it does not matter what exact kind of embedding (23) we have, i.e., only what matters is a cuspidal support of $\sigma(\phi_0)_v$. We can relate the cuspidal support of that (negative) unramified representation with the Jordan block. For simplicity of notation, assume that v is unramified place such that $\mu_{1,v} = \mu_{2,v} = \cdots = \mu_{r,v} = \chi_0$, and $\mu_{r+1,v} = \cdots = \mu_{l,v} = 1$ (here we use the notation for the basis of our inductive procedure given in the third section). Then

$$\sigma(\phi_0)_v \hookrightarrow \zeta\left(-\frac{a_{1,1}-1}{2}, \frac{a_{2,1}-1}{2}; \chi_0\right) \times \cdots \times \zeta\left(-\frac{a_{r-1,1}-1}{2}, \frac{a_{r,1}-1}{2}; \chi_0\right)$$
$$\times \zeta\left(-\frac{a_{r+1,1}-1}{2}, \frac{a_{r+2,1}-1}{2}; 1\right) \times \cdots \zeta\left(-\frac{a_{l,1}-1}{2}, -1; 1\right) \rtimes 1. \tag{24}$$

Then, part of the appropriate normalizing factor (attached to the third and fourth set of the roots) is

$$\prod_{1\le k\le i}\left(\prod_{f=1}^{l}\prod_{p=-\frac{a_{f,1}-1}{2}}^{\frac{a_{f,1}-1}{2}} \frac{L(-\frac{i-1}{2}+s+k-p, \mu_{t,v}\mu_{f,v})\varepsilon(-\frac{i-1}{2}+s+k-p-1, \mu_{t,v}\mu_{f,v})}{L(-\frac{i-1}{2}+s+k-p-1, \mu_{t,v}\mu_{f,v})}\right)$$
$$\frac{L(-\frac{i-1}{2}+s+k-1, \mu_{t,v})}{L(-\frac{i-1}{2}+s+k, \mu_{t,v})\varepsilon(-\frac{i-1}{2}+s+k-1, \mu_{t,v})}.$$

The second line above comes from the fact that in the first line we calculated $\nu^0 1_{GL_1}$ as a part of cuspidal support in (24), and it isn't, so we had to divide it by the factor in the second line. Note that this second line exactly cancels with the contribution from the first set of roots in the proof of Lemma 4.2. So, we can conclude

$$r_v(s, w_0) = \prod_{1\le k<l\le i} \frac{L(2s-i+k+l, 1_v)\varepsilon(2s-i+k+l-1, 1_v, \psi_v)}{L(2s-i+k+l-1, 1_v)}$$
$$\prod_{1\le k\le i}\left(\prod_{(\mu,a)\in \mathrm{Jord}(\phi')}\prod_{p=-\frac{a-1}{2}}^{\frac{a-1}{2}} \frac{L(-\frac{i-1}{2}+s+k-p, \mu_{t,v}\mu_v)\varepsilon(-\frac{i-1}{2}+s+k-p-1, \mu_{t,v}\mu_v)}{L(-\frac{i-1}{2}+s+k-p-1, \mu_{t,v}\mu_v)}\right). \tag{25}$$

This leads us to define

$$r(s,w_0)=\prod_v r_v(s,w_0)=\prod_{1\le k<l\le i}\frac{L(2s-i+k+l,1)\varepsilon(2s-i+k+l-1,1)}{L(2s-i+k+l-1,1)}$$
$$\prod_{1\le k\le i}\left(\prod_{(\mu,a)\in \mathrm{Jord}(\phi')}\prod_{p=-\frac{a-1}{2}}^{\frac{a-1}{2}}\frac{L(-\frac{i-1}{2}+s+k-p,\mu_t\mu)\varepsilon(-\frac{i-1}{2}+s+k-p-1,\mu_t\mu)}{L(-\frac{i-1}{2}+s+k-p-1,\mu_t\mu)}\right).$$

although this normalization is justified (for now) only for $v\notin S$.

Continuing after Theorem 5.3 and Proposition 5.4, and the discussion after it, we conclude the following (for $f_s=\otimes f_{s,v}\in\pi_s$):

$$\begin{aligned}M(w_0,\pi,s)f_s&=r(s,w_0)^{-1}(\otimes_{v\notin S}r_v(s,w_0)A(w_0,\pi_v,s)f_{v,s}\otimes_{v\in S}r_v(s,w_0)A(w_0,\pi_v,s)f_{v,s})\\&=r(s,w_0)^{-1}(\otimes_{v\notin S}N(w_0,\pi_v,s)f_{v,s}\otimes_{v\in S}N(w_0,\pi_v,s)f_{v,s}),\end{aligned}$$

where we have defined normalized operators as in the Sect. 4.1. (for $v\notin S$). So, for $v\notin S$ we can define subspaces $Y_v(\sigma(\phi'))^{\pm}$ as in Sect. 4.1. We now prove that for $v\in S$, the normalized operators also have the required properties.

Proposition 5.5 *With the notation as above,* $\lim_{s\to 0}r(s,w_0)^{-1}=1$. *For* $v\in S$ *the intertwining operator* $N(w_0,\pi_v,s)$ *is holomorphic for* $s=0$, *and the space* $\mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)}(\mu_{t,GL(a_{t,j},k_v)}\otimes\sigma(\phi')_v)$ *decomposes into two subspaces* $Y_v(\sigma(\phi'))^+$ *and* $Y_v(\sigma(\phi'))^-$, *where this operators acts as identity and minus identity, respectively.*

Proof Using the global functional equation for Hecke Grossencharacters and change of variables, similarly as in the proof of Lemma 4.2, we get that

$$\lim_{s\to 0}r(s,w_0)^{-1}=1.$$

Assume that $v\in S$ is a non-archimedean place. We briefly recall a certain involution on the set of (local) Arthur parameters for the classical groups (for the general definition of the Arthur parameters we refer to [1]). If ψ is some (local) Arthur parameter $\psi: W_{k_v}\times SL(2,\mathbb{C})\times SL(2,\mathbb{C})\to SO(2n+1,\mathbb{C})$, $\hat{\psi}$ is Arthur parameter obtained by interchanging the action of two $SL(2,\mathbb{C})$' s. The Arthur parameter is generic if it is trivial on the second copy of $SL(2,\mathbb{C})$ and it then corresponds to a tempered packet; in this paper we work with the unipotent parameter; i.e., trivial on the first $SL(2,\mathbb{C})$. By the results of Mœglin [13], which are partially global, depending on Arthur's Theorem 2.2.1 of [1], (cf. [1, section 7.1]), we know that when we apply involution on the generic Arthur parameter (to obtain a unipotent parameter), the associated representations in the unipotent Arthur packets are (Aubert–Schneider–Schuler) duals of the representations in the original generic Arthur packets. For an irreducible representation σ, we denote by $\hat{\sigma}$ its Aubert dual representation ($\hat{\sigma}$ is a genuine representation, so it is plus or minus Aubert dual representation as defined in [3]). Note that, in that case, in our notation, $\sigma(\hat{\phi}'_v)$ describes a tempered packet, and the Plancherel measure attached to the induced representation $\mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)}(|\det|_v^s\mu_{t,v}\mathrm{St}_{GL_{a_{t,j},k_v}}\otimes\sigma(\hat{\phi}')_v)$

can be calculated in terms of this (Arthur) parameter (i.e., Jordan block, cf. [14, Section 13]). Here $\mathrm{St}_{GL_{a_{t,j}},k_v}$ denotes the Steinberg representation of $GL_{a_{t,j}}(k_v)$, which is the Aubert dual of the trivial representation of $GL_{a_{t,j}}(k_v)$. The results of Ban [4, the proof of Lemma 7.1], cf. [1, section 7.1] (obtained without any restrictions) show that the Plancherel measure $\mu(s, \mu_{t,v}\mathrm{St}_{GL_{a_{t,j}},k_v} \otimes \sigma(\hat{\phi}')_v)$ attached to the representation $\mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)}(|\det|_v^s \mu_{t,v}\mathrm{St}_{GL_{a_{t,j}},k_v} \otimes \sigma(\hat{\phi}')_v)$ is the same as the Plancherel measure attached to the representation

$$\mathrm{Ind}_{P(k_v)}^{Sp_{2n'+2a_{t,j}}(k_v)}(|\det|_v^s \mu_{t,GL(a_{t,j},k_v)} \otimes \sigma(\phi')_v)$$

(here we have a particular representation $\sigma(\phi')_v$ belonging to the local Arthur packet, but the measure should be the same for any member of the packet). Because we are interested in the intertwining operator attached to the longest element w_0 of the Weyl group, we prove that (5-18) and (5-19) of [18] hold, and this is enough to prove the claim of the Proposition. By the above discussion, to prove these relations it is enough to show that

$$r_v(s, w_0) r_v(-s, w_0) = \mu(s, \mu_{t,v}\mathrm{St}_{GL_{a_{t,j}},k_v} \otimes \sigma(\hat{\phi}')_v).$$

We can use [14, Section 13] to easily obtain the above result, but only up to a non-zero constant (since [14] gives the expression for the Plancherel measure up to a non-zero constant). After adjusting our normalization by this constant, we get (5–19) of [18] and we can introduce spaces $Y_v(\sigma(\phi'))^{\pm}$. This constant is not a serious obstacle, since it is a positive real number (we know that $\mu(0, \mu_{t,v}\mathrm{St}_{GL_{a_{t,j}},k_v} \otimes \sigma(\hat{\phi}')_v) \geq 0$ and we assume $\overline{\psi_v} = \psi_v$) so introducing spaces $Y_v(\sigma(\phi'))^{\pm}$ still makes sense (even without second normalization with this constant).

Assume $v \in S$ is an archimedean place. We note that the normalization of the intertwining operators attached to the local Arthur parameter ψ_v (we use this notation for ϕ_v to avoid confusion) explained in the section 2.3 of [1] is uniform (used for all the members of (non-generic) packet ψ_v) and obtained in the following way: we introduce $\phi_{\psi_v} : W_{k_v} \to SO(2n+1, \mathbb{C})$ given by $\phi_{\psi_v}(u) = \psi_v\left(u, \begin{bmatrix} |u|^{\frac{1}{2}} & 0 \\ 0 & |u|^{-\frac{1}{2}} \end{bmatrix}\right)$. We easily see that, for a representation $\psi_v = \mu_v \otimes V_a$ of $W_{k_v} \times SL(2, \mathbb{C})$, its pull back ψ_{ϕ_v} to W_{k_v} equals $\oplus_{k=0}^{a-1} \mu_v(\cdot)| \cdot |^{-\frac{a-1}{2}+i}$. When we, in our situation, use then formulas (2.3.3), (2.3.8) and (2.3.26) and (2.3.27) of [1] (to be in the same situation as in our case), calculate these normalizing factors, we obtain the same formulas as we have for $r_v(w_0, s)$. Then Proposition 2.3.1 of [1] guarantees that our normalization satisfies the claim of this Proposition. □

Proposition 5.6 *Let S be a finite set of places, containing all the archimedean places and if $v \notin S$, $\pi_{0,v}$ is unramified (cf. (19)). Assume that $|S|$ is even. Let π be an irreducible subrepresentation of the representation $\otimes_{v \notin S} Y_v(\sigma(\phi'))^+ \otimes_{v \in S} Y_v(\sigma(\phi'))^-$. Then, the Eisenstein series from Theorem 5.3 acting on the representation π_s is non-zero on π for $s = 0$.*

Proof By the the discussion before this proposition, we saw that, for the choice $f_s = \otimes f_{s,v}$ from π, the following holds: $\lim_{s\to 0} M(w_0, \pi, s) f_s = C f_0$, where $C > 0$ so that $\lim_{s\to 0} E_{P_{\theta_0}}(s, f_s) \neq 0$, and, consequently, $\lim_{s\to 0} E(s, f_s) \neq 0$. □

5.2 Step 2

We now continue our construction, and we have the analogon of Proposition 4.3. The image of π under the action of the Eisenstein series is denoted by $E(\pi)$. We denote the space of constant terms of $E(\pi)$ along P_{θ_0} by $E_{\theta_0}(\pi)$, (in the previous proposition we proved that it is non-zero). We study the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\mu_t |det|^s \otimes E_{\theta_0}(\pi))$, and calculate Eisenstein series associated to this representation: $f_s \mapsto E(s, f_s)(g) = \sum_{\gamma \in P(k)\backslash Sp_{2n}(k)} f_s(\gamma g)$, where P is a maximal parabolic subgroup (analogous to the one in Proposition 4.3. We prove the following proposition:

Proposition 5.7 *The map*

$$\pi_s = \mathrm{Ind}_{P(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\mu_t |det|^{\frac{a_{t,j}+a_{t,j+1}}{4}} \otimes E_0(\pi)) \to A(Sp_{2n}(k)\backslash Sp_{2n}(\mathbb{A}))$$

given by

$$f_{\frac{a_{t,j}+a_{t,j+1}}{4}} \to \left(s - \frac{a_{t,j} + a_{t,j+1}}{4}\right)^2 E(s, f_s)(\cdot)|_{s=\frac{a_{t,j}+a_{t,j+1}}{4}}$$

is well-defined and non-trivial, and its image $E(\pi, \mu_t, a_{t,j+1})$ is contained in the space of square-integrable automorphic forms. Every irreducible subrepresentation of $E(\pi, \mu_t, a_{t,j+1})$ (which is semi-simple representation) is thus automorphic square-integrable, and has a global Arthur parameter equal to ϕ.

Proof Proof of this proposition will differ somewhat from the proof of Proposition 4.3. Again, to prove that the Eisenstein series has a pole of order two, we calculate the order of a pole for the constant term $E_{P_{\theta_0}}(s, f_s)(g)$ (we recall our convention about the notation of roots θ_0). Again, as in the proof of Theorem 5.3, for $f_s = \otimes f_{s,v}$ we have

$$E_{P_{\theta_0}}(s, f_s)(g) = \sum_{\substack{w\in W,\, w(\Delta\backslash\{\alpha\})>0 \\ w(\theta_0)=\theta_0}} (M(w, \pi, s) f_s)(g). \tag{26}$$

Let S be a finite set of places, such that for $v \notin S$ $\pi_{s,v}$ is unramified and $f_{s,v}$ is a spherical vector, normalized as usual. For $v \notin S$, let $\chi_{s,v}$ be a character of maximal torus in the symplectic group in question, such that $\pi_{s,v}$ is embedded in the principal series representation induced by the character $\chi_{s,v}$. Assume that for those v, ψ_v is unramified. We then have

$$M(w, \pi, s) f_s = \otimes_{v\notin S} \prod_{\substack{\beta>0 \\ w(\beta)<0}} \frac{L(0, \chi_{s,v} \circ \check{\beta})}{L(1, \chi_{s,v} \circ \check{\beta})} f_{w(s),v} \otimes_{v\in S} A(w, s, \pi_v) f_{s,v}. \tag{27}$$

Note that now $i=\frac{a_{t,j+1}-a_{t,j}}{2}$, so for $v\notin S$, s near $\frac{a_{t,j+1}+a_{t,j}}{4}$

$$\chi_{s,v}=\mu_{t,v}\nu^{s-\frac{i-1}{2}}\otimes\cdots\otimes\mu_{t,v}\nu^{s+\frac{i-1}{2}}\otimes\mu_{t,v}\nu^{-\frac{a_{t,j}-1}{2}}\otimes\cdots\otimes\mu_{t,v}\nu^{\frac{a_{t,j}-1}{2}}\otimes\lambda_{1,v}\nu^{s_1}$$
$$\otimes\cdots\otimes\lambda_{k,v}\nu^{s_k}\rtimes\sigma(\phi_0)_v),$$

cf. (22).

The difference in the proof of this proposition and Proposition 4.3 is in the fact that we calculate the order of pole of $M(w,\pi,s)f_s$ by calculating the orders of the poles of partial Hecke L-functions obtained from the previous expression. Although the representation π_s is not globally induced from Hecke characters, in the same way as in the discussion after Proposition 5.4, we can attach to it (partial L-functions of) a certain global character. As we saw in the proof of Lemma 4.4, that for given $w\in W$ as above, the set of all $\beta>0$ such that $w(\beta)<0$ can be divided in the six sets. We examine the behavior of the partial L-functions which we form when we explicitly express $\prod_{v\notin S}\frac{L(0,\chi_{s,v}\circ\check{\beta})}{L(1,\chi_{s,v}\circ\check{\beta})}$ for every set of roots.

The contribution from the first set of roots (where $w=p\epsilon,\ w\in W_j$) is

$$\prod_{k=j+1}^{i}\frac{L_S\left(-\frac{i-1}{2}+s+k-1,\mu_t\right)}{L_S\left(-\frac{i-1}{2}+s+k-1,\mu_t\right)}.$$

Note that for $s\approx\frac{a_{t,j+1}+a_{t,j}}{4}$ every factor in the numerator (and denominator) is strictly positive. Note that in the integer points (the full) Hecke L-functions are non-zero. Also note that in the strictly positive points, local Hecke L-functions (attached to quadratic characters) are non-zero and do not have a pole. We conclude that this contribution has no zeros or poles, except when $\mu_t=1,\ a_{t,j}=1,\ j=0$, when $L_S(1,\mu_t)$ has a pole of the first order (appearing in the numerator), so the whole contribution has a pole of the first order.

The contribution from the second set of roots is given by

$$\prod_{j+1\le k<l\le i}\frac{L_S(2s-i+k+l-1;1)}{L_S(2s-i+k+l;1)}.$$

As $2s-i\approx a_{t,j}$ we see that $k+l-1\ge 3$, so this expression has no zeros or poles; and if $j=i$ or $j+1=i$ there is no second set of roots at all.

Analogously, we see that the fourth set of roots contributes with expression which has no zeros or poles, and there is no contribution if $j=0$ or $j=i$.

The third set of roots does not exist if $j=i$. Otherwise, we can divide the expression

$$\prod_{\substack{j+1\le k\le i\\ i+1\le l\le n}}\otimes_{v\notin S}\frac{L(0,\chi_{s,v}\circ(\check{e}_k-\check{e}_l))}{L(1,\chi_{s,v}\circ(\check{e}_k-\check{e}_l))}$$

into a product of three factors, according whether

- $j+1 \le l \le \frac{a_{t,j+1}+a_{t,j}}{2}$ (case (a)),
- $\frac{a_{t,j+1}+a_{t,j}}{2}+1 \le l \le n-n_0$ (case (b)) and
- $n-n_0+1 \le l \le n$ (case (c)).

Contribution from case (a) equals

$$\prod_{\substack{j+1\le k\le i\\ i+1\le l\le \frac{a_{t,j+1}+a_{t,j}}{2}}} \frac{L\left(s-\frac{i-1}{2}+\frac{a_{t,j}-1}{2}+k-l+i;1\right)}{L\left(s-\frac{i-1}{2}+\frac{a_{t,j}-1}{2}+k-l+i+1;1\right)}.$$

If $j=0$ then for $k=1$ and $l=\frac{a_{t,j+1}+a_{t,j}}{2}$ there is a factor $L_S(1;1)$ in the numerator; if $j\ge 1$ there are no poles or zeros in the numerator or denominator. So, if $j=0$ there is a pole of the first order. As for the case (b), we easily again get global (partial) Hecke functions, like in the case (a), and by our assumption (Δ) on the way of adding new elements in Jordan block ($a_{t,j}$ is greater or equal to every previous member of Jordan block (associated to any appearing character)) we can conclude that there is no zeros or poles in that contribution. The case (c) is not directly related to the global characters, but we can go around it in the following way: according to the proof of Proposition 5.4, either $w(\Delta(Sp_{2n_0}))=\mathrm{id}$ or for $\sigma(\phi_0)$ relation (21) holds. In the first case we can automatically conclude that $p(l)>p(k)$ for case (c) ($p(l)=l$), but then we can combine this case with the contribution of the sixth group of roots (case (c); we divide this group of roots in cases in the same way as for the third group) and analogously as in the discussion after Proposition 5.4 we get the contribution

$$\prod_{j+1\le k\le i}\left(\prod_{r=1}^{l}\prod_{p=-\frac{a_{r,1}-1}{2}}^{\frac{a_{r,1}-1}{2}}\frac{L_S\left(s-\frac{i-1}{2}+k-1-p,\,\mu_t\mu_r\right)}{L_S\left(s-\frac{i-1}{2}+k-p,\,\mu_t\mu_r\right)}\right)\frac{L_S\left(s-\frac{i-1}{2}+k,\,\mu_t\right)}{L_S\left(s-\frac{i-1}{2}+k-1,\,\mu_t\right)}.$$

We easily see that this expression has no zeros or poles unless $\mu_t=1,\ a_{t,j}=1,\ j=0$ when it has a zero of the first order. If (21) holds for $\sigma(\phi_0)$, we proceed as follows: we divide the contribution 3 (c) into subsets

$$\prod_{\substack{j+1\le k\le i\\ n-n_0+1\le l\le n-m_0\\ p(l)>p(k)}}\frac{L_S\left(s-\frac{i-1}{2}+k-1-s_{l-(n-n_0)};\mu_t\chi_{l-(n-n_0)}\right)}{L_S\left(s-\frac{i-1}{2}-s_{l-(n-n_0)};\mu_t\chi_{l-(n-n_0)}\right)},$$

$$\prod_{\substack{j+1\le k\le i\\ n-n_0+1\le l\le n-m_0\\ p(l)<p(k)}}\frac{L_S\left(s-\frac{i-1}{2}+k-1-s_{l-(n-n_0)};\mu_t\chi_{l-(n-n_0)}\right)}{L_S\left(s-\frac{i-1}{2}-s_{l-(n-n_0)};\mu_t\chi_{l-(n-n_0)}\right)}$$

and, the last contribution, coming from embedding $\sigma_{0,v}$ in principal series (cf. Theorem 2.2 and Corollary 2.3), which we combine with analogous situation for the part of sixth (c) case (since here $p(l)=l>p(k)$):

$$\prod_{j+1\le k\le i}\left(\prod_{r=1}^{l}\prod_{p=-\frac{a_{r,1}-1}{2}}^{\frac{a_{r,1}-1}{2}}\frac{L_S\left(s-\frac{i-1}{2}+k-1-p,\mu_t\mu_r\right)}{L_S\left(s-\frac{i-1}{2}+k-p,\mu_t\mu_r\right)}\right)\frac{L_S\left(s-\frac{i-1}{2}+k,\mu_t\right)}{L_S\left(s-\frac{i-1}{2}+k-1,\mu_t\right)}$$
$$\prod_{r=1}^{k_1}\frac{L_S\left(s-\frac{i-1}{2}+k-s_r,\mu_t\chi_r\right)L_S\left(s-\frac{i-1}{2}+k+s_r,\mu_t\chi_r\right)}{L_S\left(s-\frac{i-1}{2}+k-1-s_r,\mu_t\chi_r\right)L_S\left(s-\frac{i-1}{2}+k-1+s_r,\mu_t\chi_r\right)}.$$

Again, the result is the same as for the situation $w(\Delta(Sp_{2n_0}))=\mathrm{id}$.

As for the fifth set of roots (the contribution is not there if $j=0$): again, we divide the contribution in three cases, as above. The contribution of the case (a) is

$$\prod_{\substack{1\le k\le j\\ i+1\le l\le \frac{a_{t,j+1}+a_{t,j}}{2}\\ p(l)<p(k)}}\frac{L_S\left(s-\frac{i-1}{2}+k-1+\frac{a_{t,j}-1}{2}-l+(i-1);1\right)}{L_S\left(s-\frac{i-1}{2}+k+\frac{a_{t,j}-1}{2}-l+(i-1);1\right)}.$$

We see that this expression has a pole of the first order if $k=1,\ l=\frac{a_{t,j+1}+a_{t,j}}{2}$ (and $(p(\frac{a_{t,j+1}+a_{t,j}}{2})<p(1))$. As for the contribution from the case b (l varies among the exponents attached to the adding new elements in Jordan blocks), it easily follows that, with the assumption (Δ), it does not have zeros or poles. The discussion in the case (c) again resembles the discussion about the third set of roots: if w acts as an identity on $\Delta(Sp_{2n_0})$, then there is no contribution from this set of roots, since we cannot have $l=p(l)<p(k)$ (since $(n-n_0+1\le l\le n)$). If, on the other hand, we have (21), then if $p(l)<p(k)$ for roots in the case (c), then they are attached to characters $\chi_v^{s_1},\ldots,\chi_k\nu^{s_k}$ in the notation of (21), so our assumption (Δ) then guarantees that the contribution from this roots does not have zeros or poles (analogously to the case (b)).

We are left to analyze the sixth set of roots (case (a) and (b) ; case (c) is resolved). The contribution from the case (a) is

$$\prod_{\substack{j+1\le k\le i\\ i+1\le l\le \frac{a_{t,j+1}+a_{t,j}}{2}\\ p(l)>p(k)}}\frac{L_S\left(s-\frac{i-1}{2}+k-1-\frac{a_{t,j}-1}{2}+l-(i+1);1\right)}{L_S\left(s-\frac{i-1}{2}+k-\frac{a_{t,j}-1}{2}+l-(i+1);1\right)}.$$

We note that for $j=0,\ k=1$ and $l=i+1$ we obtain factor $L_S(1;1)$ in the numerator if $p(i+1)>p(1)$ (and this is the only possible pole). There is no pole if $j\ge 1$. Again assuming (Δ), the contribution from case (b) is without zeros and poles.

We conclude: if $j\ge 1$ the expression $\otimes_{v\notin S}A(w,s,\pi_v)f_{s,v}$ has a pole of the first order (if additionally $p(1)>p(\frac{a_{t,j+1}+a_{t,j}}{2})$). If $j=0$, this expression has a pole of the first order or the pole of second order if $p(i+1)>p(1)$. But the last condition on $w\in W$ says that there is only one such w, namely $w=w_0$. To conclude: $\otimes_{v\notin S}A(w,s,\pi_v)f_{s,v}$ for $w\neq w_0$ has a pole of at most the first order for $s=\frac{a_{t,j+1}+a_{t,j}}{4}$ and for $w=w_0$ it has a pole of the second order.

Now we analyze $\otimes_{v\in S}A(w,s,\pi_v)f_{s,v}$. We prove that all these operators are holomorphic for $s=\frac{a_{t,j+1}+a_{t,j}}{4}$. This, in turn, proves that the mapping $f_{\frac{a_{t,j}+a_{t,j+1}}{4}}\mapsto (s-\frac{a_{t,j}+a_{t,j+1}}{4})^2E(f_s,\cdot)|_{s=\frac{a_{t,j}+a_{t,j+1}}{4}}$ is well-defined and non-trivial (cf. (26), (27)). Recall that

$$\pi_{s,v}\hookrightarrow \mathrm{Ind}_{P_{\theta_0}(k_v)}^{Sp_{2n}(k_v)}(\mu_{t,v}\nu^{s-\frac{i-1}{2}}\otimes\cdots\otimes\mu_{t,v}\nu^{s+\frac{i-1}{2}}\otimes\mu_{t,v}\nu^{-\frac{a_{t,j}-1}{2}}\otimes\cdots$$
$$\otimes\mu_{t,v}\nu^{\frac{a_{t,j}-1}{2}}\otimes\lambda_{1,v}\nu^{s_1}\otimes\cdots\otimes\lambda_{k,v}\nu^{s_k}\rtimes\sigma(\phi_0)_v)),$$

where $\sigma(\phi_0)_v$ is not (necessarily) a principal series subquotient, and we decompose each $A(w,s,\pi_v)$ into a product of generalized rank-one intertwining operators corresponding to the decomposition of w (cf. Lemma 2.1.2 and Theorem 2.1.1 of [20]). So it is either study of GL_2-case (with Levi isomorphic to $GL_1\times GL_1$) or study of the intertwining operator acting on a maximal Levi subgroup $GL_1\times Sp_{2n_0}$ (because $w(\theta_0)=\theta_0$). The exact description of the action of the elements of the Weyl group is given, for example, in [17, (3.15) and (3.16)], where the action is first described for certain elements $w_j\in W_j$, and then the action of all other elements in W_j is described using "shuffles", as is explained in loc.cit). According to this description, we are only interested (in $GL_1\times GL_1$ case) in intertwining operator $\chi_1\times\chi_2\to\chi_2\times\chi_1$, where χ_1 is one of the characters $\mu_{t,v}\nu^{\pm(s-\frac{i-1}{2})},\ldots,\mu_{t,v}\nu^{\pm(s+\frac{i-1}{2})}$, and χ_2 one of the characters $\mu_{t,v}\nu^{-\frac{a_{t,j}-1}{2}},\ldots,\mu_{t,v}\nu^{\frac{a_{t,j}-1}{2}},\lambda_{1,v}\nu^{s_1},\ldots,\lambda_{k,v}\nu^{s_k}$, or vice-versa, or χ_1 is one of the characters $\mu_{t,v}\nu^{s-\frac{i-1}{2}},\ldots,\mu_{t,v}\nu^{s-\frac{i-1}{2}+j-1}$ and χ_2 one of the characters $\mu_{t,v}\nu^{\pm(s-\frac{i-1}{2}+j)},\ldots,\mu_{t,v}\nu^{\pm(s+\frac{i-1}{2})}$ (or vice-versa). In both cases we have holomorphy of intertwining operators (for archimedean and non-archimedean) places. Now we are left to prove the holomorphy of intertwining operators

$$\begin{aligned}\mu_{t,v}\nu^{s-\frac{i-1}{2}+j}\rtimes\sigma(\phi_0)_v&\to\mu_{t,v}\nu^{-(s-\frac{i-1}{2}+j)}\rtimes\sigma(\phi_0)_v,\\ &\vdots\\ \mu_{t,v}\nu^{s+\frac{i-1}{2}}\rtimes\sigma(\phi_0)_v&\to\mu_{t,v}\nu^{-(s+\frac{i-1}{2})}\rtimes\sigma(\phi_0)_v.\end{aligned}\tag{28}$$

Assume now that v is non-archimedean. We prove that these intertwining operators are holomorphic, by embedding the representation $\sigma(\phi_0)_v$ into induced representation, using the cuspidal support of $\sigma(\phi_0)_v$, and then these intertwining operators are viewed as the restrictions of the intertwining operators on these induced representations. We actually observe that these (new) intertwining operators are holomorphic. We do that by estimating the cuspidal support of the representation $\sigma(\phi_0)_v$. The representation $\sigma(\hat{\phi}_0)_v$ is tempered and has the same cuspidal support as $\sigma(\phi_0)_v$. We recall that, for an irreducible admissible representation π of a symplectic group there exists a representation π' of an appropriate general linear group and the unique irreducible supercuspidal representation π_{cusp} of a symplectic group such that

$$\pi\hookrightarrow\pi'\rtimes\pi_{cusp}.\tag{29}$$

We call π_{cusp} **the partial cuspidal support** of π.

If π is tempered representation, by the classification of discrete series (and tempered representations) of classical groups by Moœglin and Tadić [14], we can attach to it a finite multiset, called the Jordan block of this tempered representation. The Jordan block consists of pairs consisting of a cuspidal representation of general linear group and a positive rational integer. Together with the partial cuspidal support (and another parameter, called the ε-function), the Jordan block describes how this tempered representation is embedded in a parabolically induced representation, similarly to the way in which Jordan blocks of negative representations describe the embedding of that unramified representation in a principal series representation (cf.(2)). We then define support of the Jordan block of a tempered representation π, denoted by supp(Jord(π)), in an analogous way to the way in which support of a Jordan block is defined for unramified representations (cf. Definition 2.1), but now the sum in this definition runs through (ρ, a) belonging to the Jordan block (ρ is an irreducible cupsidal representation, as explained above).

Although we do not know explicitly the partial cuspidal support of $\sigma(\hat{\phi}_0)_v$ just by knowing its Jordan block, we know that

$$\mathrm{supp}(\mathrm{Jord}(\sigma(\hat{\phi}_0)_{v\,cusp})) \subset \mathrm{supp}(\mathrm{Jord}(\sigma(\hat{\phi}_0)_v)).$$

Form the definition of the Jordan block (more details in [14]) it follows that if an irreducible cuspidal representation of a general linear group, say ρ, is in the cuspidal support of a representation π' in a situation of (29), then ρ is in supp(Jord)(π). This means that, when we embed $\sigma(\hat{\phi}_0)_v$ in the representation induced from the cuspidal one, all the exponents will be smaller (by an absolute value) of the exponents in $\mu_{t,v}\nu^{s-\frac{i-1}{2}+j}, \ldots, \mu_{t,v}\nu^{s+\frac{i-1}{2}}$. This means that all the intertwining operators in (28) are holomorphic.

For $v \in S$ archimedean we proceed as follows. To some local Arthur parameter ψ_v we can attach parameter ϕ_{ψ_v} like in the proof of Proposition 5.5 (so, in our notation, we form the parameter $\phi_{\sigma(\phi)_v}$). To this new parameter we attach a (non-tempered) packet, like in Proposition 7.4.1 of [1]. In the proof of that proposition, there is a claim which says that every member of Arthur packet $\sigma(\phi_0)_v$ has a (Langlands) parameter smaller (or equal) to the Langlands parameter (or linear form) of $\phi_{\sigma(\phi_0)_v}$. But exponents in this form vary from $-\frac{a_{k,1}-1}{2}, \ldots, \frac{a_{k,1}-1}{2}$, $k = 1, \ldots, l$, and are strictly smaller of exponents in $\mu_{t,v}\nu^{s-\frac{i-1}{2}+j}, \ldots, \mu_{t,v}\nu^{s+\frac{i-1}{2}}$. This means that all the intertwining operators in (28) are holomorphic at archimedean places, too. □

Now the rest of the proof (i.e., the conclusion in which we argue that the representation thus realized in the space of square-integrable automorphic forms $A(Sp_{2n}(k)\backslash Sp_{2n}(\mathbb{A}))$ really has Arthur parameter equal to $\sigma(\phi)$) is analogous to the argument in the third section.

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