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Biharmonic submanifolds into ellipsoids

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Abstract In this paper we construct proper biharmonic submanifolds into various types of ellipsoids. We also prove, in this context, some useful composition properties which can be used to produce large families of new proper biharmonic immersions.

Keywords Biharmonic maps · Biharmonic submanifols · Ellipsoids · Composition properties

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1 Introduction

Harmonic maps are critical points of the energy functional

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 dv_g, \qquad (1.1)$$

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where $\varphi : (M, g) \to (N, h)$ is a smooth map between two Riemannian manifolds M and N. In analytical terms, the condition of harmonicity is equivalent to the fact that the map φ is a solution of the Euler–Lagrange equation associated to the energy functional (1.1), i.e.

$$\text{trace}\, \nabla d\varphi = 0. \tag{1.2}$$

The left member of (1.2) is a vector field along the map φ, or, equivalently, a section of the pull-back bundle $\varphi^{-1}(TN)$: it is called tension field and denoted $\tau(\varphi)$.

A related topic of growing interest deals with the study of the so-called biharmonic maps: these maps, which provide a natural generalisation of harmonic maps, are the critical points of the bienergy functional (as suggested by Eells–Lemaire [10])

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 \, dv_g.$$

In [11] G. Jiang derived the first variation and the second variation formulas for the bienergy. In particular, he showed that the Euler–Lagrange equation associated to $E_2(\varphi)$ is

$$\tau_2(\varphi) = -J\,(\tau(\varphi)) = -\Delta\tau(\varphi) - \text{trace}\; R^N(d\varphi, \tau(\varphi))d\varphi = 0, \tag{1.3}$$

where J denotes (formally) the Jacobi operator of φ, Δ is the rough Laplacian on sections of $\varphi^{-1}(TN)$ that, for a local orthonormal frame $\{e_i\}_{i=1}^m$ on M, is defined by

$$\Delta = -\sum_{i=1}^{m} \left\{ \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} - \nabla^{\varphi}_{\nabla^M_{e_i} e_i} \right\}, \tag{1.4}$$

and

$$R^N(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

is the curvature operator on (N, h). We point out that (1.3) is a it fourth order semi-linear elliptic system of differential equations. We also note that any harmonic map is an absolute minimum of the bienergy, and so it is trivially biharmonic. Therefore, a general working plan is to study the existence of biharmonic maps which are not harmonic: these shall be referred to as it proper biharmonic maps. We refer to [12] for existence results and general properties of biharmonic maps.

An immersed submanifold into a Riemannian manifold (N, h) is called a biharmonic submanifold if the immersion is a biharmonic map. In a purely geometric context, Chen [8] defined biharmonic submanifolds $M \subset \mathbb{R}^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = (\Delta H_1, \ldots, \Delta H_n) = 0$, where $H = (H_1, \ldots, H_n)$ is the mean curvature vector as seen in $\mathbb{R}^n$ and Δ is the Beltrami–Laplace operator on M. It is important to point out that, if we apply the definition of biharmonic maps to immersions into the Euclidean space, we recover Chen's notion of biharmonic submanifolds. In this sense, our work can be regarded in the spirit of a generalization of Chen's biharmonic submanifolds.

A general result of Jiang [11] tells us that a compact, orientable, biharmonic submanifold M into a manifold N such that $\mathrm{Riem}^N \leq 0$ is necessarily minimal. Moreover, Oniciuc [15], proved that also CMC biharmonic isometric immersions into a manifold N with $\mathrm{Riem}^N \leq 0$ are necessarily minimal. In fact, it is still open the Chen's conjecture: biharmonic submanifolds into a non-positive constant sectional curvature manifold are minimal. The Chen's conjecture was generalized in [7] for biharmonic submanifolds into a Riemannian manifold with non-positive sectional curvature, although Ou and Tang found in [14] a counterexample. These facts have pushed research towards the investigation of biharmonic submanifolds of the Euclidean sphere (see [1–7] for an overview of the main results in this context). A further step is the study of biharmonic submanifolds into Euclidean ellipsoids, because these manifolds are geometrically rich and interestingly do not have constant sectional curvature: in [13] we obtained a complete classification of proper biharmonic curves into 3-dimensional ellipsoids and, more generally, into any non-degenerate quadric. In this paper, we shall focus on proper biharmonic submanifolds of dimension ≥ 2.

2 Biharmonic submanifolds into ellipsoids

We begin with the study of biharmonic submanifolds into Euclidean ellipsoids $Q^{p+q+1}(c,d)$ defined as follows:

$$Q^{p+q+1}(c,d) = \left\{ (x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^n \ : \ \frac{|x|^2}{c^2} + \frac{|y|^2}{d^2} = 1 \right\},$$

where c, d are fixed positive constants. The symmetry of $Q^{p+q+1}(c,d)$ makes it natural to look for biharmonic generalized Clifford's tori. More precisely, we shall study isometric immersions of the following type:

$$\begin{array}{cccccc} i\ : & S^p(a) & \times & S^q(b) & \longrightarrow & Q^{p+q+1}(c,d) \\ & (x_1,\ldots,x_{p+1}, & & y_1,\ldots,y_{q+1}) & \longmapsto & (x_1,\ldots,x_{p+1},y_1,\ldots,y_{q+1}), \end{array} \tag{2.1}$$

where i denotes the inclusion and the radii a, b must satisfy the following condition:

$$\frac{a^2}{c^2} + \frac{b^2}{d^2} = 1. \tag{2.2}$$

In this context, we shall prove the following result:

Theorem 2.1 *Let $i \,:\, S^p(a) \times S^q(b) \to Q^{p+q+1}(c,d)$ be an isometric immersion as in (2.1). If*

$$a^2 = c^2\,\frac{p}{p+q}; \quad b^2 = d^2\,\frac{q}{p+q} \tag{2.3}$$

then the immersion is minimal. If (2.3) does not hold and

$$a^2 = c^2\,\frac{c}{c+d}; \quad b^2 = d^2\,\frac{d}{c+d}, \tag{2.4}$$

then the immersion is proper biharmonic.

Remark 2.2 We observe that, interestingly, if $c = p$ and $d = q$, then we have generalized minimal Clifford's tori, but we do not have proper biharmonic submanifolds of the type (2.1). We also point out that, according to Theorem 2.1, the ellipsoid $Q^3(c, d)$ ($p = q = 1, c \neq d$) admits a proper biharmonic torus, while in S^3 there exists no genus 1 proper biharmonic submanifold (see [7]).

Proof We shall work essentially by using coordinates in $\mathbb{R}^n$, suitably restricted to the ellipsoid or to the torus, according to necessity. In particular, the splitting

$$(x, y) = (x_1, \ldots, x_{p+1}, y_1, \ldots, y_{q+1})$$

will be used in an obvious way, without further comments. The symbol $\langle , \rangle$ will denote the Euclidean scalar product (whether in $\mathbb{R}^{p+1}$, $\mathbb{R}^{q+1}$ or $\mathbb{R}^n$ will be clear from the context). We shall use a superscript Q for objects concerning the ellipsoid, while the letter T will appear for reference to the torus $T = S^p(a) \times S^q(b)$.

We shall need to know the algebraic conditions which ensure that a given vector field is tangent either to the torus or to the ellipsoid. More specifically, a vector field

$$W = (X, Y),$$

where

$$X = \sum_{i=1}^{p+1} X^i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_{j=1}^{q+1} Y^j \frac{\partial}{\partial y_j},$$

is tangent to $Q^{p+q+1}(c, d)$ if and only if

$$\sum_{i=1}^{p+1} \frac{1}{c^2} x_i X^i + \sum_{j=1}^{q+1} \frac{1}{d^2} y_j Y^j = 0.$$

In the same order of ideas, W is tangent to the torus T if and only if

$$\sum_{i=1}^{p+1} x_i X^i = 0 = \sum_{j=1}^{q+1} y_j Y^j. \tag{2.5}$$

To end preliminaries, we observe that the vector field

$$\eta^Q = \frac{\eta_1^Q}{|\eta_1^Q|}, \tag{2.6}$$

where

$$\eta_1^Q = \left(\frac{1}{c^2} x_1, \ldots, \frac{1}{c^2} x_{p+1}, \frac{1}{d^2} y_1, \ldots, \frac{1}{d^2} y_{q+1} \right),$$

represents a unit normal vector field on the ellipsoid $Q^{p+q+1}(c,d)$. Note, for future use, that the equality

$$|\eta_1^Q|^2 = \frac{a^2}{c^4} + \frac{b^2}{d^4} \tag{2.7}$$

holds on T. Similarly, the vector

$$\eta^T = \frac{\eta_1^T}{|\eta_1^T|}, \tag{2.8}$$

where

$$\eta_1^T = \left(\frac{c^2}{a^2}\,x_1, \ldots, \frac{c^2}{a^2}\,x_{p+1}, -\frac{d^2}{b^2}\,y_1, \ldots, -\frac{d^2}{b^2}\,y_{q+1} \right), \tag{2.9}$$

represents a unit normal vector on the torus T viewed as a submanifold of the ellipsoid $Q^{p+q+1}(c,d)$. We also note that

$$|\eta_1^T|^2 = \frac{c^4}{a^2} + \frac{d^4}{b^2} \tag{2.10}$$

on T. In order to compute the tension and the bitension fields, it is convenient to make explicit the formulas which will enable us to calculate the relevant covariant derivatives. More precisely, following, for instance [9], we know that

$$\nabla^Q_{W_1} W_2 = \nabla^{\mathbb{R}^n}_{W_1} W_2 - B^Q(W_1, W_2), \tag{2.11}$$

where $B^Q(W_1, W_2)$ denotes the second fundamental form of the ellipsoid $Q^{p+q+1}(c,d)$ into $\mathbb{R}^n$. We now need to do some work to make (2.11) more explicit:

$$\begin{aligned} B^Q(W_1, W_2) &= -\langle \nabla^{\mathbb{R}^n}_{W_1} \eta^Q,\ W_2\rangle \eta^Q \\ &= -\langle W_1\left(\frac{1}{|\eta_1^Q|}\right)\eta_1^Q + \frac{1}{|\eta_1^Q|}\nabla^{\mathbb{R}^n}_{W_1}\eta_1^Q, W_2\rangle\eta^Q \\ &= -\langle \frac{1}{|\eta_1^Q|}\nabla^{\mathbb{R}^n}_{W_1}\eta_1^Q, W_2\rangle\eta^Q. \end{aligned} \tag{2.12}$$

Next, we compute

$$\nabla^{\mathbb{R}^n}_{W_1}\eta_1^Q = \frac{1}{c^2}\sum_{i=1}^{p+1} X_1^i \frac{\partial}{\partial x_i} + \frac{1}{d^2}\sum_{j=1}^{q+1} Y_1^j \frac{\partial}{\partial y_j}. \tag{2.13}$$

Finally, using (2.13) into (2.12), we obtain:

$$B^Q(W_1, W_2) = -\frac{1}{|\eta_1^Q|}\left[\frac{1}{c^2}\langle X_1, X_2\rangle + \frac{1}{d^2}\langle Y_1, Y_2\rangle\right]\eta^Q, \tag{2.14}$$

which in (2.11) yields:

$$\nabla^Q_{W_1} W_2 = \nabla^{\mathbb{R}^n}_{W_1} W_2 + \frac{1}{|\eta_1^Q|}\left[\frac{1}{c^2}\langle X_1, X_2\rangle + \frac{1}{d^2}\langle Y_1, Y_2\rangle\right]\eta^Q.$$

Now, we are in the right position to proceed to the computation of the tension field τ of our immersion (2.1). Indeed, by definition,

$$\tau = \text{trace}\, B^T(\cdot,\cdot), \tag{2.15}$$

where

$$\begin{aligned} B^T(W_1, W_2) &= -\langle \nabla^Q_{W_1}\eta^T, W_2\rangle \eta^T \\ &= -\frac{1}{|\eta_1^T|^2}\langle \nabla^Q_{W_1}\eta_1^T, W_2\rangle \eta_1^T \\ &= -\frac{1}{|\eta_1^T|^2}\langle \nabla^{\mathbb{R}^n}_{W_1}\eta_1^T, W_2\rangle \eta_1^T \\ &= -\frac{1}{|\eta_1^T|^2}\left[\frac{c^2}{a^2}\langle X_1, X_2\rangle - \frac{d^2}{b^2}\langle Y_1, Y_2\rangle\right]\eta_1^T. \end{aligned} \tag{2.16}$$

Let now X_i , $i = 1, \dots, p$ and Y_j , $j = 1, \dots, q$, be local orthonormal bases of $S^p(a)$ and $S^q(b)$, respectively. By using (2.16) in (2.15) we find:

$$\begin{aligned} \tau &= \sum_{i=1}^{p} B^T\left((X_i, 0), (X_i, 0)\right) + \sum_{i=1}^{q} B^T\left((0, Y_j), (0, Y_j)\right) \\ &= -\frac{1}{|\eta_1^T|^2}\left[\frac{p\,c^2}{a^2} - \frac{q\,d^2}{b^2}\right]\eta_1^T \\ &= \lambda\,\eta_1^T, \end{aligned} \tag{2.17}$$

where, taking into account (2.10), we have set

$$\lambda = -\left[\frac{c^4}{a^2} + \frac{d^4}{b^2}\right]^{-1}\left[\frac{p\,c^2}{a^2} - \frac{qd^2}{b^2}\right].$$

In particular, using (2.2), it is now immediate to conclude that (2.3) is equivalent to the minimality of the immersion.

Next, we proceed to the computation of the bitension field τ_2. To this purpose, we must apply (1.3) in the case that $\varphi = i$. We begin with the computation of $\Delta\tau$. It is convenient to choose a geodesic local orthonormal frame obtained from geodesic local orthonormal frames on each factor of the torus. Under this assumption the terms $\nabla^T_{e_i} e_i$ in the formula (1.4) vanish (note that this simplification is acceptable because we shall not need to compute covariant derivatives of higher order). So the expression for the rough Laplacian (1.4) in our context reduces to:

$$\Delta\tau = -\left[\sum_{i=1}^{p} \nabla^Q_{X_i}\left(\nabla^Q_{X_i}\tau\right) + \sum_{j=1}^{q} \nabla^Q_{Y_j}\left(\nabla^Q_{Y_j}\tau\right)\right], \tag{2.18}$$

where, to simplify notation, we have written X_i for $(X_i, 0)$ and Y_j for $(0, Y_j)$. Using (2.11), (2.9) and (2.17) we find:

$$\begin{aligned}
\nabla^Q_{X_i}\tau &= \lambda\nabla^Q_{X_i}\eta_1^T \\
&= \lambda\nabla^{\mathbb{R}^n}_{X_i}\eta_1^T + \frac{\lambda}{|\eta_1^Q|^2}\left[\frac{1}{c^2}\langle X_i, \eta_1^T\rangle + \frac{1}{d^2}\langle 0, \eta_1^T\rangle\right]\eta_1^Q \\
&= \lambda\frac{c^2}{a^2}X_i + \frac{\lambda}{|\eta_1^Q|^2}[\,0\,]\,\eta_1^Q \\
&= \lambda\,\frac{c^2}{a^2}\,X_i.
\end{aligned} \tag{2.19}$$

Next, using first (2.19),

$$\begin{aligned}
\nabla^Q_{X_i}\left(\nabla^Q_{X_i}\tau\right) &= \lambda\,\frac{c^2}{a^2}\,\nabla^Q_{X_i}\,X_i \\
&= \lambda\,\frac{c^2}{a^2}\left[\nabla^T_{X_i}\,X_i + B^T\,(X_i, X_i)\right] \\
&= -\,\lambda\,\frac{c^2}{a^2}\,\frac{1}{|\eta_1^T|^2}\left[\frac{c^2}{a^2}\,\langle X_i, X_i\rangle\right]\eta_1^T,
\end{aligned} \tag{2.20}$$

where, in order to obtain the last equality, we have used the fact that our orthonormal frame is geodesic and also (2.16). Now, a very similar computation leads us to

$$\nabla^Q_{Y_j}\left(\nabla^Q_{Y_j}\tau\right) = -\,\lambda\,\frac{d^2}{b^2}\,\frac{1}{|\eta_1^T|^2}\left[\frac{d^2}{b^2}\,\langle Y_j, Y_j\rangle\right]\eta_1^T. \tag{2.21}$$

Putting together (2.18), (2.20) and (2.21) we obtain

$$\Delta\,\tau\; = \;\mu\;\eta_1^T, \tag{2.22}$$

where, taking into account (2.10), we have defined the constant μ as follows:

$$\mu = \frac{\lambda}{|\eta_1^T|^2}\left[\frac{p\,c^4}{a^4} + \frac{q\,d^4}{b^4}\right] \tag{2.23}$$

(note that, if (2.3) does not hold, then $\lambda \neq 0$, so that $\mu \neq 0$ and the immersion is not minimal).

By way of summary, the previous computations have led us to the following conclusion:

$$\tau_2 = -\left[\mu\,\eta_1^T + \text{trace}\,R^Q(d\,i,\tau)\,d\,i\right]. \tag{2.24}$$

We have to investigate for which values (if any) of $a,\ b$ the bitension τ_2 vanishes. In order to deal in an efficient way with the curvature tensor, we shall study the vanishing of normal and tangential components separately. In particular, we shall prove that the normal component of τ_2 is identically zero if and only if (2.4) holds. The proof of the theorem will then be completed by the verification that the tangential part of τ_2 vanishes for all values of a and b. So, let us first study whether, for suitable values of a and b, we can have

$$\langle \tau_2,\ \eta_1^T\rangle\ =\ 0. \tag{2.25}$$

From (2.24) and (2.23) we have:

$$-\langle \tau_2,\ \eta_1^T\rangle = \lambda\left[\left(\frac{p\,c^4}{a^4}+\frac{q\,d^4}{b^4}\right)+\sum_{i=1}^{p}\langle R^Q(X_i,\tau)X_i,\eta_1^T\rangle\right.$$
$$\left.+\sum_{i=1}^{q}\langle R^Q(Y_j,\tau)Y_j,\eta_1^T\rangle\right].$$

Next, we observe that

$$\langle R^Q(X_i,\tau)X_i,\eta_1^T\rangle = -\frac{\langle R^Q(X_i,\tau)\eta_1^T,X_i\rangle}{|\eta_1^T|^2}\,|\eta_1^T|^2 = K^Q(X_i,\eta^T)\,|\eta_1^T|^2, \tag{2.26}$$

where $K^Q(X_i,\eta^T)$ denotes sectional curvature, which (see [9]) can be expressed by means of:

$$K^Q(X_i,\eta^T) = \langle B^Q(X_i,X_i),\ B^Q(\eta^T,\eta^T)\rangle - \langle B^Q(X_i,\eta^T),\ B^Q(X_i,\eta^T)\rangle. \tag{2.27}$$

By using (2.26) in (2.27) and performing a computation which, according to (2.14), uses

$$B^Q(X_i,X_i) = -\frac{1}{|\eta_1^Q|^2}\,\frac{1}{c^2}\,\langle X_i,X_i\rangle\,\eta_1^Q,$$
$$B^Q(\eta_1^T,\eta_1^T) = -\frac{1}{|\eta_1^Q|^2}\left(\frac{c^2}{a^2}+\frac{d^2}{b^2}\right)\eta_1^Q,$$
$$B^Q(X_i,\eta^T) = 0, \tag{2.28}$$

we find:

$$\langle R^Q(X_i,\eta_1^T)X_i,\eta_1^T\rangle = -\frac{1}{|\eta_1^Q|^2}\,\frac{1}{c^2}\,\langle X_i,X_i\rangle\left(\frac{c^2}{a^2}+\frac{d^2}{b^2}\right). \tag{2.29}$$

In a very similar fashion we also compute:

$$\langle R^Q(Y_j,\eta_1^T)Y_j,\eta_1^T\rangle = -\,\frac{1}{|\eta_1^Q|^2}\,\frac{1}{d^2}\,\langle Y_j,Y_j\rangle\left(\frac{c^2}{a^2}+\frac{d^2}{b^2}\right). \tag{2.30}$$

Putting together (2.29), (2.30) and (2.24) it is easy to obtain the following conclusion:

$$\langle \tau_2,\ \eta_1^T\rangle = -\,\lambda\,\left\{\left[\,\frac{p\,c^4}{a^4}+\frac{q\,d^4}{b^4}\,\right]-\,\frac{1}{|\eta_1^Q|^2}\left[\,\frac{c^2}{a^2}+\frac{d^2}{b^2}\,\right]\left[\,\frac{p}{c^2}+\frac{q}{d^2}\,\right]\right\}. \tag{2.31}$$

Now, using (2.7) and (2.2) in (2.31), it is not difficult to check that (2.25) holds if and only if (2.4) is satisfied. At this stage, we can say that the proof of the theorem will be completed if we show that

$$\langle \tau_2,\ W\rangle\ =\ 0 \tag{2.32}$$

for any vector field W which is tangent to the torus. Taking into account (2.24), we see that (2.32) is equivalent to:

$$\langle \text{trace}\ R^Q(d\,i,\tau)\,d\,i\ ,\ W\rangle = 0.$$

Because of (2.17), it is enough to show that

$$\langle R^Q(X,\ \eta_1^T)\,X,\ W\rangle = 0$$

holds if $X,\ W$ are arbitrary vectors tangent to T. But, by the Gauss equation (see [9]), we deduce:

$$\begin{aligned}\langle R^Q(X,\ \eta_1^T)\,X,\ W\rangle &= \langle B^Q(X,W),\,B^Q(\eta_1^T,X)\rangle\\ &\quad -\langle B^Q(\eta_1^T,W),\ B^Q(X,X)\rangle = 0,\end{aligned} \tag{2.33}$$

where, for the last equality, we have used (2.28). □

Next, we study biharmonic submanifolds into Euclidean ellipsoids of revolution $Q^{p+1}(c,d)$ defined as follows:

$$Q^{p+1}(c,d) = \left\{(x,y)\in\mathbb{R}^{p+1}\times\mathbb{R}=\mathbb{R}^n\ :\ \frac{|x|^2}{c^2}+\frac{y^2}{d^2}=1\right\},$$

where $c,\ d$ are fixed positive constants. In this case, the symmetry of $Q^{p+1}(c,d)$ makes it natural to look for biharmonic hyperspheres. More precisely, we shall study isometric immersions of the following type:

$$\begin{aligned} i\ :\quad S^p(a)\quad &\times\{b\}\longrightarrow\quad Q^{p+1}(c,d)\\ (x_1,\ldots,x_{p+1},\quad &b)\ \longmapsto (x_1,\ldots,x_{p+1},b),\end{aligned} \tag{2.34}$$

where i denotes the inclusion and the constants a, b must again satisfy the condition

$$\frac{a^2}{c^2} + \frac{b^2}{d^2} = 1 \tag{2.35}$$

(note that a is a radius, so it is positive, while the only request on b is: $|b| < d$).
In this context, we shall prove the following result:

Theorem 2.3 *Let* $i : S^p(a) \times \{b\} \to Q^{p+1}(c, d)$ *be an isometric immersion as in* (2.34). *If*

$$a^2 = c^2; \quad b = 0 \tag{2.36}$$

then the immersion is minimal (this is the case of the equator hypersphere). If

$$a = c\,\sqrt{\frac{c}{c+d}}; \quad b = \pm\, d\,\sqrt{\frac{d}{c+d}}, \tag{2.37}$$

then the immersion is proper biharmonic.

Proof Again, we shall use a superscript Q for objects concerning the ellipsoid, while the letter S will appear for reference to the hypersphere $S = S^p(a) \times \{b\}$. Essentially, the proof follows the arguments of Theorem 2.1 and most of the calculations can be performed by setting $q = 0$ in the formulas above: for this reason, we limit ourselves to point out the relevant differences only. First, let us assume that $b \neq 0$: normal vectors η_1^Q, η^Q, η_1^S and η^S can be introduced precisely as in (2.6)–(2.10). We also note that, since $b \neq 0$, (2.5) implies that a tangent vector to S must be of the form

$$W = (X, 0). \tag{2.38}$$

Taking into account (2.38) we easily obtain

$$\tau = \lambda\, \eta_1^S,$$

where

$$\lambda = -\left[\frac{c^4}{a^2} + \frac{d^4}{b^2}\right]^{-1} \left[\frac{p\,c^2}{a^2}\right]$$

(note that $\lambda \neq 0$, so that in this case the hypersphere is not minimal).
In the computation of $\Delta\,\tau$ only the terms $\nabla^Q_{X_i}\left(\nabla^Q_{X_i}\,\tau\right)$ in (2.18) are relevant: this fact leads us to the expression

$$\Delta\,\tau = \mu\,\eta_1^S, \tag{2.39}$$

where now

$$\mu = \frac{\lambda}{|\eta_1^S|^2}\left[\frac{p\,c^4}{a^4}\right].$$

Also the calculation involving the curvature terms follows the lines above and leads us to

$$\langle \tau_2,\, \eta_1^S \rangle = -\,\lambda \left\{ \left[\frac{p\,c^4}{a^4} \right] - \frac{1}{|\eta_1^Q|^2} \left[\frac{c^2}{a^2} + \frac{d^2}{b^2} \right] \left[\frac{p}{c^2} \right] \right\}. \tag{2.40}$$

Now, inspection of (2.40) shows that (2.37) is equivalent to the vanishing of the normal component of the bitension. Finally, an argument as above shows that the tangential component of the bitension always vanishes, so ending the case $b \neq 0$.

In the case that $b = 0$ we observe that

$$\eta^S = (0,\, \ldots,\, 1). \tag{2.41}$$

Using (2.41) in (2.16) it is easy to conclude that, in this case, the second fundamental form of S vanishes identically, so that the equator hypersphere is totally geodesic and so minimal, a fact which ends the theorem. □

3 Composition properties

Our first result is:

Theorem 3.1 *Let $i : S^p(a) \to Q^{p+1}(c, d)$ be a proper biharmonic immersion as in Theorem 2.3, and let $\varphi : M^m \to S^p(a)$ be a minimal immersion. Then $i \circ \varphi : M^m \to Q^{p+1}(c, d)$ is a proper biharmonic immersion.*

Proof Let W_i, $i = 1, \ldots, m$, be a local orthonormal frame on M^m. To simplify notation, for a tangent vector W to M^m, we write W for both $d\varphi(W)$ and $di\,(d\varphi(W))$. The composition law for the tension field (see [10]), together with the minimality of φ, gives:

$$\begin{aligned} \tau(i \circ \varphi) &= \sum_{i=1}^{m} \nabla d\, i\, (W_i, W_i) + \tau(\varphi) \\ &= \sum_{i=1}^{m} \nabla d\, i\, (W_i, W_i) \\ &= \sum_{i=1}^{m} B^S\, (W_i, W_i)\,. \end{aligned} \tag{3.1}$$

Now, adapting the calculation of (2.16), we have:

$$B^S\,(W_i, W_i) = -\,\frac{c^2}{a^2}\,\frac{1}{|\eta_1^S|}\,\langle W_i, W_i \rangle\, \eta^S. \tag{3.2}$$

Next, using (3.2) in (3.1), we obtain

$$\tau(i \circ \varphi) = -\,m\,\frac{c^2}{a^2}\,\frac{1}{|\eta_1^S|}\,\eta^S. \tag{3.3}$$

In particular, we deduce from (3.3) that $i \circ \varphi$ is not minimal and we proceed to the computation of the bitension. For convenience, we set $\nu = m \, \frac{c^2}{a^2} \, \frac{1}{|\eta_1^S|}$.

Using (3.3) we have:

$$\begin{aligned} \tau_2(i \circ \varphi) &= -\Delta^M \tau(i \circ \varphi) - \sum_{i=1}^{m} R^Q(W_i, \tau(i \circ \varphi)) W_i \\ &= \nu \left[\Delta^M \eta^S + \sum_{i=1}^{m} R^Q(W_i, \eta^S)) W_i \right]. \end{aligned} \tag{3.4}$$

Next, we study separately the two terms in the right-hand side of (3.4). First, computing as in (2.39) (with p replaced by m), we find

$$\Delta^M \eta^S = \left[\frac{m}{|\eta_1^S|^2} \frac{c^4}{a^4} \right] \eta^S. \tag{3.5}$$

Second, using the Gauss equation as in (2.33), we obtain:

$$\langle \sum_{i=1}^{m} R^Q(W_i, \eta^S)) W_i, \eta^S \rangle = -m \frac{1}{|\eta_1^S|^2} \frac{1}{|\eta_1^Q|^2} \frac{1}{c^2} \left(\frac{c^2}{a^2} + \frac{d^2}{b^2} \right), \tag{3.6}$$

and

$$\langle \sum_{i=1}^{m} R^Q(W_i, \eta^S)) W_i, W \rangle = 0 \tag{3.7}$$

for all vector W which is tangent to S . Putting together (3.4)–(3.7) we conclude that $\tau_2(i \circ \varphi)$ is parallel to η^S and vanishes if and only if

$$\left\{ \left[\frac{c^4}{a^4} \right] - \frac{1}{|\eta_1^Q|^2} \left[\frac{c^2}{a^2} + \frac{d^2}{b^2} \right] \left[\frac{1}{c^2} \right] \right\} = 0. \tag{3.8}$$

But (3.8) is equivalent to the two conditions (2.37) and (2.35), so that the proof is completed. □

In a spirit similar to the previous theorem, we also obtain the following result:

Theorem 3.2 *Let $i : S^p(a) \times S^q(b) \to Q^{p+q+1}(c, d)$ be a proper biharmonic immersion as in Theorem 2.1, and let $\varphi_1 : M_1^{m_1} \to S^p(a)$, $\varphi_2 : M_2^{m_2} \to S^q(b)$ be two minimal immersions. Then $i \circ (\varphi_1 \times \varphi_2) : M_1^{m_1} \times M_2^{m_2} \to Q^{p+q+1}(c, d)$ is a proper biharmonic immersion.*

Proof The proof of this result is a straightforward variant of the arguments of Theorem 3.1 and so the details are omitted. □

Remark 3.3 When $c = d = 1$ the composition properties described in Theorem 3.1 and Theorem 3.2 reduce to those first proved in [6]. It is important to note that all biharmonic submanifolds into the ellipsoids constructed using the composition properties have parallel mean curvature vector field.

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