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The differentiability of Banach space-valued functions of bounded variation

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Abstract In this paper we consider Banach space-valued functions with the compact range. It is shown that if a Banach space-valued function $F : [0, 1] \to X$ is of bounded variation with respect to the Minkowski functional $||.||_F$ associated to the closed absolutely convex hull C_F of $F([0, 1])$, then F is differentiable almost everywhere on $[0, 1]$.

Keywords Banach space · Compact range · Differential · Bounded variation · Minkowski functional

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1 Introduction and preliminaries

Throughout this paper X denotes a real Banach space with its norm $||.||$. We denote by $B(x, \varepsilon)$ the open ball with center x and radius $\varepsilon > 0$ and by X^* the topological dual to X. If a function $F : [0, 1] \to X$ is given, then we denote by X_F the vector space spanned by the set C_F and by $||.||_F$ the Minkowski functional associated to the closed absolutely convex hull C_F of $F([0, 1]) = \{F(t) : t \in [0, 1]\}$. Thus, $||x||_F = \inf\{r > 0 : x \in r \cdot C_F\}$, for all $x \in X_F$.

At first, we introduce the concept of the limit average range.

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Definition 1.1 Let $F : [0, 1] \to X$ be a function and let $t \in [0, 1]$. We put

$$\Delta F(t, h) = \frac{F(t+h) - F(t)}{h} \qquad A_F(t, \delta) = \left\{\Delta F(t, h) : 0 < |h| < \delta\right\}$$

and

$$A_F(t) = \bigcap_{\delta > 0} \overline{A_F}(t, \delta),$$

where $\overline{A_F}(t, \delta)$ is the closure of $A_F(t, \delta)$. The set $A_F(t)$ is said to be *the average range* of F at the point t. Denote

$$diam(W) = \sup\{||x - y|| : x, y \in W\} \quad (W \subset X).$$

We say that F has *the limit average range* at the point t, if $A_F(t)$ is a bounded set and for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$diam(\overline{A_F}(t, \delta_\varepsilon)) < diam(A_F(t)) + \varepsilon.$$

Next, we recall the notion of differentiation, see Definition 7.3.2 in [8].

Definition 1.2 Let $F : [0, 1] \to X$ be a function and let $t \in [0, 1]$. The function F is said to be *differentiable at the point t* if there is a vector $x \in X$ such that

$$\lim_{h \to 0} ||\Delta F(t, h) - x|| = 0.$$

By $x = F'(t)$ the derivative of F at t is denoted.

We denote by $\mathcal{I}$ the family of all non-degenerate closed subintervals of [0, 1], by λ the Lebesgue measure and by $\mathcal{L}$ the family of all Lebesgue measurable subsets of [0, 1]. The intervals $I, J \in \mathcal{I}$ are said to be *nonoverlapping* if $\text{int}(I) \cap \text{int}(J) = \emptyset$, where $\text{int}(I)$ denotes the interior of I. We will identify an interval function $\widetilde{F} : \mathcal{I} \to X$ with the point function $F(t) = \widetilde{F}([0, t])$, $t \in [0, 1]$; and conversely, we will identify a point function $F : [0, 1] \to X$ with the interval function $\widetilde{F}([u, v]) = F(v) - F(u)$, $[u, v] \in \mathcal{I}$.

Assume that an interval $[a, b] \subset [0, 1]$ and a function $F : [0, 1] \to X$ are given. A finite collection $\{I_i \in \mathcal{I} : i = 1, 2, \ldots, m\}$ of pairwise nonoverlapping intervals is said to be *a partition of* $[a, b]$, if $\cup_{i=1}^m I_i = [a, b]$. We denote by $\Pi_{[a,b]}$ the family of all partitions of $[a, b]$. Let us define *the total variation* $V_{[a,b]}^{(X_F)} F$ of F on $[a, b]$, with respect to the norm $||.||_F$, by equality

$$V_{[a,b]}^{(X_F)} F = \sup\left\{\sum_{I \in \pi} ||\widetilde{F}(I)||_F : \pi \in \Pi_{[a,b]}\right\}.$$

If $c \in (a, b)$, then

$$V_{[a,b]}^{(X_F)} F = V_{[a,c]}^{(X_F)} F + V_{[c,b]}^{(X_F)} F. \tag{1.1}$$

The last equality was proven for real valued functions in [9] (p. 83), but the proof works also for vector valued functions, it is enough to change the absolute value with the norm $||.||_F$. If we have

$$V_{[a,b]}^{(X_F)} F < +\infty,$$

then F is said to be *of bounded variation on* $[a, b]$ with respect to $||.||_F$.

Functions of usual bounded variation from $[0, 1]$ into a Banach space need not have a single point of differentiability. A simple and well-known example is the function $F : [0, 1] \to L_1([0, 1])$ defined by

$$F(t) = \chi_{[0,t]} \quad \text{for all} \quad t \in [0, 1],$$

where $\chi_{[0,t]}$ is the characteristic function of $[0, t]$. The function F is of usual bounded variation on $[0, 1]$, but F is not differentiable at a single point of $[0, 1]$. This pathology does not appear in the class of Banach spaces with the Radon–Nikodym property, see the statement (3) in [3] (p. 217). A detailed study of Banach spaces possessing the RNP is presented in books [1,2] and [3].

A function $F : [0, 1] \to X$ is said to be *strongly absolutely continuous (sAC)* if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every finite collection $\{I_i \in \mathcal{I} : i = 1, 2, \dots, m\}$ of pairwise nonoverlapping intervals, we have

$$\sum_{i=1}^{m} \lambda(I_i) < \eta \Rightarrow \sum_{i=1}^{m} ||\widetilde{F}(I_i)|| < \varepsilon. \tag{1.2}$$

Replacing (1.2) by

$$\sum_{i=1}^{m} \lambda(I_i) < \eta \Rightarrow \left|\left|\sum_{i=1}^{m} \widetilde{F}(I_i)\right|\right| < \varepsilon,$$

we obtain the definition of *absolute continuity (AC)*.

We say that a function $F : [0, 1] \to X$ is *Lipschitz at* $t \in [0, 1]$ if there exist $c_t > 0$ and $\delta_t > 0$ such that

$$|h| < \delta_t \text{ and } t + h \in [0, 1] \Rightarrow ||F(t + h) - F(t)|| \leq c_t \cdot |h|.$$

2 The Differentiability of functions of bounded variation

The main result is Theorem 2.6. Let us start with some auxiliary statements. Lemma 2.1 together with Examples 2.2 and 2.3 highlights the local relation between the differential and the limit average range.

Lemma 2.1 *Let* $F : [0, 1] \to X$ *be a function and let* $t_0 \in [0, 1]$. *Then, the following statements are equivalent.*

(i) *F is differentiable at t_0 with $F'(t_0) = x_0$,*
(ii) *F has the limit average range at t_0 and*

$$A_F(t_0) = \{x_0\}. \tag{2.1}$$

Proof (i)⇒(ii) Assume that F is differentiable at t_0.

Claim 1 The equality

$$A_F(t_0) = \{F'(t_0)\} \tag{2.2}$$

holds.

First, we will show that $F'(t_0) \in A_F(t_0)$. To see this, we choose a sequence (h_k) of real numbers such that

$$0 < |h_k| < \frac{1}{k} \quad \text{for all} \quad k \in \mathbb{N}.$$

Then

$$\lim_{k\to\infty} ||\Delta F(t_0, h_k) - F'(t_0)|| = 0,$$

and since for each $n \in \mathbb{N}$, we have

$$\Delta F(t_0, h_k) \in A_F\left(t_0, \frac{1}{n}\right) \quad \text{for all} \quad k \geq n,$$

it follows that

$$F'(t_0) \in \bigcap_{n=1}^{\infty} \overline{A_F}\left(t_0, \frac{1}{n}\right) = \bigcap_{\delta>0} \overline{A_F}(t_0, \delta) = A_F(t_0).$$

Secondly, we will show that $A_F(t_0) \subset \{F'(t_0)\}$. Assume that $x \in A_F(t_0)$ is given. Then, for each $n \in \mathbb{N}$, we have

$$B\left(x, \frac{1}{n}\right) \bigcap A_F\left(t_0, \frac{1}{n}\right) \neq \emptyset.$$

Therefore, there is a sequence (h'_n) of real numbers such that

$$0 < |h'_n| < \frac{1}{n} \quad \text{and} \quad \Delta F(t_0, h'_n) \in B\left(x, \frac{1}{n}\right) \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence

$$\lim_{n\to\infty} ||\Delta F(t_0, h'_n) - x|| = 0,$$

and we infer that $x = F'(t_0)$.

Claim 2 Given $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$diam(\overline{A_F}(t_0, \delta_\varepsilon)) = diam(A_F(t_0, \delta_\varepsilon)) < diam(A_F(t_0)) + \varepsilon = \varepsilon. \tag{2.3}$$

Indeed, since F is differentiable at t_0 there is a $\delta_\varepsilon > 0$ such that

$$0 < |h| < \delta_\varepsilon \Rightarrow ||\Delta F(t_0, h) - x_0|| < \frac{\varepsilon}{3}.$$

Hence, for each $h', h'' \in \mathbb{R}$, we have

$$0 < |h'|, |h''| < \delta_\varepsilon \Rightarrow ||\Delta F(t_0, h') - \Delta F(t_0, h'')|| < \frac{2 \cdot \varepsilon}{3}.$$

The last result yields that (2.3) holds true.

Clearly, by (2.2) and (2.3), we obtain that F has the limit average range at t_0 and (2.1) holds true.

(ii)⇒(i) Assume that (ii) holds. Let $\varepsilon > 0$ be given and let δ_ε corresponds to ε by Definition 1.1. Then, since $x_0 \in \overline{A_F}(t, \delta)$ for all $\delta > 0$, we have

$$0 < |h| < \delta_\varepsilon \Rightarrow ||\Delta F(t_0, h) - x_0|| < diam(A_F(t_0)) + \varepsilon = \varepsilon.$$

This means that F is differentiable at t_0 and $F'(t_0) = x_0$. □

The following example shows that there is a function $F : [-1, 1] \to l_p$, $p > 1$, that has not the limit average range at $t = 0$ and $A_F(0) = \{(0)\}$. Hence, by Lemma 2.1, F is not differentiable at $t = 0$.

Example 2.2 Let $F : [-1, 1] \to l_p$ be a function given as follows

$$F(t) = \begin{cases} (0, \ldots, 0, \ldots) & \text{if} \quad t \neq \frac{1}{n} \\ (0, \ldots, 0, \frac{1}{n}, 0, \ldots) & \text{if} \quad t = \frac{1}{n} \end{cases} \quad t \in [-1, 1] \quad n = 1, 2, 3, \ldots$$

Since

$$\Delta F(0, h) = \begin{cases} (0, \ldots, 0, \ldots) & \text{if} \quad h \neq \frac{1}{n} \\ (0, \ldots, 0, 1, 0, \ldots) & \text{if} \quad h = \frac{1}{n} \end{cases}$$

we have

$$diam(A_F(0, \delta)) = 2^{\frac{1}{p}} \quad \text{for all} \quad \delta > 0. \tag{2.4}$$

We claim that

$$A_F(0) = \{(0, \ldots, 0, \ldots)\}. \tag{2.5}$$

Let us consider an arbitrary element $x_0 \in A_F(0)$. Since

$$A_F(0) = \bigcap_{k=1}^{\infty} \overline{A_F}\left(0, \frac{1}{k}\right),$$

there is a sequence $(h_k) \subset \mathbb{R}$ such that for each $k \in \mathbb{N}$, we have

$$0 < |h_k| < \frac{1}{k} \quad \text{and} \quad ||\Delta F(0, h_k) - x_0||_{l_p} < \frac{1}{k}.$$

Therefore

$$\lim_{k \to \infty} ||\Delta F(0, h_k) - x_0||_{l_p} = 0.$$

Hence

$$\lim_{k \to \infty} x^*(\Delta F(0, h_k)) = x^*(x_0) \quad \text{for all} \quad x^* \in (l_p)^*. \tag{2.6}$$

Fix an arbitrary $x^* \in (l_p)^*$. Since $(l_p)^* = l_q$, there is a sequence $(a_n) \in l_q$ such that

$$x^*(x) = \sum_{n=1}^{+\infty} a_n \cdot x_n \quad \text{for all} \quad x = (x_n) \in l_p,$$

and since

$$x^*(\Delta F(0, h_k)) = \begin{cases} 0 & \text{if} \quad h_k \neq \frac{1}{n} \\ a_n & \text{if} \quad h_k = \frac{1}{n} \end{cases} \quad (n > k),$$

we obtain

$$\lim_{k \to \infty} x^*(\Delta F(0, h_k)) = 0.$$

Hence, by (2.6), it follows that

$$x^*(x_0) = 0 \quad \text{for all} \quad x^* \in (l_p)^*,$$

because x^* was arbitrary. Therefore, we obtain by Hahn–Banach Theorem that

$$x_0 = (0, \ldots, 0, \ldots)$$

and consequently (2.5) holds true.

Clearly, by (2.4) and (2.5), we obtain that F has not the limit average range at $t = 0$.

By Lemma 2.1, if a function $F : [0, 1] \to X$ is differentiable at a point $t \in [0, 1]$, then F has the limit average range at this point, but the converse does not hold. The next example shows that there is a function $F : [-1, 1] \to l_\infty$ which has the limit average range at $t = 0$, but F is not differentiable at this point.

Example 2.3 Let $F : [-1, 1] \to l_\infty$ be a function given as follows

$$F(t) = \begin{cases} (0, \ldots, 0, \ldots) & \text{if} \quad t \neq \frac{1}{n} \\ \left(\frac{1}{|n|}, \ldots, \frac{1}{|n|}, \ldots\right) & \text{if} \quad t = \frac{1}{n} \end{cases} \quad t \in [-1, 1] \quad n = \pm 1, \pm 2, \pm 3, \ldots.$$

Since

$$\Delta F(0, h) = \begin{cases} (0, \ldots, 0, \ldots) & \text{if} \quad h \neq \frac{1}{n} \\ (1, \ldots, 1, \ldots) & \text{if} \quad h = \frac{1}{n}, n > 0 \\ (-1, \ldots, -1, \ldots) & \text{if} \quad h = \frac{1}{n}, n < 0 \end{cases}$$

we have

$$A_F(0, \delta) = \{(0), (+1), (-1)\} \quad \text{for all} \quad \delta > 0.$$

It follows that F has the limit average range at $t = 0$ and

$$A_F(0) = \{(0), (+1), (-1)\}.$$

Note that

$$\lim_{n \to +\infty} ||\Delta F\left(0, \frac{1}{n}\right) - (1)||_{l_\infty} = 0 \quad \text{and} \quad \lim_{n \to -\infty} \left\|\Delta F\left(0, \frac{1}{n}\right) - (-1)\right\|_{l_\infty} = 0.$$

This means that the function F is not differentiable at $t = 0$.

Lemma 2.4 *Let $F : [0, 1] \to X$ be a function. If F is sAC and has the limit average range almost everywhere on* $[0, 1]$*, then F is differentiable almost everywhere on* $[0, 1]$.

Proof Since F has the limit average range almost everywhere on $[0, 1]$, there exists $Z \subset [0, 1]$ with $\lambda(Z) = 0$ such that F has the limit average range at all $t \in [0, 1]\setminus Z$.

Claim 1 We have

$$A_F(t) \neq \emptyset \quad \text{for all} \quad t \in [0, 1]\setminus Z \tag{2.7}$$

To see this, fix an arbitrary $t \in [0, 1]\setminus Z$ and assume by contradiction that

$$\bigcap_{\delta > 0} \overline{A_F}(t, \delta) = \emptyset. \tag{2.8}$$

Then, by Definition 1.1, there is a decreasing sequence (δ_n) of real numbers such that for each $n \in \mathbb{N}$, we have

$$0 < \delta_n < \frac{1}{n} \quad \text{and} \quad diam(A_F(t, \delta_n)) < \frac{1}{n}.$$

Define a sequence $(\Delta F(t, h_n))$ by choosing a $\Delta F(t, h_n) \in A_F(t, \delta_n)$ for all $n \in \mathbb{N}$. Hence, we get that $(\Delta F(t, h_n))$ is a Cauchy sequence. Therefore, there is a $x_0 \in X$ such that

$$\lim_{n \to \infty} \Delta F(t, h_n) = x_0.$$

Since

$$\Delta F(t, h_k) \in A_F(t, \delta_n) \quad \text{for all} \quad k \geq n \quad \text{and} \quad n \in \mathbb{N},$$

we obtain that $x_0 \in \overline{A_F}(t, \delta_n)$, for all $n \in \mathbb{N}$, and since

$$\bigcap_{n=1}^{\infty} \overline{A_F}(t, \delta_n) = \bigcap_{\delta > 0} \overline{A_F}(t, \delta),$$

it follows that

$$x_0 \in \bigcap_{\delta > 0} \overline{A_F}(t, \delta).$$

This contradicts (2.8) and consequently, $A_F(t) \neq \emptyset$. Since t was arbitrary, we obtain that (2.7) holds true.

Now, we choose $x_t \in A_F(t)$, for each $t \in [0, 1] \backslash Z$, and define the function f : $[0, 1] \to X$ as follows

$$f(t) = \begin{cases} x_t & t \in [0, 1] \backslash Z \\ 0 & t \in Z \end{cases}. \tag{2.9}$$

Claim 2 The function f is Pettis integrable on [0, 1]. It is easy to see that

$$\overline{x^*(A_F(t))} \subset A_{x^* \circ F}(t) \quad \text{for all} \quad x^* \in X^* \quad \text{and} \quad t \in [0, 1] \backslash Z. \tag{2.10}$$

We also have that each $x^* \circ F$ is sAC. It follows that each function $x^* \circ F$ is differentiable almost everywhere on [0, 1]. Thus, for each $x^* \in X^*$ there exists $Z^{(x^*)} \subset [0, 1]$ with $\lambda(Z^{(x^*)}) = 0$ such that $(x^* \circ F)'(t)$ exists for all $t \in [0, 1] \backslash Z^{(x^*)}$. Hence, by Lemma 2.1, we obtain

$$\{(x^* \circ F)'(t)\} = A_{x^* \circ F}(t) \quad \text{for all} \quad t \in [0, 1] \backslash Z^{(x^*)}.$$

The last equality together with (2.10) and (2.9) yields

$$(x^* \circ F)'(t) = (x^* \circ f)(t) \quad \text{for all} \quad t \in [0,1] \setminus (Z \cup Z^{(x^*)}).$$

This means that f is a scalar derivative of F on [0, 1], see Definition 2.1(b) in [6]. Then, since F is also AC, we obtain by Theorem 5.1 in [6] that f is Pettis integrable on [0, 1] and

$$\widetilde{F}(I) = (P) \int_I f \, d\lambda \quad \text{for all} \quad I \in \mathcal{I}.$$

Claim 3 The function f is strongly measurable. Since F is sAC the function F is continuous on [0, 1], and because this the set $\{F(t) : t \in [0,1]\} \subset X$ is compact and therefore separable. If $Y \subset X$ is the closed linear subspace spanned by the set $\{F(t) : t \in [0,1]\}$, then Y is separable. Since $\Delta F(t,h) \in Y$ for all $t \in [0,1]$ and $h \neq 0$, we obtain by Definition 1.1 that $A_F(t) \subset Y$ for all $t \in [0,1] \setminus Z$. Thus, we have $f(t) \in Y$ for all $t \in [0,1]$. This means that f is almost everywhere separable valued. Hence by the Pettis measurability theorem, Theorem II.1.2 in [3], the function f is strongly measurable.

Claim 4 The function f is Bochner integrable on [0, 1]. We set

$$\nu(E) = (P) \int_E f \, d\lambda \quad \text{for all} \quad E \in \mathcal{L}.$$

Let (E_k) be a sequence of disjoint members of $\mathcal{L}$ such that $\cup_{k=1}^{+\infty} E_k = [0,1]$. Since

$$\nu(I) = \widetilde{F}(I) \quad \text{for all} \quad I \in \mathcal{I} \tag{2.11}$$

and F is sAC we obtain by Caratheodory–Hahn–Kluvanek extension theorem in [5] that ν is of bounded variation. Then

$$(L) \int_{\cup_{k=1}^{n} E_k} ||f(t)|| d\lambda \leq |\nu|([0,1]) < +\infty$$

for all $n \in \mathbb{N}$. Hence, by the Monotone Convergence Theorem, the function $||f(.)||$ is Lebesgue integrable on [0, 1]. Therefore, we obtain by Theorem II.2.2 in [3] that f is Bochner integrable on [0, 1]. Since the Bochner and Pettis integrals coincide whenever they coexist, we have

$$\nu(E) = (B) \int_E f \, d\lambda \quad \text{for every } E \in \mathcal{L}.$$

The last result together with (2.11) and Theorem II.2.9 in [3] yields that F is differentiable a.e. on [0, 1] and the proof is finished. □

Lemma 2.5 *Let $F : [0, 1] \to X$ be a function and let $t_0 \in [0, 1]$. If there is a $\delta_0 > 0$ such that $\overline{A_F}(t_0, \delta_0)$ is a compact set, then F has the limit average range at t_0.*

Proof Assume by contradiction that F has not the limit average range at t_0. Then, there exists $\varepsilon_0 > 0$ and the sequences (δ_n), (h'_n), (h''_n) such that $\lim_{n\to\infty} \delta_n = 0$ and for each $n \in \mathbb{N}$, we have

(a) $0 < \delta_{n+1} < \delta_n < \delta_0$,
(b) $0 < |h'_n|, |h''_n| < \delta_n$,
(c) $||\Delta F(t_0, h'_n) - \Delta F(t_0, h''_n)|| \geq diam(A_F(t_0)) + \varepsilon_0$.

Since $\overline{A_F}(t_0, \delta_0)$ is a compact set, there exist subsequences $(\Delta F(t_0, h'_{n_k}))$ and $(\Delta F(t_0, h''_{n_k}))$ of sequences $(\Delta F(t_0, h'_n))$ and $(\Delta F(t_0, h''_n))$, respectively, such that

$$\lim_{k\to\infty} \Delta F(t_0, h'_{n_k}) = x'_0 \quad \text{and} \quad \lim_{k\to\infty} \Delta F(t_0, h''_{n_k}) = x''_0$$

where

$$x'_0, x''_0 \in \overline{A_F}(t_0, \delta_0).$$

The last result together with (c) yields

$$||x'_0 - x''_0|| \geq diam(A_F(t_0)) + \varepsilon_0. \tag{2.12}$$

On the other hand, we have

$$x'_0, x''_0 \in \bigcap_{k=1}^{\infty} \overline{A_F}(t_0, \delta_{n_k}) = A_F(t_0)$$

and therefore

$$||x'_0 - x''_0|| < diam(A_F(t_0)) + \varepsilon_0. \tag{2.13}$$

This contradicts (2.12). Consequently, the function F has the limit average range at t_0 and the proof is finished. □

Now, we are ready to present the main result.

Theorem 2.6 *Let $F : [0, 1] \to X$ be a function with compact range. If F is of bounded variation on $[0, 1]$ with respect to $||.||_F$, then F is differentiable almost everywhere on $[0, 1]$.*

Proof First of all, we claim that F has the limit average range almost everywhere on $[0, 1]$. To see this, we define the function $\varphi : [0, 1] \to [0, +\infty)$ by $\varphi(0) = 0$ and $\varphi(t) = V_{[0,t]}^{(X_F)} F$ for all $t \in (0, 1]$. Since φ increases on $[0, 1]$, φ is differentiable almost everywhere on $[0, 1]$. Thus, there exists $Z_\varphi \subset [0, 1]$ with $\lambda(Z_\varphi) = 0$ such that $\varphi'(t)$ exists at all $t \in [0, 1]\setminus Z_\varphi$.

Fix an arbitrary point $t \in [0, 1] \setminus Z_\varphi$. Then, given ε there exists $\delta_\varepsilon > 0$ such that

$$0 < |h| < \delta_\varepsilon \Rightarrow \Delta\varphi(t, h) \in (\varphi'(t) - \varepsilon, \varphi'(t) + \varepsilon). \tag{2.14}$$

Note that the inequality

$$||F(t+h) - F(t)||_F \leq |\varphi(t+h) - \varphi(t)|$$

implies

$$F(t+h) - F(t) \in (\varphi(t+h) - \varphi(t)) \cdot C_F.$$

It follows that

$$A_F(t, \delta_\varepsilon) \subset A_\varphi(t, \delta_\varepsilon) \cdot C_F$$

and since

$$A_\varphi(t, \delta_\varepsilon) \cdot C_F \subset [\varphi'(t) - \varepsilon, \varphi'(t) + \varepsilon] \cdot C_F = C'_F$$

we obtain

$$A_F(t, \delta_\varepsilon) \subset C'_F.$$

By the corollary of Theorem III.6.5 in [7] we have that C_F is a compact set. Hence, we obtain by Theorems 5.13 and 5.8 in [4] that C'_F is also a compact set. Further, $\overline{A_F}(t, \delta_\varepsilon)$ is a compact set and therefore we obtain by Lemma 2.5 that F has the limit average range at t. Since t was arbitrary F has the limit average range at all $t \in [0, 1] \setminus Z_\varphi$. We set

$$K = [0, 1] \setminus Z_\varphi$$

and denote by L the set of all points $t \in [0, 1]$ at which F is Lipschitz.

Claim 1 The function F is Lipschitz at all $t \in K$. Fix an arbitrary $t \in K$. By Definition 1.1, given $\varepsilon = 1$ there exists $\delta_t > 0$ such that

$$diam(\overline{A_F}(t, \delta_t)) < diam(A_F(t)) + 1,$$

and since $A_F(t)$ is a bounded set there exists $r_t > 0$ such that

$$diam(\overline{A_F}(t, \delta_t)) < r_t + 1 = c_t,$$

whence

$$||F(t+h) - F(t)|| \leq c_t \cdot |h| \quad \text{for all} \quad |h| < \delta_t.$$

This means that t is a Lipschitz point of F, and since t was arbitrary the function F is Lipschitz at all $t \in K$. It follows that $\lambda(L \backslash K) = 0$ and $\lambda([0, 1] \backslash L) = 0$.

Denote by L_n the set of all $t \in L$ such that

$$|h| < \frac{1}{n} \Rightarrow ||F(t+h) - F(t)|| \leq n \cdot |h| \qquad (n \in \mathbb{N}). \tag{2.15}$$

It is easy to see that each L_n is closed and $L = \cup_{n=1}^{\infty} L_n$. Fix an arbitrary L_n with $\lambda(L_n) > 0$ and let

$$(0, 1) \setminus L_n = \bigcup_{k=1}^{\infty} (a_k^{(n)}, b_k^{(n)}).$$

Define the function $F_n : [0, 1] \to X$ by $F_n(t) = F(t)$ for all $t \in L_n$, $F_n(0) = F(0)$, $F_n(1) = F(1)$ and

$$F_n(t) = F(a_k^{(n)}) + \frac{F(b_k^{(n)}) - F(a_k^{(n)})}{b_k^{(n)} - a_k^{(n)}} \cdot (t - a_k^{(n)}), \tag{2.16}$$

for all $t \in [a_k^{(n)}, b_k^{(n)}]$ and $k \in \mathbb{N}$.

Claim 2 There is a real number $M_n \geq 1$ such that

$$\frac{||F(b_k^{(n)}) - F(a_k^{(n)})||}{(b_k^{(n)} - a_k^{(n)})} \leq M_n \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.17}$$

Indeed, since $\sum_{k=1}^{\infty} (b_k^{(n)} - a_k^{(n)}) \leq 1$, there exists $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} (b_k^{(n)} - a_k^{(n)}) < \frac{1}{n}$. Hence, we obtain by (2.15) that

$$\frac{||F(b_k^{(n)}) - F(a_k^{(n)})||}{(b_k^{(n)} - a_k^{(n)})} \leq n \quad \text{for } k = N+1, N+2, \ldots$$

and therefore

$$0 \leq T_n = \sup \left\{ \frac{||F(b_k^{(n)}) - F(a_k^{(n)})||}{(b_k^{(n)} - a_k^{(n)})} : k \in \mathbb{N} \right\} < +\infty.$$

Then, $M_n = \max\{M_n^{(0)}, 1\}$ is the desired real number.

Claim 3 The function F_n is sAC. To see this, we assume that an arbitrary $0 < \varepsilon < 1$ is given. Then, we choose $\eta = \frac{\varepsilon}{9 \cdot (M_n + n)}$ and consider a finite collection α of pairwise nonoverlapping subintervals in $\mathcal{I}$ such that $\sum_{I \in \alpha} \lambda(I) < \eta$. Sort the intervals $[u, v]$ in α into three collections α_1, α_2 and α_3 as follows

(i) $u \in L_n$ or $v \in L_n$
(ii) $[u, v] \subset (a_k^{(n)}, b_k^{(n)})$,
(iii) $u \in (a_k^{(n)}, b_k^{(n)})$ and $v \in (a_{k'}^{(n)}, b_{k'}^{(n)})$, where $k < k'$.

In case (iii), we have

$$[u, v] = [u, b_k^{(n)}] \cup [b_k^{(n)}, a_{k'}^{(n)}] \cup [a_{k'}^{(n)}, v]$$

and therefore

$$[u, b_k^{(n)}], [b_k^{(n)}, a_{k'}^{(n)}], [a_{k'}^{(n)}, v] \in \alpha_1.$$

Then

$$\begin{aligned}\sum_{I\in\alpha} ||\widetilde{F_n}(I)|| &\le \sum_{I\in\alpha_1} ||\widetilde{F_n}(I)|| + \sum_{I\in\alpha_2} ||\widetilde{F_n}(I)|| + \sum_{I\in\alpha_3} ||\widetilde{F_n}(I)|| \le \\ & n \cdot \sum_{I\in\alpha_1} \lambda(I) + M_n \cdot \sum_{I\in\alpha_2} \lambda(I) + \sum_{I\in\alpha_3} ||\widetilde{F_n}(I)|| \le \\ & n \cdot \frac{\varepsilon}{9\cdot(M_n+n)} + M_n \cdot \frac{\varepsilon}{9\cdot(M_n+n)} + 3\cdot\left(n\cdot\frac{\varepsilon}{9\cdot(M_n+n)}\right) < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.\end{aligned} \tag{2.18}$$

This means that F_n is sAC.

It is easy to see that the distance function $g_n(t) = \text{dist}(t, L_n)$ is sAC. Therefore, it is differentiable almost everywhere on [0, 1]. Thus, there exists $Z_n \subset [0, 1]$ with $\lambda(Z_n) = 0$ such that $g_n'(t)$ exists at all $t \in [0, 1]\setminus Z_n$. In particular, we have $g_n'(t) = 0$ for all $t \in L_n\setminus Z_n$. If we set

$$S_n = L_n\setminus(Z_n \cup Z_\varphi),$$

then for each $t \in S_n$ we have that $g_n'(t) = 0$ and F has the limit average range at t. Fix an arbitrary $t_0 \in S_n$.

Claim 4 The equality

$$\lim_{h\to 0} \frac{F_n(t_0+h) - F(t_0+h)}{h} = 0 \tag{2.19}$$

holds true.

Since $g_n'(t_0) = 0$, given $0 < \varepsilon < 1$ there exists $0 < \delta_\varepsilon^{(d)} < 1$ such that for each $h \in \mathbb{R}$, we have

$$0 < |h| < \delta_\varepsilon^{(d)} \Rightarrow \frac{\text{dist}(t_0+h, L_n)}{|h|} < \frac{\varepsilon}{3\cdot n\cdot M_n}$$

and therefore for each $h \in \mathbb{R}$ with $0 < |h| < \delta_\varepsilon^{(d)}$ there exists $\overline{h} \in \mathbb{R}$ such that

$$t_0 + \overline{h} \in L_n \text{ and } \frac{|h - \overline{h}|}{|h|} < \frac{\varepsilon}{3 \cdot n \cdot M_n}. \tag{2.20}$$

Fix an arbitrary $h \in \mathbb{R}$ with $0 < |h| < \delta_\varepsilon^{(d)}$. If $(t_0 + h) \in L_n$ then $F_n(t_0 + h) - F(t_0 + h) = 0$. Otherwise $t_0 + h \in (a_k^{(n)}, b_k^{(n)})$, for some $k \in \mathbb{N}$. Then, we choose $\overline{h} \in \mathbb{R}$ so that (2.20) is satisfied. We have $t_0 + \overline{h} \notin (a_k^{(n)}, b_k^{(n)})$. If $t_0 + \overline{h} \leq a_k^{(n)}$, then

$$\begin{aligned} &\left\| \frac{F_n(t_0 + h) - F(t_0 + h)}{h} \right\| \\ &\quad \leq \left\| \frac{F_n(t_0 + h) - F_n(a_k^{(n)})}{h} \right\| + \left\| \frac{F_n(a_k^{(n)}) - F_n(t_0 + \overline{h})}{h} \right\| \\ &\quad + \left\| \frac{F_n(t_0 + \overline{h}) - F(t_0 + h)}{h} \right\| = A_a + B_a + C_a, \end{aligned} \tag{2.21}$$

otherwise if $t_0 + \overline{h} \geq b_k^{(n)}$, we have

$$\begin{aligned} &\left\| \frac{F_n(t_0 + h) - F(t_0 + h)}{h} \right\| \\ &\quad \leq \left\| \frac{F_n(t_0 + h) - F_n(b_k^{(n)})}{h} \right\| + \left\| \frac{F_n(b_k^{(n)}) - F_n(t_0 + \overline{h})}{h} \right\| \\ &\quad + \left\| \frac{F_n(t_0 + \overline{h}) - F(t_0 + h)}{h} \right\| = A_b + B_b + C_b. \end{aligned} \tag{2.22}$$

Let us evaluate the right hand side of the last inequality. By (2.16) we obtain

$$A_b = \left\| \frac{F_n(t_0 + h) - F_n(b_k^{(n)})}{h} \right\| \leq \frac{|t_0 + h - b_k^{(n)}|}{|h|} \cdot M_n \leq \frac{|h - \overline{h}|}{|h|} \cdot M_n < \frac{\varepsilon}{3}. \tag{2.23}$$

Since

$$|(t_0 + \overline{h}) - b_k^{(n)}| \leq |h - \overline{h}| \qquad |h - \overline{h}| < \frac{1}{n}$$

and $t_0 + \overline{h} \in L_n$, we obtain by the definition of L_n that

$$B_b = \left\| \frac{F(b_k^{(n)}) - F(t_0 + \overline{h})}{h} \right\| \leq n \cdot \frac{|t_0 + \overline{h} - b_k^{(n)}|}{|h|} < n \cdot \frac{|\overline{h} - h|}{|h|} < \frac{\varepsilon}{3}, \tag{2.24}$$

and

$$C_b = \left\| \frac{F(t_0 + \overline{h}) - F(t_0 + h)}{h} \right\| \leq n \cdot \frac{|\overline{h} - h|}{|h|} < \frac{\varepsilon}{3}. \tag{2.25}$$

The inequalities (2.23), (2.24) and (2.25) together with (2.22) yield

$$\left\| \frac{F_n(t_0 + h) - F(t_0 + h)}{h} \right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \tag{2.26}$$

It is proved by the same manner as above that the last inequality holds also for the case when $t_0 + \overline{h} \leq a_k^{(n)}$. Consequently, since h was arbitrary, the inequality (2.26) holds whenever $0 < |h| < \delta_\varepsilon^{(d)}$. This means that (2.19) holds true.

Claim 5 The equality

$$A_{F_n}(t_0) = A_F(t_0) \tag{2.27}$$

holds. First, we will show

$$A_{F_n}(t_0) \subset \overline{A_F}(t_0, \delta) \quad \text{for each} \quad \delta > 0. \tag{2.28}$$

To see this, we assume that an arbitrary $x \in A_{F_n}(t_0)$ and an arbitrary $\delta_0 > 0$ are given. By (2.19), given an arbitrary $\varepsilon > 0$, there exists $0 < \delta_\varepsilon < \delta_0$ such that for each $h \in \mathbb{R}$, we have

$$0 < |h| < \delta_\varepsilon \Rightarrow \left\| \frac{F_n(t_0 + h) - F(t_0 + h)}{h} \right\| < \frac{\varepsilon}{2}.$$

Since $x \in \overline{A_{F_n}}(t_0, \delta_\varepsilon)$, there is $h_\varepsilon \in \mathbb{R}$ such that

$$0 < |h_\varepsilon| < \delta_\varepsilon \quad \text{and} \quad ||x - \Delta F_n(t_0, h_\varepsilon)|| < \frac{\varepsilon}{2},$$

and since

$$\Delta F(t_0, h_\varepsilon) = \Delta F_n(t_0, h_\varepsilon) + \frac{F(t_0 + h_\varepsilon) - F_n(t_0 + h_\varepsilon)}{h_\varepsilon},$$

we obtain

$$||x - \Delta F(t_0, h_\varepsilon)|| < \varepsilon.$$

Since ε was arbitrary, the last result yields that $x \in \overline{A_F}(t_0, \delta_0)$, and since x and δ_0 have been taken arbitrarily it follows that (2.28) holds for all $\delta > 0$.

Secondly, by the same manner as above, we get

$$A_F(t_0) \subset \overline{A_{F_n}}(t_0, \delta) \quad \text{for all} \quad \delta > 0. \tag{2.29}$$

Clearly, (2.28) together with (2.29) yields that (2.27) holds true.

Claim 6 Given $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that

$$diam(\overline{A_{F_n}}(t_0, \delta_\varepsilon)) = diam(A_{F_n}(t_0, \delta_\varepsilon)) < diam(A_F(t_0)) + \varepsilon. \tag{2.30}$$

Since F has the limit average range at the point t_0 there is a $\delta_\varepsilon^{(1)} > 0$ such that for each $h', h'' \in \mathbb{R}$, we have

$$0 < |h'|, |h''| < \delta_\varepsilon^{(1)} \Rightarrow ||\Delta F(t_0, h') - \Delta F(t_0, h'')|| < diam(A_F(t_0)) + \frac{\varepsilon}{4}. \tag{2.31}$$

By (2.19), there is a $\delta_\varepsilon^{(2)} > 0$ such that for each $h \in \mathbb{R}$, we have

$$0 < |h| < \delta_\varepsilon^{(2)} \Rightarrow \left\| \frac{F_n(t_0 + h) - F(t_0 + h)}{h} \right\| < \frac{\varepsilon}{4}.$$

Choose $\delta_\varepsilon = \min\{\delta_\varepsilon^{(1)}, \delta_\varepsilon^{(2)}\}$. Then, for each $h', h'' \in \mathbb{R}$ such that $0 < |h'|, |h''| < \delta_\varepsilon$, we have

$$\begin{aligned}
&||\Delta F_n(t_0, h') - \Delta F_n(t_0, h'')|| \\
&\leq ||\Delta F_n(t_0, h') - \Delta F(t_0, h')|| + ||\Delta F(t_0, h') - \Delta F(t_0, h'')|| \\
&+||\Delta F(t_0, h'') - \Delta F_n(t_0, h'')|| \\
&= \left\| \frac{F_n(t_0 + h') - F(t_0 + h')}{h'} \right\| + ||\Delta F(t_0, h') - \Delta F(t_0, h'')|| \\
&+ \left\| \frac{F_n(t_0 + h'') - F(t_0 + h'')}{h''} \right\| \\
&< diam(A_F(t_0)) + \frac{3 \cdot \varepsilon}{4}.
\end{aligned}$$

Hence, we obtain

$$diam(A_{Fn}(t_0, \delta_\varepsilon)) \leq diam(A_F(t_0)) + \frac{3 \cdot \varepsilon}{4} < diam(A_F(t_0)) + \varepsilon.$$

Therefore, we infer that (2.30) holds true.

Now, we obtain by (2.30) and (2.27) that F_n has the limit average range at t_0 and $A_{F_n}(t_0) = A_F(t_0)$. Since t_0 has been taken arbitrarily, we have that F_n has the limit average range at t and

$$A_{F_n}(t) = A_F(t) \quad \text{for all} \quad t \in S_n. \tag{2.32}$$

Claim 7 The function F is differentiable almost everywhere on L_n. Indeed, we have that the function F_n has the limit average range at all $t \in S_n$, and since $\lambda(S_n) = \lambda(L_n)$, F_n has the limit average range almost everywhere on L_n. Clearly, the function

F_n has the limit average range at all $t \in (a_k^{(n)}, b_k^{(n)})$ and $k \in \mathbb{N}$. Thus, the function F_n has the limit average range almost everywhere on $[0, 1]$, and since F_n is also sAC we obtain by Lemma 2.4 that F_n is differentiable almost everywhere on $[0, 1]$. Then, there is a subset $Z_{F_n} \subset [0, 1]$ with $\lambda(Z_{F_n}) = 0$ such that $F_n'(t)$ exists for all $t \in [0, 1]\backslash Z_{F_n}$. Hence, we obtain by Lemma 2.1 that

$$\{F_n'(t)\} = A_{F_n}(t) \quad \text{for all} \quad t \in [0, 1]\backslash Z_{F_n}.$$

The last equality together with (2.32) and Lemma 2.1 yields that $F'(t)$ exists and $F'(t) = F_n'(t)$ for all $t \in S_n\backslash Z_{F_n}$, and since

$$\lambda(S_n\backslash Z_{F_n})) = \lambda(S_n) = \lambda(L_n),$$

the function F is differentiable almost everywhere on L_n.

Finally, since L_n has been taken arbitrarily, F is differentiable almost everywhere on $L = \cup_{n=1}^{\infty} L_n$, and since $\lambda(L) = \lambda([0, 1])$, the function F is differentiable almost everywhere on $[0, 1]$ and the proof is finished. □

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