Monatsh Math (2014) 173:593–603
DOI 10.1007/s00605-013-0506-1

Integral representations for the solutions of infinite order of the stationary Schrödinger equation in a cone

Lei Qiao · Yudong Ren

Received: 31 January 2013 / Accepted: 23 April 2013 / Published online: 7 May 2013

Abstract Each solution of infinite order of the stationary Schrödinger equation defined in a smooth cone and continuous in the closure can be represented in terms of the modified Poisson integral and an infinite series vanishing continuously on the boundary.

Keywords Integral representation · Stationary Schrödinger equation · Generalized harmonic function · Cone

Mathematics Subject Classification (2000) 31B10 · 31C05

1 Introduction and results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n (n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set $\mathbf{S}$ in $\mathbf{R}^n$ are denoted by $\partial\mathbf{S}$ and $\overline{\mathbf{S}}$, respectively.

Communicated by A. Jüngel.

This work is supported by the National Natural Science Foundation of China (Grant No. 11226093).

L. Qiao · Y. Ren (✉)
Department of Mathematics and Information Science,
Henan University of Economics and Law, Zhengzhou 450002, China
e-mail: yudong83@163.com

L. Qiao
e-mail: qiaocqu@163.com

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(X, x_n) = (x_1, x_2, \ldots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$.

For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with center at P and radius r in $\mathbf{R}^n$. $S_r = \partial B(O, r)$. The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Lambda, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half space $\mathbf{R}_+ \times \mathbf{S}_+^{n-1} = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by $\mathbf{T}_n$.

By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}$. We call it a cone. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega; r)$ we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$ which is $\partial C_n(\Omega) - \{O\}$.

Furthermore, we denote by $d\sigma_Q$ (resp. dS_r) the $(n-1)$-dimensional volume elements induced by the Euclidean metric on $\partial C_n(\Omega)$ (resp. S_r) and by dw the elements of the Euclidean volume in $\mathbf{R}^n$.

Let $\mathscr{A}_a$ denote the class of nonnegative radial potentials $a(P)$, i.e. $0 \le a(P) = a(r)$, $P = (r, \Theta) \in C_n(\Omega)$, such that $a \in L^b_{loc}(C_n(\Omega))$ with some $b > n/2$ if $n \ge 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

This article is devoted to the stationary Schrödinger equation

$$Sch_a u(P) = -\Delta u(P) + a(P)u(P) = 0 \quad \text{for} \quad P \in C_n(\Omega),$$

where Δ is the Laplace operator and $a \in \mathscr{A}_a$. These solutions are called a-harmonic functions or generalized harmonic functions (g.h.f.s) associated with the operator Sch_a. Note that they are classical harmonic functions in the case $a = 0$. Under these assumptions the operator Sch_a can be extended in the usual way from the space $C_0^\infty(C_n(\Omega))$ to an essentially self-adjoint operator on $L^2(C_n(\Omega))$ (see [11, Ch. 13]). We will denote it Sch_a as well. This last one has a Green function $G(\Omega, a)(P, Q)$ which is positive on $C_n(\Omega)$ and its inner normal derivative $\partial G(\Omega, a)(P, Q)/\partial n_Q \ge 0$, where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$. We denote this derivative $P(\Omega, a)(P, Q)$, which is called the Poisson a-kernel with respect to $C_n(\Omega)$.

Let Δ^* be a Laplace-Beltrami operator (spherical part of the Laplace) on $\Omega \subset \mathbf{S}^{n-1}$ and λ_j $(j = 1, 2, 3\ldots, 0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots)$ be the eigenvalues of the eigenvalue problem for Δ^* on Ω (see, e.g., [12, p. 41])

$$\Delta^*\varphi(\Theta) + \lambda\varphi(\Theta) = 0 \quad \text{in} \quad \Omega,$$
$$\varphi(\Theta) = 0 \quad \text{on} \quad \partial\Omega.$$

Corresponding eigenfunctions are denoted by $\varphi_j(\Theta)$. We set $\lambda_0 = 0$, norm the eigenfunctions in $L^2(\Omega)$ and $\varphi_1(\Theta) > 0$.

In order to ensure the existences of λ_j $(j = 1, 2, 3, \ldots)$. We put a rather strong assumption on Ω: if $n \ge 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see [5, p. 88–89] for the definition of $C^{2,\alpha}$-domain), $\varphi_j \in C^2(\overline{\Omega})$ $(j = 1, 2, 3, \dots)$ and $\partial\varphi_1/\partial n > 0$ on $\partial\Omega$ (here and below, $\partial/\partial n$ denotes differentiation along the interior normal).

Here well-known estimates (see, e.g., [2], and also [4, p. 120 and p. 126–128] imply the following inequalities:

$$M_1 j^{\frac{2}{n-1}} \leq \lambda_j \quad (j = 1, 2, 3, \dots) \tag{1.1}$$

and

$$|\varphi_j(\Theta)| \leq M_2 j^{\frac{1}{2}} \quad (\Theta \in \Omega,\ j = 1, 2, 3, \dots), \tag{1.2}$$

where M_1 and M_2 are two positive constants.

Let $V_j(r)$ and $W_j(r)$ stand, respectively, for the increasing and non-increasing, as $r \to +\infty$, solutions of the equation

$$-Q''(r) - \frac{n-1}{r}Q'(r) + \left(\frac{\lambda_j}{r^2} + a(r)\right)Q(r) = 0, \quad 0 < r < \infty, \tag{1.3}$$

normalized under the condition $V_j(1) = W_j(1) = 1$.

We will also consider the class $\mathscr{B}_a$, consisting of the potentials $a \in \mathscr{A}_a$ such that there exists the finite limit $\lim_{r\to\infty} r^2 a(r) = k \in [0, \infty)$, moreover, $r^{-1}|r^2a(r) - k| \in L(1, \infty)$. If $a \in \mathscr{B}_a$, then the g.h.f.s are continuous (see [13]).

In the rest of paper, we assume that $a \in \mathscr{B}_a$ and we shall suppress this assumption for simplicity. Meanwhile, we use the standard notations $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$ and $[d]$ is the integer part of d, where d is a positive real number.

Denote

$$\iota^{\pm}_{j,k} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(k + \lambda_j)}}{2} \quad (j = 0, 1, 2, 3 \dots).$$

The solutions to Eq. (1.3) have the asymptotic (see [6])

$$V_j(r) \sim M_3 r^{\iota^{+}_{j,k}},\ W_j(r) \sim M_4 r^{\iota^{-}_{j,k}}, \text{ as } r \to \infty, \tag{1.4}$$

where M_3 and M_4 are some positive constants.

Further, we have

$$\iota^{+}_{j,k} \geq \iota^{+}_{j,0} > M_5 j^{\frac{1}{n-1}} \quad (j = 1, 2, 3 \dots) \tag{1.5}$$

from (1.1), where M_5 is a positive constant independent of j.

If $a \in \mathscr{A}_a$, it is known that the following expansion for the Green function $G(\Omega, a)(P, Q)$ (see [3, Ch. 11])

$$G(\Omega, a)(P, Q) = \sum_{j=0}^{\infty} \frac{1}{\chi'(1)} V_j(\min\{r, t\}) W_j(\max\{r, t\}) \varphi_j(\Theta)\varphi_j(\Phi),$$

where $P = (r, \Theta)$, $Q = (t, \Phi)$, $r \neq t$ and $\chi'(t) = w\,(W_1(r), V_1(r))\,|_{r=1}$ is their Wronskian. This series converges uniformly if either $r \leq st$ or $t \leq sr$ $(0 < s < 1)$. In the case $a = 0$, this expansion coincides with the well-known result by Lelong-Ferrand (see [9]).

For a nonnegative integer m and two points $P = (r, \Theta)$, $Q = (t, \Phi) \in C_n(\Omega)$, we put

$$K(\Omega, a, m)(P, Q) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ \widetilde{K}(\Omega, a, m)(P, Q) & \text{if } 1 \leq t < \infty, \end{cases}$$

where

$$\widetilde{K}(\Omega, a, m)(P, Q) = \sum_{j=0}^{m} \frac{1}{\chi'(1)} V_j(r) W_j(t) \varphi_j(\Theta) \varphi_j(\Phi).$$

To obtain the modified Poisson integral representation for the Schrödinger operator in a cone, we use the following modified kernel function defined by

$$G(\Omega, a, m)(P, Q) = G(\Omega, a)(P, Q) - K(\Omega, a, m)(P, Q)$$

for two points $P = (r, \Theta)$, $Q = (t, \Phi) \in C_n(\Omega)$.

Write

$$U(\Omega, a, m; u)(P) = \int_{S_n(\Omega)} P(\Omega, a, m)(P, Q) u(Q) d\sigma_Q,$$

where

$$P(\Omega, a, m)(P, Q) = \frac{\partial G(\Omega, a, m)(P, Q)}{\partial n_Q}, \quad P(\Omega, a, 0)(P, Q) = P(\Omega, a)(P, Q)$$

and $u(Q)$ is a continuous function on $\partial C_n(\Omega)$.

Now we define the function $\rho(R)$ under consideration. Hereafter, the function $\rho(R)$ (≥ 1) is always supposed to be nondecreasing and continuously differentiable on the interval $[0, +\infty)$. We assume further that

$$\epsilon_0 = \limsup_{R \to \infty} \frac{(\iota^+_{[\rho(R)]+1,k})' R \ln R}{\iota^+_{[\rho(R)]+1,k}} < 1. \tag{1.6}$$

Remark $\iota^+_{[\rho(R)]+1,k}$ in (1.6) is not the function $V_j(R)$. For any ϵ $(0 < \epsilon < 1 - \epsilon_0)$, there exists a sufficiently large positive number R_ϵ such that $R > R_\epsilon$, by (1.5) and (1.6) we have

$$M(\rho(R))^{\frac{1}{n-1}} < \iota^+_{[\rho(R)]+1,k} < \iota^+_{[\rho(e)]+1,k} (\ln R)^{\epsilon_0 + \epsilon},$$

where M is a positive constant.

For positive real numbers β, we denote $\mathcal{C}_{\Omega,\beta,a}$ the class of all measurable functions $f(t,\Phi)$ $(Q=(t,\Phi)\in C_n(\Omega))$ satisfying the following inequality

$$\int_{C_n(\Omega)} \frac{|f(t,\Phi)|\varphi_1}{1+V_{[\rho(t)]+1}(t)t^{n+\beta-1}}dw<\infty \tag{1.7}$$

and the class $\mathcal{D}_{\Omega,\beta,a}$, consists of all measurable functions $g(t,\Phi)$ $(Q=(t,\Phi)\in S_n(\Omega))$ satisfying

$$\int_{S_n(\Omega)} \frac{|g(t,\Phi)|V_1(t)W_1(t)}{1+\chi'(t)V_{[\rho(t)]+1}(t)t^{n+\beta-2}}\frac{\partial\varphi_1}{\partial n}d\sigma_Q<\infty, \tag{1.8}$$

where $\chi'(t)=w\,(W_1(r),V_1(r))\,|_{r=t}$ is their Wronskian.

We will also consider the class of all continuous functions $u(t,\Phi)$ $((t,\Phi)\in\overline{C_n(\Omega)})$ generalized harmonic in $C_n(\Omega)$ with $u^+(t,\Phi)\in\mathcal{C}_{\Omega,\beta,a}$ $((t,\Phi)\in C_n(\Omega))$ and $u^+(t,\Phi)\in\mathcal{D}_{\Omega,\beta,a}$ $((t,\Phi)\in S_n(\Omega))$ is denoted by $\mathcal{E}_{\Omega,\beta,a}$.

Next we define the order of g.h.f, which is similar to the F. Riesz' definition for the order of classical harmonic function (see [7, Definition 4.1]). We shall say that a g.h.f.-$u(P)$ $(P=(r,\Theta)\in C_n(\Omega))$ is of order λ if

$$\lambda=\limsup_{r\to\infty}\frac{\log\left(\sup_{C_n(\Omega)\cap S_r}|u|\right)}{\log r}.$$

If $\lambda<\infty$, then u is said to be of finite order.

In case $\lambda<\infty$, about the solutions of the Dirichlet problem for the Schrödinger operator with continuous data in $\mathbf{T}_n$, we refer the readers to the paper by Kheyfits (see [8]).

Motivated by Kheyfits's conclusions, we prove the following results for the g.h.f.s of infinite order. In the case $a=0$, we refer readers to the paper by Qiao (see [10]).

Theorem 1 *If $u\in\mathcal{E}_{\Omega,\beta,a}$, then $u\in\mathcal{D}_{\Omega,\beta,a}$.*

Theorem 2 *If $u\in\mathcal{E}_{\Omega,\beta,a}$, then the following properties hold:*

(I) *$U(\Omega,a,[\rho(t)];u)(P)$ is a g.h.f. on $C_n(\Omega)$ and can be continuously extended to $\overline{C_n(\Omega)}$ such that $U(\Omega,a,[\rho(t)];u)(P)=u(P)$ for $P=(r,\Theta)\in S_n(\Omega)$.*

(II) *There exists an infinite series $h(P)=\sum_{j=1}^{\infty}A_jV_j(r)\varphi_j(\Theta)$ vanishing continuously on $\partial C_n(\Omega)$ such that*

$$u(P) = U(\Omega, a, [\rho(t)]; u)(P) + h(P)$$

for $P = (r, \Theta) \in C_n(\Omega)$*, where* A_j $(j = 1, 2, 3, \ldots)$ *is a constant.*

2 Lemmas

The following Lemma generalizes the Carleman's formula (referring to the holomorphic functions in the half space) (see [1]) to the g.h.f.s in a cone, which is due to Levin and Kheyfits (see [3, Ch. 11]).

Lemma 1 *If* $u(t, \Phi)$ *is a g.h.f. on a domain containing* $C_n(\Omega; (1, R))$*, then*

$$m_+(R) + \int_{S_n(\Omega;(1,R))} u^+\Psi(t)\frac{\partial\varphi_1}{\partial n}d\sigma_Q + M_6 + \frac{W_1(R)}{V_1(R)}M_7 = m_-(R) + \int_{S_n(\Omega;(1,R))} u^-\Psi(t)\frac{\partial\varphi_1}{\partial n}d\sigma_Q,$$

where

$$\Psi(t) = W_1(t) - \frac{W_1(R)}{V_1(R)}V_1(t), \quad m_\pm(R) = \int_{S_n(\Omega;R)} \frac{\chi'(R)}{V_1(R)}u^\pm\varphi_1 dS_R,$$

$$M_6 = \int_{S_n(\Omega;1)} u\varphi_1 W_1'(1) - W_1(1)\varphi_1\frac{\partial u}{\partial n}dS_1 \quad \text{and} \quad M_7 = \int_{S_n(\Omega;1)} V_1(1)\varphi_1\frac{\partial u}{\partial n} - u\varphi_1 V_1'(1)dS_1.$$

Lemma 2 (see [3, Ch. 11]) *For a non-negative integer m, we have*

$$|P(\Omega, a, m)(P, Q)| \leq M_8 V_{m+1}(2r)\frac{W_{m+1}(t)}{t}\varphi_1(\Theta)\frac{\partial\varphi_1(\Phi)}{\partial n_\Phi} \tag{2.1}$$

for any $P = (r, \Theta) \in C_n(\Omega)$ *and* $Q = (t, \Phi) \in S_n(\Omega)$ *satisfying* $2r \leq t$*, where* M_8 *is a constant depending only n.*

Lemma 3 *If* $h(r, \Theta)$ *is a g.h.f. in* $C_n(\Omega)$ *vanishing continuously on* $\partial C_n(\Omega)$*, then*

$$h(r, \Theta) = \sum_{j=1}^{\infty} B_j V_j(r)\varphi_j(\Theta), \tag{2.2}$$

where the series converges uniformly and absolutely in any compact set of $\overline{C_n(\Omega)}$*, and* B_j $(j = 1, 2, 3, \ldots)$ *is a constant satisfying*

$$B_j V_j(r) = \int_{\Omega} h(r, \Theta)\varphi_j(\Theta) dS_1 \tag{2.3}$$

for every r $(0 < r < \infty)$.

Proof Set

$$y_j(r) = \int_{\Omega} h(r, \Theta)\varphi_j(\Theta) dS_1 \quad (j = 1, 2, 3, \ldots).$$

Making use of the assumptions on h and self-adjoint of the Laplace-Beltrami operator Δ^*, one can check directly (by differentiating under the integral sign) that the functions y_j $(j = 1, 2, 3, \ldots)$ satisfy the Eq. (1.3). This equation has a general solution $y_j(r) = B_j V_j(r) + D_j W_j(r)$, where B_j and D_j are constants independent of r $(j = 1, 2, 3, \ldots)$. We note that $h(r, \Theta)$ converges uniformly to zero as $r \to 0$ and hence $\lim_{r\to 0} y_j(r) = 0$ $(j = 1, 2, 3, \ldots)$. Thus we see that $D_j = 0$ $(j = 1, 2, 3, \ldots)$. Since $y_j(r)$ takes the value $y_j(r_1)$ at $r = r_1$, we have

$$y_j(r) = \frac{V_j(r)}{V_j(r_1)} y_j(r_1)$$

for any r and r_1 $(0 < r, r_1 < R_1)$, where $0 < R_1 \leq +\infty$. In particular, if $R_1 = \infty$, then

$$\lim_{r\to\infty} \frac{y_j(r)}{V_j(r)} = B_j \quad (j = 1, 2, 3, \ldots) \tag{2.4}$$

exists.

Since $y_j(r_1) \to y_j(R_1)$ as $r_1 \to R_1$, we see that

$$y_j(r) = \frac{V_j(r)}{V_j(R_1)} y_j(R_1) \quad (j = 1, 2, 3, \ldots), \tag{2.5}$$

which gives

$$|y_j(r)| \leq M_2 w_n \left(\frac{r}{R_1}\right)^{\iota^+_{j,k}} j^{\frac{1}{2}} \sup_{\Theta\in\Omega} |h(R_1, \Theta)|$$

from (1.2) and (1.4), where w_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

Now we set

$$J = \max\left\{j;\ \iota^+_{j,k} - \iota^+_{l+1,k} < \frac{1}{2} M_5 j^{\frac{1}{n-1}}\right\} \quad (l = 1, 2, 3, \ldots).$$

The existence of this J is known from (1.5). Hence if we put

$$M_9 = w_n M_2^2 \left\{ \sum_{j=l+1}^{J} j \left(\frac{1}{2}\right)^{\iota^+_{j,k}-\iota^+_{m+1,k}} + \sum_{j=J+1}^{\infty} j \left(\frac{1}{2}\right)^{2^{-1} M_5 j^{\frac{1}{n-1}}} \right\}$$

and use (1.2), then from the completeness of $\{\varphi_j(\Theta)\}$ we can expand $h(r, \Theta)$ into the Fourier series

$$h(r, \Theta) = \sum_{j=1}^{\infty} y_j(r)\varphi_j(\Theta)$$

satisfying

$$\sum_{j=l+1}^{\infty} |y_j(r)||\varphi_j(\Theta)| \le M_9 \left(\frac{r}{R_1}\right)^{\iota^+_{l+1,k}} \sup_{\Theta\in\Omega} |h(R_1, \Theta)| \ \ (l = 1, 2, 3, \ldots) \tag{2.6}$$

on $C_n(\Omega; (0, \frac{R_1}{2}))$, where M_9 is a positive constant independent of r and R_1.

Take any compact H, $H \subset \overline{C_n(\Omega)}$ and a number R_1 satisfying $R_1 > 2\max\{r; (r, \Theta) \in H\}$. So we can represent $h(r, \Theta)$ as

$$h(r, \Theta) = \sum_{j=1}^{\infty} y_j(r)\varphi_j(\Theta), \tag{2.7}$$

where (r, Θ) is a point in H. Hence we observe in (2.5) that $y_j(r)$ is a number independent of R_1. Hence as $R_1 \to \infty$, we see from (2.4) that $y_j(r) = B_j V_j(r)$, which is (2.3). This and (2.7) give (2.2).

To prove the absolute and uniform convergence of (2.7) on H, see from (2.6) that

$$\sum_{j=l+1}^{\infty} |y_j(r)||\varphi_j(\Theta)| \le M_9 2^{-\iota^+_{l+1,k}} \sup_{\Theta\in\Omega} |h(R_1, \Theta)|,$$

which converges to 0 as $l \to \infty$. Then Lemma 3 is proved. □

3 Proof of Theorem 1

Since $u \in \mathcal{E}_{\Omega,\beta,a}$, we obtain by (1.7)

$$\int_1^{\infty} \frac{m_+(R)V_1(R)}{\chi'(R)V_{[\rho(R)]+1}(R)R^{n+\beta-1}} dR \le 2 \int_{C_n(\Omega)} \frac{u^+\varphi_1}{1 + V_{[\rho(t)]+1}(t)t^{n+\beta-1}} dw < \infty. \tag{3.1}$$

From (1.4) and (1.8), we conclude that

$$\begin{aligned}
&\int_1^\infty \frac{V_1(R)}{\chi'(R)V_{[\rho(R)]+1}(R)R^{n+\beta-1}} \int_{S_n(\Omega;(1,R))} u^+\Psi(t)\frac{\partial\varphi_1}{\partial n}d\sigma_Q dR \\
&\quad\le \frac{2\chi_{1,k}}{(\chi_{1,k}+\beta)\beta}\int_{S_n(\Omega)} \frac{u^+V_1(t)W_1(t)}{1+\chi'(t)V_{[\rho(t)]+1}(t)t^{n+\beta-2}}\frac{\partial\varphi_1}{\partial n}d\sigma_Q \\
&\quad< \infty,
\end{aligned} \tag{3.2}$$

where $\chi_{1,k} = \iota^+_{1,k} - \iota^-_{1,k}$.

It follows from (1.4), Remark and the L'hospital's rule

$$\lim_{t\to\infty}\frac{\chi'(t)V_{[\rho(t)]+1}(t)t^{n+\beta-2}}{W_1(t)}\int_t^\infty \frac{V_1(R)}{\chi'(R)V_{[\rho(R)]+1}(R)R^{n+\frac{\beta}{2}-1}}\left(\frac{W_1(t)}{V_1(t)}-\frac{W_1(R)}{V_1(R)}\right)dR = +\infty,$$

which yields that there exists a positive constant M_{10} such that for any $t \ge 1$,

$$\int_t^\infty \frac{V_1(R)}{\chi'(R)V_{[\rho(t)]+1}(R)R^{n+\frac{\beta}{2}-1}}\Psi(t)dR \ge \frac{M_{10}V_1(t)W_1(t)}{\chi'(t)V_{[\rho(t)]+1}(t)t^{n+\beta-2}}.$$

From (3.1), (3.2) and Lemma 1 we see that

$$\begin{aligned}
&M_{10}\int_{S_n(\Omega;(1,\infty))} \frac{u^-V_1(t)W_1(t)}{\chi'(t)V_{[\rho(t)]+1}(t)t^{n+\beta-2}}\frac{\partial\varphi_1}{\partial n}d\sigma_Q \\
&\le \int_{S_n(\Omega;(1,\infty))} u^-\int_t^\infty \frac{V_1(R)}{\chi'(R)V_{[\rho(t)]+1}(R)R^{n+\frac{\beta}{2}-1}}\Psi(t)dR\frac{\partial\varphi_1}{\partial n}d\sigma_Q \\
&< \infty.
\end{aligned}$$

Then Theorem 1 is proved from $|u| = u^+ + u^-$.

4 Proof of Theorem 2

Let l_1 be any positive number such that $l_1 \ge 2\beta$. For any fixed $P = (r,\Theta) \in C_n(\Omega)$, take a number σ satisfying $\sigma > \sigma_r = \max\{[2r]+1, \vartheta_r\}$, where $\vartheta_r = \exp(\frac{l_1}{\beta}\iota^+_{[\rho(e)]+1,k}2^{1+\epsilon_0+\epsilon}\ln 2r)^{\frac{1}{1-\epsilon_0-\epsilon}}$.

From the Remark we see that there exists a constant $M(r)$ dependent only on r such that $M(r) \ge (2r)^{\iota^+_{[\rho(i+1)]+1,k}}i^{-\frac{\beta}{l_1}}$ from $\sigma \ge \vartheta_r$.

By (1.4), (1.8), (2.1) and Theorem 1, we have

$$\begin{aligned}
&\int\limits_{S_n(\Omega;(\sigma,\infty))} |P(\Omega,a,[\rho(t)])(P,Q)||u(Q)|d\sigma_Q \\
&\leq M_8\varphi_1(\Theta)\sum_{i=\sigma_r}^{\infty}\int\limits_{S_n(\Omega;[i,i+1))}\frac{(2r)^{\iota^+_{[\rho(t)]+1,k}}}{t^{\frac{\beta}{l_1}}}\frac{|u(t,\Phi)|}{V_{[\rho(t)]+1}(t)t^{n-2+\frac{\beta}{l_1}}}d\sigma_Q \\
&\leq M_8\sum_{i=\sigma_r}^{\infty}\frac{(2r)^{\iota^+_{[\rho(i+1)]+1,k}}}{i^{\frac{\beta}{l_1}}}\int\limits_{S_n(\Omega;[i,i+1))}\frac{|u(t,\Phi)|}{V_{[\rho(t)]+1}(t)t^{n-2+\frac{\beta}{l_1}}}d\sigma_Q \\
&\leq M_8M(r)\varphi_1(\Theta)\int\limits_{S_n(\Omega;[\sigma_r,\infty))}\frac{|u(t,\Phi)|}{V_{[\rho(t)]+1}(t)t^{n-2+\frac{\beta}{l_1}}}d\sigma_Q \\
&< \infty.
\end{aligned}$$

Hence $U(\Omega,a,[\rho(t)];u)(P)$ is absolutely convergent and finite for any $P \in C_n(\Omega)$. Thus $U(\Omega,a,[\rho(t)];u)(P)$ is generalized harmonic on $C_n(\Omega)$.

Now we study the boundary behavior of $U(\Omega,a,[\rho(t)];u)(P)$. Let $Q' = (t',\Phi') \in \partial C_n(\Omega)$ be any fixed point and l_2 be any positive number such that $l_2 > t' + 1$.

Set $\chi_{S(l_2)}$ is the characteristic function of $S(l_2) = \{Q = (t,\Phi) \in \partial C_n(\Omega), t \leq l_2\}$ and write

$$U(\Omega,a,[\rho(t)];u)(P) = U'(P) - U''(P) + U'''(P),$$

where

$$\begin{aligned}
U'(P) &= \int\limits_{S_n(\Omega;(0,2l_2])} P(\Omega,a)(P,Q)u(Q)d\sigma_Q, \\
U''(P) &= \int\limits_{S_n(\Omega;(1,2l_2])} \frac{\partial K(\Omega,a,[\rho(t)])(P,Q)}{\partial n_Q}u(Q)d\sigma_Q
\end{aligned}$$

and

$$U'''(P) = \int\limits_{S_n(\Omega;(2l_2,\infty))} P(\Omega,a,[\rho(t)])(P,Q)u(Q)d\sigma_Q.$$

Notice that $U'(P)$ is the Poisson integral of $u(Q)\chi_{S(2l_2)}$, we have $\lim_{P\in C_n(\Omega),P\to Q'} U'(P) = u(Q')$. Since $\lim_{\Theta\to\Phi'}\varphi_j(\Theta) = 0$ $(j = 1,2,3\ldots)$ as $P = (r,\Theta) \to Q' = (t',\Phi') \in S_n(\Omega)$, we have $\lim_{P\in C_n(\Omega),P\to Q'} U''(P) = 0$ from the definition of the kernel function $K(\Omega,a,[\rho(t)])(P,Q)$. $U'''(P) = O(M(r)\varphi_1(\Theta))$ and therefore tends to zero.

So the function $U(\Omega, a, [\rho(t)]; u)(P)$ can be continuously extended to $\overline{C_n(\Omega)}$ such that

$$\lim_{P \in C_n(\Omega), P \to Q'} U(\Omega, a, [\rho(t)]; u)(P) = u(Q')$$

for any $Q' = (t', \Phi') \in \partial C_n(\Omega)$ from the arbitrariness of l_2.

So (I) is proved. Finally (I) and Lemma 3 give the conclusion of (II). Then we complete the proof of Theorem 2.

References

1. Carleman, T.: Über die Approximation analytischer Funktionen durch lineare Aggregate von vorgegebenen Potenzen. Ark. för Mat. Astr. och Fysik **17**, 1–30 (1923)
2. Cheng, S.Y., Li, P.: Heat kernel estimates and lower bound of eigenvalues, Comment. Math. Helvetici **5**6, 327–338 (1981)
3. Escassut, A., Tutschke, W., Yang, C.C.: Some topics on value distribution and differentiability in complex and P-adic analysis. In: Mathematics Monograph Series, vol. 11. Science Press Beijing, Beijing, pp. 323–397 (2008)
4. Essén, M., Lewis, J.L.: The generalized Ahlfors-Heins theorems in certain d-dimensional cones. Math. Scand. **3**3, 111–129 (1973)
5. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1977)
6. Hartman, P.: Ordinary Differential Equations. Wiley, New York (1964)
7. Hayman, W.K., Kennedy, P.B.: Subharmonic Functions, vol. 1. Academic Press, London (1976)
8. Kheyfits, A.: Dirichlet problem for the Schrödinger operator in a half-space with boundary data of arbitrary growth at infinity. Differ. Integr. Equ. **10**, 153–164 (1997)
9. Lelong-Ferrand, J.: Etude des fonctions subharmoniques positives dans un cylindre ou dans un cone. C. R. Acad. Sci. Paris, Ser A. **2**29(5), 340–341 (1949)
10. Qiao, L.: Integral representations for harmonic functions of infinite order in a cone. Results Math. **61**, 63–74 (2012)
11. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, vol. 3. Academic Press, London (1970)
12. Rosenblum, G., Solomyak, M., Shubin, M.: Spectral Theory of Differential Operators. VINITI, Moscow (1989)
13. Simon, B.: Schrödinger semigroups. Bull. Am. Math. Soc. **7**, 447–526 (1982)