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Metric discrepancy results for alternating geometric progressions

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Abstract The law of the iterated logarithm for discrepancies of $\{\theta^k x\}$ is proved for $\theta < -1$. When θ is not a power root of rational number, the limsup equals to 1/2. When θ is an odd degree power root of rational number, the limsup constants for ordinary discrepancy and star discrepancy are not identical.

Keywords Discrepancy · Lacunary sequence · Law of the iterated logarithm

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1 Introduction

A sequence $\{a_k\}$ of real numbers is said to be uniformly distributed mod 1 if

$$\frac{1}{N}{}^{\#}\{k \le N \mid \langle a_k \rangle \in [a', a)\} \to a - a', \quad (N \to \infty),$$

for all $0 \le a' < a < 1$, or equivalently

$$\frac{1}{N}{}^{\#}\{k \le N \mid \langle a_k \rangle \in [0, a)\} \to a, \quad (N \to \infty),$$

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for all $0 \le a < 1$, where $\langle x \rangle$ denotes the fractional part $x - [x]$ of a real number x. We can easily see that these convergences are uniform in a' and a, and hence we use the following discrepancies $D_N\{a_k\}$ and $D_N^*\{a_k\}$ to measure speed of convergence:

$$D_N\{a_k\} = \sup_{0 \le a' < a < 1} \left| \frac{1}{N} {}^\#\{k \le N \mid \langle a_k \rangle \in [a', a)\} - (a - a') \right|,$$
$$D_N^*\{a_k\} = \sup_{0 \le a < 1} \left| \frac{1}{N} {}^\#\{k \le N \mid \langle a_k \rangle \in [0, a)\} - a \right|.$$

One of the most well known results on asymptotic behavior of discrepancies is Chung–Smirnov theorem [13,29], which asserts the law of the iterated logarithm

$$\varlimsup_{N\to\infty} \frac{N D_N\{U_k\}}{\sqrt{2N \log\log N}} = \varlimsup_{N\to\infty} \frac{N D_N^*\{U_k\}}{\sqrt{2N \log\log N}} = \frac{1}{2}, \quad \text{a.s.},$$

for $[0, 1]$-valued uniformly distributed i.i.d. $\{U_k\}$. By various studies on lacunary series, it is known that a sequence $\{n_k x\}$ behaves like uniformly distributed i.i.d. when $\{n_k\}$ diverges rapidly. Actually Philipp [28] proved the following bounded law of the iterated logarithm by assuming the Hadamard gap condition $n_{k+1}/n_k \ge q > 1$:

$$\frac{1}{4\sqrt{2}} \le \varlimsup_{N\to\infty} \frac{N D_N\{n_k x\}}{\sqrt{2N \log\log N}} \le \frac{1}{\sqrt{2}} \left(166 + \frac{664}{q^{1/2} - 1} \right), \quad \text{a.e.}$$

Beside of a result by Dhompongsa [14] stating that the limsup equals to $\frac{1}{2}$ when $\{n_k\}$ satisfies very strong gap condition

$$\frac{\log(n_{k+1}/n_k)}{\log\log k} \to \infty, \quad (k \to \infty),$$

any concrete value of limsup for exponentially growing sequence was not determined before the recent result below on divergent positive geometric progressions $\{\theta^k x\}$.

Theorem 1 [15,16,19] *For $\theta > 1$, there exists a constant Σ_θ such that*

$$\varlimsup_{N\to\infty} \frac{N D_N\{\theta^k x\}}{\sqrt{2N \log\log N}} = \varlimsup_{N\to\infty} \frac{N D_N^*\{\theta^k x\}}{\sqrt{2N \log\log N}} = \Sigma_\theta, \quad \text{a.e.}$$

When θ satisfies

$$\theta^n \notin \mathbf{Q} \quad (n \in \mathbf{N}), \tag{1}$$

then

$$\Sigma_\theta = \frac{1}{2}.$$

When $\theta > 1$ does not satisfy the condition, we write θ in the following way:

$$\theta = \sqrt[r]{\frac{p}{q}} \quad \textit{where } r = \min\{n \in \mathbf{N} \mid \theta^n \in \mathbf{Q}\}, \quad p, q \in \mathbf{N}, \ \textit{and} \ \gcd(p, q) = 1. \tag{2}$$

In this case we have

$$\frac{1}{2} < \Sigma_\theta \leq \frac{1}{2}\sqrt{\frac{pq+1}{pq-1}}. \tag{3}$$

We can evaluate Σ_θ in the following cases:

$$\Sigma_\theta = \begin{cases} \frac{1}{2}\sqrt{\frac{pq+1}{pq-1}}, & \textit{if } p \textit{ and } q \textit{ are both odd;} \\ \frac{1}{2}\sqrt{\frac{p+1}{p-1}}, & \textit{especially if } p \textit{ is odd and } q = 1; \\ \frac{1}{2}\sqrt{\frac{(p+1)p(p-2)}{(p-1)^3}}, & \textit{if } p \geq 4 \textit{ is even and } q = 1; \\ \frac{\sqrt{42}}{9}, & \textit{if } p = 2 \textit{ and } q = 1; \\ \frac{\sqrt{22}}{9}, & \textit{if } p = 5 \textit{ and } q = 2. \end{cases}$$

The proof of the above theorem is given in [15] except for the proof of the inequality $\frac{1}{2} < \Sigma_\theta$ in (3), which is proved in [19].

In this paper we consider a sequence $\{\theta^k x\}$ for $\theta < -1$, i.e., a divergent alternating geometric progression. When $\theta < -1$ does not satisfy (1), we can write θ in the following way:

$$\theta = -\sqrt[r]{\frac{p}{q}} \quad \text{where } r = \min\{n \in \mathbf{N} \mid \theta^n \in \mathbf{Q}\}, \ p, q \in \mathbf{N}, \ \text{and} \ \gcd(p, q) = 1. \tag{4}$$

Now we are in a position to state our result.

Theorem 2 *For $\theta < -1$, there exist constants Σ_θ and Σ_θ^* such that*

$$\varlimsup_{N\to\infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta, \quad \textit{and} \quad \varlimsup_{N\to\infty} \frac{ND_N^*\{\theta^k x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta^*, \quad \textit{a.e.}$$

When θ satisfies (1), then

$$\Sigma_\theta = \Sigma_\theta^* = \frac{1}{2}. \tag{5}$$

Suppose that θ is given by (4). then we have

$$\frac{1}{2} < \Sigma_\theta \leq \Sigma_{|\theta|}, \tag{6}$$

where $\Sigma_{|\theta|}$ is a limsup constant for the law of the iterated logarithm for discrepancies of $\{|\theta|^k x\}$ whose existence is proved in Theorem 1.

Moreover we can evaluate Σ_θ and Σ_θ^* in the following cases.

1. *If r is even, then*

$$\Sigma_\theta = \Sigma_\theta^* = \Sigma_{|\theta|}. \tag{7}$$

2. *If r, p, and q are odd, then*

$$\Sigma_\theta = \Sigma_{|\theta|}. \tag{8}$$

3. *If r is odd, $p \geq 4$ is even, and $q = 1$, then we have (8).*
4. *If r is odd, $p = 5$, and $q = 2$, then we have (8).*
5. *If r is odd, p is odd, and $q = 1$, then we have*

$$\Sigma_\theta^* = \frac{1}{2}\sqrt{\frac{p(p^3 + 2p^2 - p + 2)}{(p-1)(p+1)^3}}. \tag{9}$$

   It is bigger than $\frac{1}{2}$ if $p = 3$, and less than $\frac{1}{2}$ otherwise.
6. *If r is odd and pq is even, then we have*

$$\Sigma_\theta^* = \frac{1}{2}. \tag{10}$$

7. *If r, p, and $q \geq 3$ are odd, then we have*

$$\Sigma_\theta^* < \frac{1}{2}. \tag{11}$$

We can easily derive a simple fact below.

Corollary 1 *Suppose that $\theta < -1$. We have $\Sigma_\theta^* \neq \Sigma_\theta$ if and only if θ is given by (4) with odd r.*

Our results show the first examples of sequences for which two limsups in the law of the iterated logarithm for ordinary discrepancy and star discrepancy are distinct constants.

We already have examples given by Aistleitner [3,4] in which two limsups are not equal on a set of positive measure and at lease one of these is not a constant. If we consider Erdős–Fortet sequence $\{(2^k-1)x\}$, it is known [24] that these limsups are not equal a.e. and the limsup for star-discrepancy is not a constant a.e. These sequences $\{n_k x\}$ has strong dependence which make the limsup non-constant.

In our theorem a geometric progression $\{\theta^k x\}$ is asymptotically stationary and its dependence is rather regular. This is the reason why we have constant limsups for both discrepancies. It is very interesting, even in the case of such regular dependence, we have possibility that these limsups are distinct.

We can also find a construction of irregular sequences with non-constant and identical limsups [18].

As to sufficient conditions to make two limsups equal to $\frac{1}{2}$, the Chung–Smirnov constant, Aistleitner [1,2] gave almost optimal results. We can also prove existence of a sequence $\{n_k x\}$ of arbitrarily slow divergence speed of $n_{k+1} - n_k$, for which we have the Chung–Smirnov constant $\frac{1}{2}$.

There are various sequences having constant limsups different from $\frac{1}{2}$. They are mainly given as variations of geometric progressions, and limsup constants are given as modifications of constants for geometric progressions. If we randomize the common ratio of geometric progression or replace the common ratio by periodic sequence, we [25] still can prove the law of the iterated logarithm for discrepancies and investigate limsup constants. It is also possible to investigate the union of finitely many geometric progressions.

In [26], Hardy–Littlewood–Pólya sequences are investigated. In [22], it is shown that the set of constants for arbitrary subsequence of positive diverging geometric progression $\{\theta^k x\}$ coincides with the interval $[\frac{1}{2}, \Sigma_\theta]$.

In [27], it is proved that any real number bigger than or equal to $\frac{1}{2}$ can be a limsup constant for some sequence satisfying the Hamadard's gap condition. It is also proved in [19] that any positive number less than $\frac{1}{2}$ can be a limsup constant for some sequence with bounded gaps. By these two results, we see that any positive number can be a limsup constant for some strictly monotonously increasing sequence of integers.

Before closing introduction, we mention results relating to permutations of sequences. In [17] it was found that the limsups are not invariant under permutations of sequences, and this phenomenon is studied extensively by Aistleitner–Berkes–Tichy [5–9]. See also [10–13].

2 Preliminary

Suppose that f is a real valued function defined on $\mathbf{R}$ satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x)\,dx = 0 \quad \int_0^1 f^2(x)\,dx < \infty, \tag{12}$$

and suppose that f is of bounded variation over $[0, 1]$. Put

$$\sigma^2(f,\theta)=\int_0^1 f^2(x)\,dx,$$

if θ satisfy (1),

$$\sigma^2(f,\theta)=\int_0^1 f^2(x)\,dx+2\sum_{k=1}^{\infty}\int_0^1 f(p^kx)f(q^kx)\,dx,$$

if θ is given by (2), and

$$\sigma^2(f,\theta)=\int_0^1 f^2(x)\,dx+2\sum_{k=1}^{\infty}\int_0^1 f((-1)^{rk}p^kx)f(q^kx)\,dx,$$

if θ is given by (4). We can verify that $\sigma^2(f,\theta)$ is well defined and

$$\lim_{d\to\infty}\sigma^2(f_d,\theta)=\sigma^2(f,\theta), \tag{13}$$

where f_d is a dth subsum of the Fourier series of f. If f is a trigonometric polynomial satisfying (12), we can prove

$$\overline{\lim_{N\to\infty}}\frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^{N}f(\theta^kx)\right|=\sigma(f,\theta),\quad \text{a.e.} \tag{14}$$

We can prove these by simplifying the proof given in [22]. The full proof of these also can be found in [21].

For $a, a'\in\mathbf{R}$ satisfying $0\le a-a'<1$, we put

$$\mathbf{I}_{a',a}(x)=\sum_{n\in\mathbf{Z}}\mathbf{1}_{[a',a)}(x+n)\quad\text{and}\quad \widetilde{\mathbf{I}}_{a',a}(x)=\mathbf{I}_{a',a}(x)-(a-a'),$$

where $\mathbf{1}_{[a',a)}$ is the indicator function of $[a',a)$. If $0\le a'<a<1$, we see $\mathbf{I}_{a',a}(x)=\mathbf{1}_{[a',a)}(\langle x\rangle)$. By this notation, we can write the discrepancies as below:

$$D_N\{a_k\}=\sup_{0\le a'<a<1}\left|\frac{1}{N}\sum_{k=1}^{N}\widetilde{\mathbf{I}}_{a',a}(a_k)\right|,\quad D_N^*\{a_k\}=\sup_{0\le a<1}\left|\frac{1}{N}\sum_{k=1}^{N}\widetilde{\mathbf{I}}_{0,a}(a_k)\right|.$$

We use the following result in case when $\varpi(n)=n$. It can be proved by modifying the method of Takahashi [30] and Philipp [28].

Proposition 1 [23] *Let* $\{n_k\}$ *be a sequence of real numbers satisfying*

$$n_1\neq 0,\qquad |n_{k+1}/n_k|>q>1\quad (k=1,2,\ldots), \tag{15}$$

and ϖ be a permutation of $\mathbf{N}$*, i.e., a bijection* $\mathbf{N} \to \mathbf{N}$*. Then for any dense countable set* $S \subset [0, 1)$*, we have*

$$\begin{aligned}
\varlimsup_{N\to\infty} \frac{ND_N\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} &= \sup_{S\ni a'<a\in S} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{a',a}(n_{\varpi(k)}x) \right| \\
&= \sup_{0\le a'<a<1} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{a',a}(n_{\varpi(k)}x) \right|, \\
\varlimsup_{N\to\infty} \frac{ND_N^*\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} &= \sup_{a\in S} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{0,a}(n_{\varpi(k)}x) \right| \\
&= \sup_{0\le a<1} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{0,a}(n_{\varpi(k)}x) \right|,
\end{aligned} \tag{16}$$

for almost every $x \in \mathbf{R}$*. If we denote the d-th subsum of the Fourier series of* $\widetilde{\mathbf{I}}_{a',a}$ *by* $\widetilde{\mathbf{I}}_{a',a;d}$*, we have*

$$\begin{aligned}
&\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{a',a}(n_{\varpi(k)}x) \right| \\
&= \lim_{d\to\infty} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{a',a;d}(n_{\varpi(k)}x) \right|
\end{aligned}$$

for almost every $x \in \mathbf{R}$*.*

We prove later that $\sigma(\widetilde{\mathbf{I}}_{a',a}, \theta)$ is continuous with respect to $(a', a) \in \mathbf{T}^2$. By Proposition 1 and relations (13) and (14), we can prove

$$\begin{aligned}
\varlimsup_{N\to\infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N\log\log N}} &= \Sigma_\theta := \sup_{0\le a'<a<1} \sigma(\widetilde{\mathbf{I}}_{a',a}, \theta), \quad \text{a.e.}, \\
\varlimsup_{N\to\infty} \frac{ND_N^*\{\theta^k x\}}{\sqrt{2N\log\log N}} &= \Sigma_\theta^* := \sup_{0\le a<1} \sigma(\widetilde{\mathbf{I}}_{0,a}, \theta), \quad \text{a.e.}
\end{aligned} \tag{17}$$

We here verify next two important properties of $\widetilde{\mathbf{I}}_{a',a}$. When $0 \le a - a' < 1$, we have

$$\widetilde{\mathbf{I}}_{a',a}(x) = \begin{cases} \widetilde{\mathbf{I}}_{\langle a'\rangle,\langle a\rangle}(x), & \text{if } \langle a'\rangle \le \langle a\rangle, \\ -\widetilde{\mathbf{I}}_{\langle a\rangle,\langle a'\rangle}(x), & \text{if } \langle a'\rangle > \langle a\rangle, \end{cases} \tag{18}$$

$$\widetilde{\mathbf{I}}_{a',a}(x) = \widetilde{\mathbf{I}}_{0,\langle a\rangle}(x) - \widetilde{\mathbf{I}}_{0,\langle a'\rangle}(x). \tag{19}$$

Actually, by $0 \le a - a' < 1$, there exists an integer m satisfying $m \le a' \le a < m+1$ or $m - 1 \le a' < m \le a < m + 1$. In the first case we have $\langle a'\rangle \le \langle a\rangle$ and $\widetilde{\mathbf{I}}_{a',a}(x) =$

$\widetilde{\mathbf{I}}_{\langle a'\rangle,\langle a\rangle}(x)$. In the second case we have $\langle a'\rangle > \langle a\rangle$ and $\sum_{n\in\mathbf{Z}}\mathbf{1}_{[a',a)}(x+n) = 1 - \sum_{n\in\mathbf{Z}}\mathbf{1}_{[a,a'+1)}(x+n)$ which produces $\widetilde{\mathbf{I}}_{a',a}(x) = -\widetilde{\mathbf{I}}_{a,a'+1}(x) = -\widetilde{\mathbf{I}}_{\langle a\rangle,\langle a'\rangle}(x)$ by $m \le a < a'+1 < m+1$. Here we have verified (18). The formula (19) can be easily verified by (18).

To evaluate $\sigma(\widetilde{\mathbf{I}}_{a',a},\theta)$, we use the following functions. For $x, y, \xi, \eta \in [0, 1)$, put

$$V(x,\xi) = x\wedge\xi - x\xi,$$
$$\widetilde{V}(x,y,\xi,\eta) = V(x,\xi) + V(y,\eta) - V(x,\eta) - V(y,\xi).$$

Clearly we have

$$\widetilde{V}(x,y,\xi,\eta) = \widetilde{V}(\xi,\eta,x,y) = -\widetilde{V}(y,x,\xi,\eta) = -\widetilde{V}(x,y,\eta,\xi) \tag{20}$$

and

$$0 \le \widetilde{V}(0,y,0,\eta) = V(y,\eta) \le V(\eta,\eta) \le \frac{1}{4}. \tag{21}$$

We have proved the next lemma in [15] in case $a = b$ and $a' = b'$. Although we can prove the next version in the same way, we give a proof for reader's convenience.

Lemma 1 *Let μ and ν are relatively prime positive integers. Then we have*

$$\int_0^1 \widetilde{\mathbf{I}}_{a',a}(\mu t)\widetilde{\mathbf{I}}_{b',b}(\nu t)\,dt = \frac{\widetilde{V}(\langle \mu a'\rangle, \langle \mu a\rangle, \langle \nu b'\rangle, \langle \nu b\rangle)}{\mu\nu} \tag{22}$$

for a, a', b, b' with $0 \le a - a' < 1$ and $0 \le b - b' < 1$.

Proof First we prove in the case $a' = b' = 0$, in which we have $0 \le a, b < 1$. Since the integrand has period 1, we have

$$\int_0^1 \widetilde{\mathbf{I}}_{0,a}(\mu t)\widetilde{\mathbf{I}}_{0,b}(\nu t)\,dt$$
$$= \frac{1}{\mu\nu}\sum_{k=0}^{\nu-1}\sum_{j=0}^{\mu-1}\int_0^1 \widetilde{\mathbf{I}}_{0,a}(\mu(t+j/\mu+k/\nu))\widetilde{\mathbf{I}}_{0,b}(\nu(t+j/\mu+k/\nu))\,dt.$$

By $\widetilde{\mathbf{I}}_{0,a}(\mu(t+j/\mu+k/\nu)) = \widetilde{\mathbf{I}}_{0,a}(\mu(t+k/\nu))$ and $\widetilde{\mathbf{I}}_{0,b}(\nu(t+j/\mu+k/\nu)) = \widetilde{\mathbf{I}}_{0,b}(\nu(t+j/\mu))$, we have $\int_0^1 \widetilde{\mathbf{I}}_{0,a}(\mu t)\widetilde{\mathbf{I}}_{0,b}(\nu t)\,dt = \int_0^1 \Gamma(t)\Delta(t)\,dt/\mu\nu$, where $\Gamma(t) = \sum_{k=0}^{\nu-1}\widetilde{\mathbf{I}}_{0,a}(\mu(t+k/\nu))$ and $\Delta(t) = \sum_{j=0}^{\mu-1}\widetilde{\mathbf{I}}_{0,b}(\nu(t+j/\mu))$. Since μ and ν are relatively prime, the transforms $k \mapsto \mu k$ on $\mathbf{Z}/\nu\mathbf{Z}$ and $j \mapsto \nu j$ on $\mathbf{Z}/\mu\mathbf{Z}$ are bijective. We therefore have

$$\Gamma(t) = \sum_{k=0}^{\nu-1} \widetilde{\mathbf{I}}_{0,a}(\mu t + k/\nu) \quad \text{and} \quad \Delta(t) = \sum_{j=0}^{\mu-1} \widetilde{\mathbf{I}}_{0,b}(\nu t + j/\mu).$$

Note that both of $1/\mu$ and $1/\nu$ are periods of $\Gamma\Delta$. Since μ and ν are relatively prime, there exist integers P and Q such that $1 = P\mu + Q\nu$. Thus $1/\mu\nu = P/\nu + Q/\mu$ is also a period, and hence

$$\int_0^1 \widetilde{\mathbf{I}}_{0,a}(\mu t)\widetilde{\mathbf{I}}_{0,b}(\nu t)\, dt = \int_0^{1/\mu\nu} \Gamma(t)\Delta(t)\, dt.$$

By $0 \le t < 1/\mu\nu$ and $0 \le k \le \nu - 1$, we have $0 \le \mu t + k/\nu < 1$. Hence $\mathbf{I}_{0,a}(\mu t + k/\nu) = 1$ if and only if $\mu t + k/\nu \in [0, a)$, i.e., $0 \le \mu\nu t + k < \nu a = [\nu a] + \langle \nu a \rangle$. The last condition holds if and only if $0 \le k < [\nu a\,]$, or $k = [\nu a]$ and $\mu\nu t < \langle \nu a \rangle$. Therefore

$$\Gamma(t) = [\nu a] + \mathbf{1}_{[0,\langle \nu a \rangle/\mu\nu)}(t) - \nu a = \mathbf{1}_{[0,\langle \nu a \rangle/\mu\nu)}(t) - \langle \nu a \rangle, \quad (0 \le t < 1/\mu\nu).$$

In the same way we can prove

$$\Delta(t) = \mathbf{1}_{[0,\langle \nu b \rangle/\mu\nu)}(t) - \langle \nu b \rangle, \quad (0 \le t < 1/\mu\nu).$$

By integrating $\Gamma\Delta$ over $[0, 1/\mu\nu)$, we have

$$\int_0^1 \widetilde{\mathbf{I}}_{0,a}(\mu t)\widetilde{\mathbf{I}}_{0,b}(\nu t)\, dt = \frac{V(\langle \mu a \rangle, \langle \nu b \rangle)}{\mu\nu}.$$

By using (19) and by noting $\langle \mu \langle a \rangle \rangle = \langle \mu a \rangle$ etc., we can verify (22) as below:

$$\begin{aligned}\int_0^1 \widetilde{\mathbf{I}}_{a',a}(\mu t)\widetilde{\mathbf{I}}_{b',b}(\nu t)\, dt &= \int_0^1 \widetilde{\mathbf{I}}_{0,\langle a \rangle}(\mu t)\widetilde{\mathbf{I}}_{0,\langle b \rangle}(\nu t)\, dt + \int_0^1 \widetilde{\mathbf{I}}_{0,\langle a' \rangle}(\mu t)\widetilde{\mathbf{I}}_{0,\langle b' \rangle}(\nu t)\, dt \\ &\quad - \int_0^1 \widetilde{\mathbf{I}}_{0,\langle a \rangle}(\mu t)\widetilde{\mathbf{I}}_{0,\langle b' \rangle}(\nu t)\, dt - \int_0^1 \widetilde{\mathbf{I}}_{0,\langle a' \rangle}(\mu t)\widetilde{\mathbf{I}}_{0,\langle b \rangle}(\nu t)\, dt \\ &= \frac{V(\langle \mu a \rangle, \langle \nu b \rangle) + V(\langle \mu a' \rangle, \langle \nu b' \rangle) - V(\langle \mu a \rangle, \langle \nu b' \rangle) - V(\langle \mu a' \rangle, \langle \nu b \rangle)}{\mu\nu}.\end{aligned}$$

If $0 \le a - a' < 1$ and $0 \le b - b' < 1$, by (22) we have

$$\int_0^1 \widetilde{\mathbf{I}}_{a',a}(t)\widetilde{\mathbf{I}}_{b',b}(t)\, dt = \widetilde{V}(\langle a' \rangle, \langle a \rangle, \langle b' \rangle, \langle b \rangle). \tag{23}$$

Hence we have

$$\widetilde{V}(\langle a'\rangle, \langle a\rangle, \langle a'\rangle, \langle a\rangle) = \int_0^1 \widetilde{\mathbf{I}}^2_{a',a}(t)\,dt = \int_0^1 \widetilde{\mathbf{I}}^2_{0,a-a'}(t)\,dt = V(a-a', a-a'),$$

if $0 \le a - a' < 1$, and

$$\widetilde{V}(\langle a'\rangle, \langle a\rangle, \langle a'\rangle, \langle a\rangle) = V(\langle a-a'\rangle, \langle a-a'\rangle) \qquad (a, a' \in \mathbf{R}),$$

since we have $\langle a\rangle = \big\langle a' + \langle a - a'\rangle\big\rangle$.

By noting $\widetilde{V}(x, y, x, y) = \widetilde{V}(y, x, y, x)$, we can show

$$\widetilde{V}(x, y, x, y) = V(|x-y|, |x-y|) = |x-y| - |x-y|^2, \quad (x, y \in [0, 1)).$$

Although we essentially proved the next lemma in [15], we give here a simple proof for it.

Lemma 2 *For any $x, y, \xi, \eta \in [0, 1)$, we have*

$$\widetilde{V}(x, y, \xi, \eta) \le V(\langle y-x\rangle, \langle \eta-\xi\rangle), \tag{24}$$

and

$$|\widetilde{V}(x, y, \xi, \eta)| \le \frac{1}{4}.$$

Proof Let us assume that $0 \le a - a' < 1$ and $0 \le b - b' < 1$. Since $\mathbf{I}_{a',a} = 1$ holds on $[0, 1)$ with measure $a - a'$, and $\mathbf{I}_{b',b} = 1$ holds on $[0, 1)$ with measure $b - b'$, we have

$$\int_0^1 \mathbf{I}_{a',a}(t)\mathbf{I}_{b',b}(t)\,dt \le (a-a') \wedge (b-b').$$

By adding $-(a-a')(b-b')$ to both sides and by noting (23), we have

$$\widetilde{V}(a', a, b', b) = \int_0^1 \widetilde{\mathbf{I}}_{a',a}(t)\widetilde{\mathbf{I}}_{b',b}(t)\,dt \le V(a-a', b-b')$$

Hence, the case $0 \le x \le y < 1$ and $0 \le \xi \le \eta < 1$ is already proved. If $0 \le y \le x < 1$ and $0 \le \xi \le \eta < 1$, we see that $0 \le y + 1 - x < 1$ and $\langle y+1\rangle = y$, and hence by (22) and the above inequality we have

$$\widetilde{V}(x, y, \xi, \eta) = \int_0^1 \widetilde{\mathbf{I}}_{x,y+1}(t)\widetilde{\mathbf{I}}_{\xi,\eta}(t)\,dt \le V(y+1-x, \xi-\eta) = V(\langle y-x\rangle, \langle \xi-\eta\rangle).$$

The other cases can be proved in the same way. The second inequality is clear from (20), (21), and (24).

By denoting $\sigma(\widetilde{\mathbf{I}}_{a',a},\theta)$ by $\sigma_{\theta;a',a}$, we here give a series expansion of $\sigma^2_{\theta;a',a}$. Firstly, let us consider the case when θ satisfies (1). Then we have

$$\sigma^2_{\theta;a',a} = \int_0^1 \widetilde{\mathbf{I}}^2_{a',a}(t)\,dt = |a-a'| - |a-a'|^2 \le \frac{1}{4},$$

where the equality holds if and only if $a - a' = \frac{1}{2}$. Clearly $\sigma_{\theta;a',a}$ is continuous with respect to $(a',a) \in \mathbf{T}^2$. Hence by (17), we have (5).

Secondly, let us consider the case when θ is given by (4).

If r is even, then we have $\sigma_{\theta;a',a} = \sigma_{|\theta|;a',a}$. We have already proved in [15] that $\sigma_{|\theta|;a',a}$ is continuous with respect to $(a',a) \in \mathbf{T}^2$. Thus the equality (7) follows from (17).

From now on we assume that r is odd.

Since we have $\mathbf{I}_{a',a}(-t) = \mathbf{I}_{1-a,1-a'}(t)$ if $\langle -t\rangle \ne \langle a\rangle, \langle a'\rangle$, we have $\widetilde{\mathbf{I}}_{a',a}(-t) = \widetilde{\mathbf{I}}_{1-a,1-a'}(t)$ a.e. and hence we have

$$\begin{aligned}\int_0^1 \widetilde{\mathbf{I}}_{a',a}(-p^{2k-1}x)\widetilde{\mathbf{I}}_{a',a}(q^{2k-1}x)\,dx &= \int_0^1 \widetilde{\mathbf{I}}_{1-a,1-a'}(p^{2k-1}x)\widetilde{\mathbf{I}}_{a',a}(q^{2k-1}x)\,dx \\ &= \frac{1}{(pq)^{2k-1}}\widetilde{V}(\langle -p^{2k-1}a\rangle, \langle -p^{2k-1}a'\rangle, \langle p^{2k-1}a'\rangle, \langle p^{2k-1}a\rangle),\end{aligned}$$

and

$$\begin{aligned}\sigma^2_{\theta;a',a} &= \widetilde{V}(a',a,a',a) \\ &\quad + 2\sum_{k=1}^{\infty}\Bigg(\frac{1}{p^{2k-1}q^{2k-1}}\widetilde{V}(\langle -p^{2k-1}a\rangle, \langle -p^{2k-1}a'\rangle, \langle q^{2k-1}a'\rangle, \langle q^{2k-1}a\rangle) \\ &\quad + \frac{1}{p^{2k}q^{2k}}\widetilde{V}(\langle p^{2k}a'\rangle, \langle p^{2k}a\rangle, \langle q^{2k}a'\rangle, \langle q^{2k}a\rangle)\Bigg). \end{aligned} \tag{25}$$

Therefore by applying (24) and by noting $\langle\langle A\rangle - \langle B\rangle\rangle = \langle A - B\rangle$ for $A, B \in \mathbf{R}$, we have

$$\begin{aligned}\sigma^2_{\theta;a',a} &\le V(a-a', a-a') + 2\sum_{k=1}^{\infty}\frac{1}{p^kq^k}V(\langle p^k(a-a')\rangle, \langle q^k(a-a')\rangle) \\ &= \sigma^2_{|\theta|;0,a-a'} \le \Sigma^2_{|\theta|},\end{aligned}$$

and by taking the supremum for a and a', we have $\Sigma_\theta \le \Sigma_{|\theta|}$, i.e., the upper bound estimate part of the inequality (6). Since $\widetilde{V}(x,y,\xi,\eta)$ is bounded and

$\widetilde{V}(\langle(-p)^k a'\rangle, \langle(-p)^k a\rangle, \langle q^k a'\rangle, \langle q^k a\rangle)$ is uniformly continuous with respect to $(a', a) \in \mathbf{T}^2$, we see that $\sigma^2_{\theta;a',a}$ is also uniformly continuous in (a', a).

By putting $a' = 0$, and by noting

$$\widetilde{V}(\langle -p^{2k-1}a\rangle, 0, 0, \langle q^{2k-1}a\rangle) = -V(\langle -p^{2k-1}a\rangle, \langle q^{2k-1}a\rangle)$$

and

$$\widetilde{V}(0, \langle p^{2k}a\rangle, 0, \langle q^{2k}a\rangle) = V(\langle p^{2k}a\rangle, \langle q^{2k}a\rangle),$$

we have

$$\sigma^2_{\theta;0,a} = V(a, a) + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{p^k q^k} V(\langle(-p)^k a\rangle, \langle q^k a\rangle). \tag{26}$$

For $0 \le x, y < 1$, we can verify $V(x, y) = V(1 - x, 1 - y)$. Hence we have

$$\begin{aligned} V(\langle(-p)^k(1-a)\rangle, \langle q^k(1-a)\rangle) &= V(1 - \langle(-p)^k a\rangle, 1 - \langle q^k a\rangle) \\ &= V(\langle(-p)^k a\rangle, \langle q^k a\rangle), \end{aligned}$$

and thereby we have

$$\sigma_{\theta;0,a} = \sigma_{\theta;0,1-a} \tag{27}$$

We here prepare a lemma.

Lemma 3 *Let $p \ge 2$ be an integer. For $0 < a < \frac{1}{2}$ and $0 \le t < 1$, it holds*

$$-V(t, a) + \frac{1}{p} V(\langle -pt\rangle, a) \le \frac{a(1-2a)}{p},$$

where equality holds if and only if $t = 1 - \frac{a}{p}$.

Proof Put $h(t) = -pV(t, a) + V(\langle -pt\rangle, a)$. For $t < a$, we have

$$h'(t) = -p + pa + pa + \begin{cases} -p \\ 0 \end{cases} \le p(2a - 1) < 0.$$

Thus $h(t)$ strictly increases on $[0, a]$ and $h(t) < h(a)$ for $t < a$. For $t > a$, we have

$$h'(t) = pa + pa + \begin{cases} -p \\ 0 \end{cases} = \begin{cases} (2a-1)p < 0, & \langle -pt\rangle < a, \\ 2pa > 0, & \langle -pt\rangle > a. \end{cases}$$

Hence $h'(t) < 0$ on $(1 - \frac{a}{p}, 1)$ and $h'(t) > 0$ on $(1 - \frac{1}{p}, 1 - \frac{a}{p})$. Hence we have $h(t) \le h(1 - \frac{a}{p}) = a(1 - 2a)$ on $(1 - \frac{1}{p}, 1)$ and have equality only if $t = 1 - \frac{a}{p}$.

Assume that $a \le t < 1 - \frac{1}{p}$ and take a positive integer k satisfying $1 - \frac{1}{p} \le t + \frac{k}{p} < 1$. Since $-pV(t, a)$ strictly increases in $t \in [a, 1)$, we have $-pV(t, a) < -pV(t + \frac{k}{p}, a)$. Clearly we have $V(\langle -pt \rangle, a) = V(\langle -p(t + \frac{k}{p}) \rangle, a)$ and hence $h(t) \le h(t + \frac{k}{p}) \le a(1 - 2a)$.

By assuming that p and q are coprime positive integers, we put

$$B_{p,q;l}(a) = \frac{-1}{(pq)^{2l+1}} V(\langle -p^{2l+1}a \rangle, \langle q^{2l+1}a \rangle) + \frac{1}{(pq)^{2l+2}} V(\langle p^{2l+2}a \rangle, \langle q^{2l+2}a \rangle).$$

Then we have

$$\sigma^2_{-p/q;0,a} = V(a, a) + 2 \sum_{l=0}^{\infty} B_{p,q;l}(a). \tag{28}$$

It is clear from $0 \le V(a', a) \le \frac{1}{4}$ that

$$B_{p,q;l}(a) \le \frac{1}{4p^{2l+2}q^{2l+2}}. \tag{29}$$

Therefore we have

$$\sigma^2_{-p/q;0,a} \le a(1 - a) + 2B_{p,q;0}(a) + \cdots + 2B_{p,q;L-1}(a) + \frac{1}{2p^{2L}q^{2L}(p^2q^2 - 1)}, \tag{30}$$

for $L = 0, 1, 2, \ldots$, and especially we have

$$\sigma^2_{-p/q;0,a} \le a(1 - a) + \frac{1}{2(p^2q^2 - 1)} = h_{\mathrm{I}}(a), \tag{31}$$

$$\sigma^2_{-p/q;0,a} \le a(1 - a) + 2B_{p,q;0} + \frac{1}{2p^2q^2(p^2q^2 - 1)}. \tag{32}$$

When $q = 1$, by applying Lemma 3, we have

$$B_{p,1;l}(a) \le \frac{1}{p^{2l+2}} a(1 - 2a), \quad (a \in (0, \tfrac{1}{2})), \tag{33}$$

where the equality holds if and only if $\langle -p^{2l+1}a \rangle = 1 - \frac{a}{p}$. Thus we have

$$\sigma^2_{-p;0,a} \le a(1 - a) + 2B_{p,1;0}(a) + \cdots + 2B_{p,1;L-1}(a) + \frac{2}{p^{2L}(p^2 - 1)} a(1 - 2a), \quad (a \in (0, \tfrac{1}{2})) \tag{34}$$

for $L = 0, 1, 2, \ldots$, where the equality holds if $\langle -p^{2l+1}a\rangle = 1 - \frac{a}{p}$ holds for every l. Especially, for $a \in (0, \frac{1}{2})$, we have

$$\sigma^2_{-p;0,a} \le a(1-a) + \frac{2}{p^2-1}a(1-2a), \tag{35}$$

$$\sigma^2_{-p;0,a} \le a(1-a) + 2B_{p,1;0}(a) + \frac{2}{p^2(p^2-1)}a(1-2a). \tag{36}$$

3 Proof of the equality (8)

Firstly suppose that θ satisfies (4) with $q = 1$, even $p \ge 4$, and odd r. In this case, we have $\sigma_{\theta;a',a} = \sigma_{-p;a',a}$. It is proved in [15] that

$$\Sigma_p = \sigma_{p;0,a_p}, \quad \text{where } a_p = \frac{p-2}{2(p-1)}.$$

Note that $\langle p^k a_p\rangle = a_p$. If we put

$$\tilde{a}'_p = \frac{p}{2(p^2-1)}, \quad \tilde{a}_p = \frac{p}{2(p^2-1)} + \frac{p-2}{2(p-1)} = \frac{1}{2} - \frac{1}{2(p^2-1)},$$

we can verify

$$\langle -p\tilde{a}'_p\rangle = \tilde{a}_p, \quad \langle -p\tilde{a}_p\rangle = \tilde{a}'_p, \quad \tilde{a}_p - \tilde{a}'_p = a_p,$$

and

$$\widetilde{V}(\langle -p^{2k-1}\tilde{a}_p\rangle, \langle -p^{2k-1}\tilde{a}'_p\rangle, \langle \tilde{a}'_p\rangle, \langle \tilde{a}_p\rangle) = \widetilde{V}(\tilde{a}'_p, \tilde{a}_p, \tilde{a}'_p, \tilde{a}_p) = V(a_p, a_p),$$
$$\widetilde{V}(\langle p^{2k}\tilde{a}'_p\rangle, \langle p^{2k}\tilde{a}_p\rangle, \langle \tilde{a}'_p\rangle, \langle \tilde{a}_p\rangle) = \widetilde{V}(\tilde{a}'_p, \tilde{a}_p, \tilde{a}'_p, \tilde{a}_p) = V(a_p, a_p).$$

Hence by (25), we have

$$\sigma^2_{-p;\tilde{a}'_p,\tilde{a}_p} = V(a_p, a_p) + 2\sum_{k=1}^{\infty} \frac{1}{p^k} V(a_p, a_p) = \sigma^2_{p;0,a_p} = \Sigma^2_p.$$

Since we have already verified $\Sigma_{-p} \le \Sigma_p$, we see that

$$\Sigma_{-p} = \sigma_{-p;\tilde{a}'_p,\tilde{a}_p} = \sigma_{p;0,a_p} = \Sigma_p.$$

Secondly, suppose that p, q, and r are all odd. It is shown in [15] that $\Sigma_{p/q} = \sigma_{p/q;0,1/2}$. We have

$$\begin{aligned}(-p)^k &= (-p-q+q)^k \\ &= q^k + kq^{k-1}(-p-q) + \sum_{i=0}^{k-2}\binom{k}{i} q^i(-p-q)^{k-i} \\ &\equiv q^k - kq^{k-1}(p+q) \mod 2(p+q)\end{aligned}$$

since $p+q$ is even. Put

$$\tilde{a}'_{pq} = \frac{1}{2(p+q)} \quad \text{and} \quad \tilde{a}_{pq} = \frac{1}{2} + \frac{1}{2(p+q)}.$$

Because of $\tilde{a}_{pq} - \tilde{a}'_{pq} = \frac{1}{2}$,

$$\langle q^k(\tilde{a}_{pq} - \tilde{a}'_{pq})\rangle = \frac{1}{2}$$

is clear. We have

$$\frac{p^{2k}}{2(p+q)} \equiv \frac{q^{2k}}{2(p+q)} - kq^{2k-1} \equiv \frac{q^{2k}}{2(p+q)} \mod 1,$$

thereby $\langle p^{2k}\tilde{a}'_{pq}\rangle = \langle q^{2k}\tilde{a}'_{pq}\rangle$ and $\langle p^{2k}\tilde{a}_{pq}\rangle = \langle q^{2k}\tilde{a}_{pq}\rangle$.

Similarly,

$$\frac{-p^{2k-1}}{2(p+q)} \equiv \frac{q^{2k-1}}{2(p+q)} - \frac{(2k-1)q^{2k}}{2} \equiv \frac{q^{2k-1}}{2(p+q)} + \frac{q^{2k-1}}{2} \mod 1,$$

and thereby $\langle -p^{2k-1}\tilde{a}'_{pq}\rangle = \langle q^{2k-1}\tilde{a}_{pq}\rangle$ and $\langle -p^{2k-1}\tilde{a}_{pq}\rangle = \langle q^{2k-1}\tilde{a}'_{pq}\rangle$.

Hence we have

$$\begin{aligned}&\widetilde{V}(\langle -p^{2k-1}\tilde{a}_{pq}\rangle, \langle -p^{2k-1}\tilde{a}'_{pq}\rangle, \langle q^{2k-1}\tilde{a}'_{pq}\rangle, \langle q^{2k-1}\tilde{a}_{pq}\rangle) \\ &\quad = \widetilde{V}(\langle q^{2k-1}\tilde{a}'_{pq}\rangle, \langle q^{2k-1}\tilde{a}_{pq}\rangle, \langle q^{2k-1}\tilde{a}'_{pq}\rangle, \langle q^{2k-1}\tilde{a}_{pq}\rangle) \\ &\quad = V(\langle q^{2k-1}(\tilde{a}_{pq} - \tilde{a}'_{pq})\rangle, \langle q^{2k-1}(\tilde{a}_{pq} - \tilde{a}'_{pq})\rangle) = V(\tfrac{1}{2}, \tfrac{1}{2}) \\ &\quad = V(\langle p^{2k-1}\tfrac{1}{2}\rangle, \langle q^{2k-1}\tfrac{1}{2}\rangle) \\ &\widetilde{V}(\langle p^{2k}\tilde{a}'_{pq}\rangle, \langle p^{2k}\tilde{a}_{pq}\rangle, \langle q^{2k}\tilde{a}'_{pq}\rangle, \langle q^{2k}\tilde{a}_{pq}\rangle) \\ &\quad = \widetilde{V}(\langle q^{2k}\tilde{a}'_{pq}\rangle, \langle q^{2k}\tilde{a}_{pq}\rangle, \langle q^{2k}\tilde{a}'_{pq}\rangle, \langle q^{2k}\tilde{a}_{pq}\rangle) \\ &\quad = V(\langle q^{2k}(\tilde{a}_{pq} - \tilde{a}'_{pq})\rangle, \langle q^{2k}(\tilde{a}_{pq} - \tilde{a}'_{pq})\rangle) = V(\tfrac{1}{2}, \tfrac{1}{2}) \\ &\quad = V(\langle p^{2k}\tfrac{1}{2}\rangle, \langle q^{2k}\tfrac{1}{2}\rangle)\end{aligned}$$

which yields

$$\sigma^2_{-p/q;\tilde{a}'_{pq},\tilde{a}_{pq}} = V(\tfrac{1}{2}, \tfrac{1}{2}) + 2\sum_{k=1}^{\infty}\frac{1}{p^k q^k} V(\langle p^k \tfrac{1}{2}\rangle, \langle q^k \tfrac{1}{2}\rangle) = \sigma^2_{p/q;0,1/2} = \Sigma^2_{p/q}.$$

Since we have already verified $\Sigma_{-p/q} \le \Sigma_{p/q}$, we see that

$$\Sigma_{-p/q} = \sigma_{-p/q;\tilde{a}'_{pq},\tilde{a}_{pq}} = \sigma_{p/q;0,1/2} = \Sigma_{p/q}.$$

Lastly, suppose that r is odd, $p = 5$, and $q = 2$. In this case $\sigma_{\theta;a',a} = \sigma_{-5/2;a',a}$. It is shown in [15] that $\Sigma_{5/2} = \sigma_{5/2;0,1/3}$. Note that

$$V(\langle 5^{2k}\tfrac{1}{3}\rangle, \langle 2^{2k}\tfrac{1}{3}\rangle) = V(\langle 5^{2k+1}\tfrac{1}{3}\rangle, \langle 2^{2k+1}\tfrac{1}{3}\rangle) = \tfrac{2}{9}.$$

Put $\tilde{a}' = \frac{1}{21}$ and $\tilde{a} = \frac{8}{21}$. Then we have $\langle(-5)^3\tilde{a}\rangle = \tilde{a}$, $\langle(-5)^3\tilde{a}'\rangle = \tilde{a}'$, $\langle 2^3\tilde{a}\rangle = \tilde{a}'$, $\langle 2^3\tilde{a}'\rangle = \tilde{a}$, and see that all values below equal to $\frac{2}{9}$:

$$\begin{aligned}
\widetilde{V}(\langle(-5)^{6k}\tilde{a}'\rangle, \langle(-5)^{6k}\tilde{a}\rangle, \langle 2^{6k}\tilde{a}'\rangle, \langle 2^{6k}\tilde{a}\rangle) &= \widetilde{V}\left(\tfrac{1}{21}, \tfrac{8}{21}, \tfrac{1}{21}, \tfrac{8}{21}\right),\\
\widetilde{V}(\langle(-5)^{6k+1}\tilde{a}\rangle, \langle(-5)^{6k+1}\tilde{a}'\rangle, \langle 2^{6k+1}\tilde{a}'\rangle, \langle 2^{6k+1}\tilde{a}\rangle) &= \widetilde{V}\left(\tfrac{2}{21}, \tfrac{16}{21}, \tfrac{2}{21}, \tfrac{16}{21}\right),\\
\widetilde{V}(\langle(-5)^{6k+2}\tilde{a}'\rangle, \langle(-5)^{6k+2}\tilde{a}\rangle, \langle 2^{6k+2}\tilde{a}'\rangle, \langle 2^{6k+2}\tilde{a}\rangle) &= \widetilde{V}\left(\tfrac{4}{21}, \tfrac{11}{21}, \tfrac{4}{21}, \tfrac{11}{21}\right),\\
\widetilde{V}(\langle(-5)^{6k+3}\tilde{a}\rangle, \langle(-5)^{6k+3}\tilde{a}'\rangle, \langle 2^{6k+3}\tilde{a}'\rangle, \langle 2^{6k+3}\tilde{a}\rangle) &= \widetilde{V}\left(\tfrac{8}{21}, \tfrac{1}{21}, \tfrac{8}{21}, \tfrac{1}{21}\right),\\
\widetilde{V}(\langle(-5)^{6k+4}\tilde{a}'\rangle, \langle(-5)^{6k+4}\tilde{a}\rangle, \langle 2^{6k+4}\tilde{a}'\rangle, \langle 2^{6k+4}\tilde{a}\rangle) &= \widetilde{V}\left(\tfrac{16}{21}, \tfrac{2}{21}, \tfrac{16}{21}, \tfrac{2}{21}\right),\\
\widetilde{V}(\langle(-5)^{6k+5}\tilde{a}\rangle, \langle(-5)^{6k+5}\tilde{a}'\rangle, \langle 2^{6k+5}\tilde{a}'\rangle, \langle 2^{6k+5}\tilde{a}\rangle) &= \widetilde{V}\left(\tfrac{11}{21}, \tfrac{4}{21}, \tfrac{11}{21}, \tfrac{4}{21}\right).
\end{aligned}$$

Therefore we have $\sigma_{-5/2;0,1/3} = \sigma_{5/2;0,1/3} = \Sigma_{5/2}$. Since we have already proved $\Sigma_{-5/2} \le \Sigma_{5/2}$, we have $\Sigma_{-5/2} = \Sigma_{5/2}$.

4 Proof of the evaluation (9)

Let r be odd, $p \ge 3$ be an odd integer, and $q = 1$. In this case we have $\Sigma_\theta = \Sigma_{-p}$. Put

$$b_p = \frac{p}{2(p+1)} = \frac{1}{2} - \frac{1}{2(p+1)} \quad \text{and} \quad b'_p = \frac{2p+1}{2(p+1)}.$$

We can easily verify that $\langle -pb_p\rangle = b'_p$ and $\langle -pb'_p\rangle = b_p$. We have

$$\begin{aligned}
V(b_p, b_p) = \frac{p^2+2p}{4(p+1)^2}, \quad V(b'_p, b_p) = \frac{p}{4(p+1)^2},\\
p^{2l+1}B_{p,1;l}(b_p) = -V(b'_p, b_p) + \frac{1}{p}V(\langle -pb'_p\rangle, b_p) = \frac{2}{4(p+1)^2}.
\end{aligned}$$

We prove

$$\Sigma_{-p}^{*}{}^2 = \sigma_{-p;0,b_p}^2 = V(b_p, b_p) + 2\sum_{l=0}^{\infty} B_{p,1;l}(b_p) = \frac{p(p^3+2p^2-p+2)}{4(p-1)(p+1)^3}$$
$$= \frac{1}{4} + \frac{-p^2+4p+1}{4(p-1)(p+1)^3} \begin{cases} < \frac{1}{4}, & p \geq 5, \\ > \frac{1}{4}, & p = 3. \end{cases}$$

By (27) and continuity of $\sigma_{-p;0,a}^2$ in a, it is enough to prove $\sigma_{-p;0,a}^2 \leq \sigma_{-p;0,b_p}^2$ for all $0 \leq a < \frac{1}{2}$. By applying (35), we have

$$\sigma_{-p;0,a}^2 \leq \frac{1}{p^2-1}((p^2+1)a - (p^2+3)a^2) = h_{\mathrm{II}}(a).$$

Because of $\langle(-p)^{2l+1}b_p\rangle = b'_p = 1 - \frac{b_p}{p}$, the equality holds in the above inequality when $a = b_p$, i.e., $\sigma_{-p;0,b_p}^2 = h_{\mathrm{II}}(b_p)$.

Since $h_{\mathrm{II}}(a)$ is increasing for $a < \frac{p^2+1}{2(p^2+3)} = \frac{1}{2} - \frac{2}{2(p^2+3)}$ and since we have

$$\left(\frac{1}{2} - \frac{2}{2(p^2+3)}\right) - b_p = \frac{(p-1)^2}{2(p^2+3)(p+1)} > 0,$$

we see that $h_{\mathrm{II}}(a)$ is increasing for $a < b_p$. Hence we have

$$\sigma_{-p;0,a}^2 \leq h_{\mathrm{II}}(a) \leq h_{\mathrm{II}}(b_p) = \sigma_{-p;0,b_p}^2, \quad (a \leq b_p).$$

Note that $\frac{p-1}{2p} < b_p$. If $\frac{p-1}{2p} < a < \frac{1}{2}$, we have

$$-\frac{p+1}{2} < -\frac{p}{2} < -pa < -\frac{p-1}{2} = -\frac{p+1}{2} + 1.$$

Hence $[-pa] = -\frac{1}{2}(p+1)$ and $\langle -pa\rangle = -pa + \frac{1}{2}(p+1)$. By $\langle -pa\rangle - a = \frac{1}{2}(1-2a)(p+1) > 0$, we have $V(\langle -pa\rangle, a) = a(pa - \frac{1}{2}(p-1))$ and

$$B_{p,1;0} \leq -a\left(a - \frac{p-1}{2p}\right) + \frac{1}{p^2}a(1-a),$$

where the equality holds for $a = b_p$. Hence for $\frac{p-1}{2p} < a < \frac{1}{2}$, by (36) we have

$$\sigma^2_{-p;0,a} \le a(1-a) - 2a\left(a - \frac{p-1}{2p}\right) + \frac{2}{p^2}a(1-a) + \frac{2}{p^2(p^2-1)}a(1-2a)$$
$$= \frac{1}{p^2(p^2-1)}((2p^4 - p^3 + p)a - (3p^4 - p^2 + 2)a^2) = h_{\mathrm{III}}(a),$$

where the equality holds for $a = b_p$. Note that $3p^4 - p^2 + 2 > 0$. Since $h_{\mathrm{III}}(a)$ is decreasing for $a > \frac{2p^4 - p^3 + p}{2(3p^4 - p^2 + 2)}$ and since we have

$$\frac{p-1}{2p} - \frac{2p^4 - p^3 + p}{2(3p^4 - p^2 + 2)} = \frac{p^3(p^2 - 2p - 1) + 2(p^2 - 1)}{2p(3p^4 - p^2 + 2)} > 0$$

for $p \ge 3$, we see that $h_{\mathrm{III}}(a)$ is decreasing for $\frac{p-1}{2p} < a < \frac{1}{2}$, and hence for $b_p \le a < \frac{1}{2}$. Thereby we have

$$\sigma^2_{-p;0,a} \le h_{\mathrm{III}}(a) \le h_{\mathrm{III}}(b_p) = \sigma^2_{-p;0,b_p}, \quad (b_p \le a < \tfrac{1}{2}).$$

5 Proof of (10)

When $2 \mid pq$, we have $\langle (-p)^k \frac{1}{2} \rangle = 0$ or $\langle q^k \frac{1}{2} \rangle = 0$ for $k \ge 1$, and thereby $\sigma^2_{-p/q;0,1/2} = V(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$. Therefore $\Sigma^*_{-p/q} \ge \frac{1}{2}$ is trivial, and it is sufficient to show $\Sigma^*_{-p/q} \le \frac{1}{2}$ to prove (10). We divide the proof into three parts. Put

$$I_k = \left(\frac{1}{2} - \frac{1}{p^k \vee (2q^k)}, \frac{1}{2}\right) \quad \text{and} \quad J_k = \left(\frac{1}{2} - \frac{1}{2(p^k + q^k)}, \frac{1}{2}\right).$$

Clearly we have $I_k \supset I_{k+1}$ and $I_k \supset J_k$.

5.1 The case when $p \ge 4$ is even and $q \ge 3$ is odd

By using $p \ge 4$, we have $2(p^k + q^k) < 4p^k \le p^{k+1} \vee (2q^{k+1})$ and $J_k \supset I_{k+1}$. For $k \ge 1$, we have

$$\langle -p^k a \rangle = -p^k a + \frac{p^k}{2}, \quad \langle p^k a \rangle = p^k a - \frac{p^k}{2} + 1, \quad \langle q^k a \rangle = q^k a - \frac{q^k - 1}{2}, \quad (a \in I_k).$$

We can verify $\langle -p^{2l+1} a \rangle < \langle q^{2l+1} a \rangle$ on J_{2l+1}. Therefore we have

$$\frac{-1}{(pq)^{2l+1}} V(\langle -p^{2l+1} a \rangle, \langle q^{2l+1} a \rangle) = -\left(-a + \frac{1}{2}\right)\left(\frac{1}{2} + \frac{1}{2q^{2l+1}} - a\right)$$

for $a \in J_{2l+1}$, and hence for $a \in I_{2l+2}$. By applying $V(a', a) \le a(1-a')$, we have

$$\begin{aligned}\frac{1}{(pq)^{2l+2}} V(\langle p^{2l+2}a\rangle, \langle q^{2l+2}a\rangle) &\le \frac{\langle q^{2l+2}a\rangle(1-\langle p^{2l+2}a\rangle)}{(pq)^{2l+2}} \\ &= \left(a - \frac{1}{2} + \frac{1}{2q^{2l+2}}\right)\left(\frac{1}{2} - a\right)\end{aligned}$$

for $a \in I_{2l+2}$. Hence we have

$$B_{p,q;l}(a) \le -2\left(\frac{1}{2} + \frac{1}{4q^{2l+1}} - \frac{1}{4q^{2l+2}} - a\right)\left(\frac{1}{2} - a\right) < 0, \quad (a \in I_{2l+2}).$$

Suppose that $L \ge 1$. For $a \in I_{2L}$ we therefore have $B_{p,q;1}(a) + \cdots + B_{p,q;L-1}(a) < 0$ and, by (30) we have

$$\begin{aligned}\sigma^2_{-p/q;0,a} &\le a(1-a) - 4\left(\frac{1}{2} - a\right)\left(\frac{1}{2} + \frac{1}{4q} - \frac{1}{4q^2} - a\right) + \frac{1}{2p^{2L}q^{2L}(p^2q^2-1)} \\ &= h_{\mathrm{IV}}(a).\end{aligned}$$

Note that we have

$$h_{\mathrm{IV}}\left(\frac{1}{2} - \frac{1}{p^{2L+2}}\right) - \frac{1}{4} = -\frac{5}{p^{4L+4}} + \frac{-2(q-1)q^{2L-2}(p^2q^2-1) + p^2}{2p^{2L+2}q^{2L}(p^2q^2-1)} < 0$$

by $-2(q-1)(p^2q^2-1) + p^2 \le -4(p^2q^2-1) + p^2 < -2p^2q^2 + p^2 < 0$, and we have

$$h_{\mathrm{IV}}\left(\frac{1}{2} - \frac{1}{2q^{2L+2}}\right) - \frac{1}{4} = -\frac{5}{4q^{4L+4}} + \frac{-(q-1)p^{2L}(p^2q^2-1) + q^4}{2p^{2L}q^{2L+4}(p^2q^2-1)} < 0$$

by $-(q-1)p^2(p^2q^2-1) + q^4 \le -2p^2(p^2q^2-1) + q^4 \le -p^4q^2 + q^4 < 0$. Hence we have verified

$$h_{\mathrm{IV}}\left(\frac{1}{2} - \frac{1}{p^{2L+2} \vee (2q^{2L+2})}\right) < \frac{1}{4}.$$

Since h_{IV} is increasing on I_{2L}, we see that $\sigma^2_{-p/q;0,a} \le h_{\mathrm{IV}}(a) < \frac{1}{4}$ for $a \in I_{2L} \setminus I_{2L+2}$. By taking union for $L = 1, 2, \ldots$, we have $\sigma^2_{-p/q;0,a} \le h_{\mathrm{IV}}(a) < \frac{1}{4}$ for $a \in I_2$.

We have

$$h_{\mathrm{I}}\left(\frac{1}{2} - \frac{1}{2(p+q)}\right) - \frac{1}{4} = -\frac{1}{4(p+q)^2} + \frac{1}{2(p^2q^2-1)} < 0$$

by $q \geq 3$ and $2(p+q)^2 \leq 8p^2 = 9p^2 - p^2 \leq p^2q^2 - 1$. Since h_{I} is increasing on $(0, \frac{1}{2})$, we verified that

$$\sigma^2_{-p/q;0,a} \leq h_{\mathrm{I}}(a) < \frac{1}{4} \quad \text{for } a \in \left(0, \frac{1}{2} - \frac{1}{2(p+q)}\right].$$

On $(\frac{1}{2} - \frac{1}{2(p+q)}, \frac{1}{2}) = J_1$, we have $\langle -pa \rangle = -pa + \frac{p}{2} < qa - \frac{q-1}{2} = \langle qa \rangle$ and $B_{p,q;0} \leq (a - \frac{1}{2})(\frac{1}{2} + \frac{1}{2q} - a) + \frac{1}{4p^2q^2}$. Hence by (32), we have

$$\sigma^2_{-p/q;0,a} \leq a(1-a) + 2\left(a - \frac{1}{2}\right)\left(\frac{1}{2} + \frac{1}{2q} - a\right) + \frac{1}{2(p^2q^2-1)} = h_{\mathrm{V}}(a),$$

for $a \in J_1$. Recall that $I_2 \subset J_1$. Note that we have

$$h_{\mathrm{V}}\left(\frac{1}{2} - \frac{1}{p^2}\right) - \frac{1}{4} = -\frac{3}{p^4} - \frac{1}{p^2q} + \frac{1}{2(p^2q^2-1)} < 0$$

by $2(p^2q^2 - 1) \geq 6p^2q - 2 \geq p^2q$, and have

$$h_{\mathrm{V}}\left(\frac{1}{2} - \frac{1}{2q^2}\right) - \frac{1}{4} = -\frac{3}{4q^4} - \frac{1}{2q^3} + \frac{1}{2(p^2q^2-1)} < 0$$

by $2(p^2q^2 - 1) \geq 8pq^2 - 2 \geq 8q^3 - 2 \geq 2q^3$. Since we have verified

$$h_{\mathrm{V}}\left(\frac{1}{2} - \frac{1}{p^2 \vee (2q^2)}\right) < \frac{1}{4},$$

and since h_{V} is increasing in $a < \frac{1}{2}$, we have

$$\sigma^2_{-p/q;0,a} \leq h_{\mathrm{V}}(a) < \frac{1}{4} \quad \text{for } a \in \left(\frac{1}{2} - \frac{1}{2(p+q)}, \frac{1}{2} - \frac{1}{p^2 \vee (2q^2)}\right].$$

Hence we have verified $\Sigma^*_\theta = \frac{1}{2}$ in this case.

5.2 The case when p is even and $q = 1$

Since $[p^{2l}a] \leq \frac{1}{2}p^{2l} - 1$ for $a < \frac{1}{2}$ and $l \geq 1$, we have $\langle p^{2l}a \rangle \geq p^{2l}a - \frac{1}{2}p^{2l} + 1$. Hence we have

$$\frac{1}{p^{2l}} V(\langle p^{2l}a \rangle, a) \leq \frac{1}{p^{2l}} a\left(1 - \langle p^{2l}a \rangle\right) \leq a\left(\frac{1}{2} - a\right), \quad (a < \tfrac{1}{2},\ l \geq 1). \tag{37}$$

Note that

$$I_k = \left(\frac{1}{2} - \frac{1}{p^k}, \frac{1}{2}\right), \quad J_k = \left(\frac{1}{2} - \frac{1}{2(p^k+1)}, \frac{1}{2}\right), \quad \text{and} \quad J_k \subset I_k.$$

For $a \in I_{2l+1}$, we have $\langle -p^{2l+1}a\rangle = -p^{2l+1}a + \frac{1}{2}p^{2l+1}$. Since we have $\langle -p^{2l+1}a\rangle < a$ for $a \in J_{2l+1}$, we have

$$-\frac{1}{p^{2l+1}}V(\langle -p^{2l+1}a\rangle, a) = -\left(\frac{1}{2} - a\right)(1-a), \quad (a \in J_{2l+1}),$$

and $B_{p,1;l}(a) \leq -\frac{1}{2}(1-2a)^2 < 0 (a \in J_{2l+1})$. Hence we have $B_{p,1;1}(a) + \cdots + B_{p,1;L-1}(a) < 0$ on J_{2L-1}, and by (34) we have

$$\sigma^2_{-p;0,a} \leq a(1-a) - (1-2a)^2 + \frac{2}{p^{2L}(p^2-1)}a(1-2a) = a(1-a) + h_{\mathrm{VI}}(a)$$

for $a \in J_{2L-1}$ and $L \geq 1$. We have

$$h_{\mathrm{VI}}(a) = \frac{2 + 2p^{2L}(p^2-1)}{p^{2L}(p^2-1)}(1-2a)(a-c_L) \quad \text{where } c_L = \frac{1}{2} - \frac{1}{2(1+p^{2L}(p^2-1))}$$

and thereby

$$\sigma^2_{-p;0,c_L} \leq c_L(1-c_L) + h_{\mathrm{VI}}(c_L) \leq \frac{1}{4}.$$

Since $a(1-a) + h_{\mathrm{VI}}(a)$ is increasing in $(0, c_L)$, we have

$$\sigma^2_{-p;0,a} \leq a(1-a) + h_{\mathrm{VI}}(a) \leq \frac{1}{4} \quad (a \in J_{2L-1} \cap (0, c_L)).$$

We have

$$c_L - \left(\frac{1}{2} - \frac{1}{2(1+p^{2L+1})}\right) = \frac{p^{2L}(p^2-p-1)}{2(1+p^{2L}(p^2-1))(1+p^{2L+1})} > 0$$

by $p^2 - p - 1 > 0$ for $p \geq 2$. Thus we have $\sigma^2_{-p;0,a} \leq \frac{1}{4}$ for $a \in J_{2L-1} \setminus J_{2L+1}$ By taking a union for $L = 1, 2, \ldots$, we have $\sigma^2_{-p;0,a} \leq \frac{1}{4}(a \in J_1)$.

By applying (35), we have

$$\sigma^2_{-p;0,a} \leq \frac{1}{p^2-1}((p^2+1) - (p^2+3)a)a = h_{\mathrm{VII}}(a), \quad (a < \tfrac{1}{2}).$$

Note that h_{VII} is increasing for $a \leq \frac{p^2+1}{2(p^2+3)} = \frac{1}{2} - \frac{2}{2(p^2+3)}$. Because of

$$\left(\frac{1}{2} - \frac{2}{2(p^2+3)}\right) - \left(\frac{1}{2} - \frac{1}{2(p+1)}\right) = \frac{p^2-2p+1}{2(p^2+3)(p+1)} > 0,$$

h_{VII} is verified to be increasing for $a \le \frac{1}{2} - \frac{1}{2(p+1)}$. By noting

$$h_{\text{VII}}\left(\frac{1}{2} - \frac{1}{2(p+1)}\right) - \frac{1}{4} = \frac{-p^2+4p+1}{4(p+1)^3(p-1)} < 0$$

for $p \ge 6$, we have $\sigma^2_{-p;0,a} < \frac{1}{4}$ for $a \in (0, \frac{1}{2}) \setminus J_1$ and $p \ge 6$. Thus we have $\Sigma^*_{-p} = \frac{1}{2}$ for even $p \ge 6$.

In case $p = 4$, we have

$$h_{\text{VII}}\left(\frac{1}{4}\right) = \frac{49}{60}\frac{1}{4} < \frac{1}{4},$$

and hence we have $\sigma^2_{-4;0,a} < \frac{1}{4}$ for $a \in (0, \frac{1}{4}\,]$. Since we have already proved $\sigma^2_{-4;0,a} < \frac{1}{4}$ for $a \in J_1 = (\frac{2}{5}, \frac{1}{2})$, it is enough to prove $\sigma^2_{-4;0,a} < \frac{1}{4}$ for $a \in (\frac{1}{4}, \frac{2}{5}\,]$. For $a \in (\frac{1}{4}, \frac{2}{5}\,]$, we have $[-4a] = -2$, $\langle -4a \rangle = -4a + 2$, and $\langle -4a \rangle \ge a$. Hence we have $V(\langle -4a \rangle, a) = a(4a-1)$ and $B_{4,1;0}(a) \le -\frac{1}{4}a(4a-1) + \frac{1}{16}a(1-a)$. By using (36), we have

$$\begin{aligned}\sigma^2_{-4;0,a} &\le a(1-a) - \frac{1}{2}a(4a-1) + \frac{1}{8}a(1-a) + \frac{1}{120}a(1-2a)\\ &= \frac{1}{120}\left(-377\left(a - \frac{98}{377}\right)^2 + \frac{98^2}{377}\right) \le \frac{49^2}{30 \cdot 377} < \frac{1}{4}\end{aligned}$$

for $a \in (\frac{1}{4}, \frac{2}{5}\,]$. Therefore we have $\Sigma^*_{-4} = \frac{1}{2}$.

In case $p = 2$, we have $\frac{1}{2} - \frac{1}{2(p+1)} = \frac{1}{3}$ and $\sigma^2_{-2;0,a} < \frac{1}{4}$ for $\frac{1}{3} < a < \frac{1}{2}$. It is enough to prove $\sigma^2_{-2;0,a} < \frac{1}{4}$ for $a \le \frac{1}{3}$.

For $0 < a \le \frac{1}{3}$, we have $\langle -2a \rangle \ge a$ and $V(\langle -2a \rangle, a) = a(1 - \langle -2a \rangle) = 2a^2$. Hence we have $B_{2,1;0}(a) \le -a^2 + \frac{1}{4}a(1-a)$, and by (36), we have

$$\begin{aligned}\sigma^2_{-2;0,a} &\le a(1-a) - 2a^2 + \frac{1}{2}a(1-a) + \frac{1}{6}a(1-2a)\\ &= \frac{1}{6}(-23a^2 + 10) = \frac{1}{6}\left(-23\left(a - \frac{5}{23}\right)^2 + \frac{25}{23}\right) \le \frac{25}{23 \cdot 6} \le \frac{30}{20 \cdot 6} = \frac{1}{4}.\end{aligned}$$

Hence we have proved $\Sigma^*_{-2} = \frac{1}{2}$.

5.3 The case when p is odd, q is even

In this case, by $q < p$, we have

$$\left(\frac{1}{2} - \frac{1}{q^k}, \frac{1}{2}\right) \supset \left(\frac{1}{2} - \frac{1}{2p^k}, \frac{1}{2}\right) = I'_k \supset J_k$$

and hence

$$\langle -p^k a\rangle = -p^k a + \frac{p^k+1}{2}, \quad \langle p^k a\rangle = p^k a - \frac{p^k-1}{2},$$
$$\langle q^k a\rangle = q^k a - \frac{q^k}{2} + 1, \quad (a \in I'_k,\ k \geq 1).$$

Since $\langle -p^{2l+1}a\rangle < \langle q^{2l+1}a\rangle$ for $a \in J_{2l+1}$, we have

$$-\frac{1}{(pq)^{2l+1}} V(\langle -p^{2l+1}a\rangle, \langle q^{2l+1}a\rangle) = -\left(-a + \frac{1}{2} + \frac{1}{2p^{2l+1}}\right)\left(\frac{1}{2} - a\right),$$
$$(a \in J_{2l+1}).$$

If $a \in J_{2l+1}$, then $a \in I'_{2l+1}$ and $\frac{1}{2}(p^{2l+2} - p) < p^{2l+2}a < \frac{1}{2}p^{2l+2}$. Thus we have $\frac{1}{2}(p^{2l+2}-p) \leq [p^{2l+2}a] \leq \frac{1}{2}(p^{2l+2}-1)$ and $\langle p^{2l+2}a\rangle = p^{2l+2}a - \frac{1}{2}(p^{2l+2}-j)(j = 1, 3, \ldots, p)$. Since $a < \frac{1}{2}$, we have $q^{2l+1}a < \frac{1}{2}q^{2l+1}$, $[q^{2l+1}a] \leq \frac{1}{2}q^{2l+1} - 1$, and $\langle q^{2l+1}a\rangle \geq q^{2l+1}a - \frac{1}{2}q^{2l+1} + 1$ in turn. Therefore by applying $V(a, a') \leq a(1-a')$, we have

$$\frac{1}{(pq)^{2l+2}} V(\langle p^{2l+2}a\rangle, \langle q^{2l+2}a\rangle) \leq \left(a - \frac{1}{2} + \frac{j}{2p^{2l+2}}\right)\left(\frac{1}{2} - a\right), \quad (a \in J_{2l+1}).$$

By combining these and by noting $\frac{1}{4p^{2l+1}} - \frac{j}{4p^{2l+2}} \geq 0$ we have

$$B_{p,q;l}(a) \leq -2\left(-a + \frac{1}{2} + \frac{1}{4p^{2l+1}} - \frac{j}{4p^{2l+2}}\right)\left(\frac{1}{2} - a\right) < 0, \quad (a \in J_{2l+1}).$$

Hence $B_{p,q;1}(a) + \cdots + B_{p,q;L-1}(a) < 0$ for $a \in J_{2L-1}$. If $a \in I'_2$, then

$$\frac{1}{p^2q^2} V(\langle p^2 a\rangle, \langle q^2 a\rangle) \leq \left(a - \frac{1}{2} + \frac{1}{2p^2}\right)\left(\frac{1}{2} - a\right).$$

Because of $J_1 \supset I'_2$, we see

$$-\frac{1}{pq} V(\langle -pa\rangle, \langle qa\rangle) = -\left(-a + \frac{1}{2} + \frac{1}{2p}\right)\left(\frac{1}{2} - a\right), \quad (a \in I'_2),$$

and thereby we can conclude

$$B_{p,q;0}(a) \leq -2\left(-a + \frac{1}{2} + \frac{1}{4p} - \frac{1}{4p^2}\right)\left(\frac{1}{2} - a\right), \quad (a \in I'_2).$$

By (30) we have

$$\sigma^2_{-p/q;0,a} \le a(1-a) - 4\left(-a + \frac{1}{2} + \frac{1}{4p} - \frac{1}{4p^2}\right)\left(\frac{1}{2} - a\right) + \frac{1}{2p^{2L}q^{2L}(p^2q^2-1)}$$
$$= h_{\mathrm{VIII}}(a)$$

for $a \in J_{2L-1} \cap I_2'$. Note that we have

$$h_{\mathrm{VIII}}\left(\frac{1}{2} - \frac{1}{2(p^{2L+1}+q^{2L+1})}\right) - \frac{1}{4} = -\frac{5}{2(p^{2L+1}+q^{2L+1})^2}$$
$$+ \frac{p^{2L+1} + q^{2L+1} + p^{2L-1}q^{2L} + p^{2L}q^{2L+2} - p^{2L+1}q^{2L+2} - p^{2L-2}q^{2L}}{2p^{2L}q^{2L}(p^2q^2-1)} < 0$$

by $p^{2L+1} \le \frac{1}{4}p^{2L+1}q^{2L+2}$, $q^{2L+1} \le \frac{1}{6}p^{2L+1}q^{2L+2}$, $p^{2L-1}q^{2L} \le \frac{1}{36}p^{2L+1}q^{2L+2}$, $p^{2L}q^{2L+2} \le \frac{1}{3}p^{2L+1}q^{2L+2}$, and $\frac{1}{4} + \frac{1}{6} + \frac{1}{36} + \frac{1}{3} = \frac{7}{9} < 1$. Since $h_{\mathrm{VIII}}(a)$ increases on J_{2L-1}, we see $h_{\mathrm{VIII}}(a) < \frac{1}{4}$ or $\sigma^2_{-p/q;0,a} \le \frac{1}{4}$ for $a \in (J_{2L-1} \setminus J_{2L+1}) \cap I_2'$. By taking a union for $L = 1, 2, \ldots$, and by noting that $J_1 \supset I_2' \supset J_3 \supset J_5 \supset \cdots$, we have $\sigma^2_{-p/q;0,a} \le \frac{1}{4}$ for $a \in I_2'$.

Let $a \in J_1$. we have $p^2a > \frac{1}{2}p^2 - \frac{p^2}{2(p+q)}$, $[p^2a] \ge \frac{1}{2}p^2 - \frac{p^2}{2(p+q)} - 1$, and $\langle p^2a\rangle \le p^2a - \frac{1}{2}p^2 + \frac{p^2}{2(p+q)} + 1$. Since we have $\langle q^2a\rangle \ge q^2a - \frac{1}{2}q^2 + 1$ as before, we have

$$\frac{1}{p^2q^2}V(\langle p^2a\rangle, \langle q^2a\rangle) \le \left(a - \frac{1}{2} + \frac{1}{2(p+q)} + \frac{1}{p^2}\right)\left(\frac{1}{2} - a\right),$$

and

$$B_{p,q;0}(a) \le -2\left(-a + \frac{1}{2} + \frac{1}{4p} - \frac{1}{4(p+q)} - \frac{1}{2p^2}\right)\left(\frac{1}{2} - a\right), \quad (a \in I_2').$$

By applying (32) and these estimates, we have

$$\sigma^2_{-p/q;0,a} \le a(1-a) - 4\left(-a + \frac{1}{2} + \frac{1}{4p} - \frac{1}{4(p+q)} - \frac{1}{2p^2}\right)\left(\frac{1}{2} - a\right)$$
$$+ \frac{1}{2p^2q^2(p^2q^2-1)}$$
$$= h_{\mathrm{IX}}(a), \quad (a \in J_1).$$

Note that we have

$$h_{\mathrm{IX}}\left(\frac{1}{2} - \frac{1}{2p^2}\right) - \frac{1}{4} \le -\frac{1}{4p^2} - \frac{q^3(p^2q^2-1) - p(p+q)}{2p^3q^2(p+q)(p^2q^2-1)} < 0$$

by $q^3(p^2q^2-1)-p(p+q)\geq 4p^2q^3-q^3-p^2-pq=(p^2q^3-q^3)+(p^2q^3-p^2)+(2p^2q^3-pq)>0$. Since $h_{\mathrm{IX}}(a)$ increases on J_1, we see that $\sigma^2_{-p/q;0,a}\leq \frac{1}{4}$ on $J_1\cap(0,\frac{1}{2}-\frac{1}{2p^2}]$ and hence on $(\frac{1}{2(p+q)},\frac{1}{2})$.

We here use (31). Assume that $p/q\neq 3/2$. In this case we have $2(p+q)^2\leq p^2q^2-1$. Actually, when $q\geq 3$, then we have $2(p+q)^2\leq 8p^2<p^2q^2$, and when $q=2$, $2(p+2)^2-(p^22^2-1)=-2p^2+8p+9\leq 0$ for $p\geq 5$.

Since h_{I} is increasing on $(0,\frac{1}{2})$ and satisfies

$$h_{\mathrm{I}}\left(\frac{1}{2}-\frac{1}{2(p+q)}\right)-\frac{1}{4}=\frac{-(p^2q^2-1)+2(p+q)^2}{4(p+q)^2(p^2q^2-1)}<0,$$

we see that $\sigma^2_{-p/q;0,a}\leq h_{\mathrm{I}}(a)\leq\frac{1}{4}$ for $a\in(0,\frac{1}{2})\backslash J_1$. Therefore we have $\sigma^2_{-p/q;0,a}\leq\frac{1}{4}$ for all $a\in(0,\frac{1}{2})$ and $\Sigma^*_{-p/q}=\frac{1}{2}$.

Lastly we consider the case $p/q=3/2$. In this case we have

$$h_{\mathrm{I}}\left(\frac{1}{2}-\frac{1}{2p}\right)-\frac{1}{4}=\frac{-(p^2q^2-1)+2p^2}{4p^2(p^2q^2-1)}=\frac{-35+18}{36\cdot 35}<0$$

and since $h_{\mathrm{I}}(a)$ is increasing in $(0,\frac{1}{2})$, we have $\sigma^2_{-3/2;0,a}\leq h_{\mathrm{I}}(a)\leq\frac{1}{4}$ for $a\in(0,\frac{1}{2})\backslash I_1'$. For $a\in I_1'\backslash J_1$, we have $\langle qa\rangle\leq\langle -pa\rangle$ and hence

$$-\frac{1}{pq}V(\langle -pa\rangle,\langle qa\rangle)=-\left(a-\frac{1}{2}+\frac{1}{q}\right)\left(a-\frac{1}{2}+\frac{1}{2p}\right)=-a\left(a-\frac{1}{3}\right).$$

Thus we have $B_{3,2;0}\leq -a(a-\frac{1}{3})+\frac{1}{4}\cdot 3^2\cdot 2^2$, and by (32) we have

$$\sigma^2_{-3/2;0,a}\leq a(1-a)-2a\left(a-\frac{1}{3}\right)+\frac{1}{2(3^22^2-1)}=-3a^2+\frac{5}{3}a+\frac{1}{70}=h_{\mathrm{X}}(a).$$

Since $h_{\mathrm{X}}(a)$ is decreasing for $a>\frac{5}{18}$, hence for $a\in(\frac{1}{3},\frac{2}{5}]=I_1'\backslash J_1$. Hence

$$h_{\mathrm{X}}(a)\leq h_{\mathrm{X}}\left(\frac{1}{3}\right)=\frac{149}{630}<\frac{1}{4}\quad(a\in I_1'\backslash J_1).$$

Therefore $\sigma^2_{-3/2;0,a}\leq\frac{1}{4}$ for $a<\frac{1}{2}$ and $\Sigma^*_{-3/2}=\frac{1}{2}$.

6 Proof of the inequality (11)

We assume that $p>q\geq 3$ are odd numbers. Since h_{I} increases on $(0,\frac{1}{2})$, by (31) and

$$h_{\mathrm{I}}\left(\frac{1}{2}-\frac{1}{2p}\right)-\frac{1}{4}=\frac{-(p^2q^2-1)+2p^2}{4p^2(p^2q^2-1)}\leq\frac{-9p^2+1+2p^2}{4p^2(p^2q^2-1)}<0,$$

we see that $\sigma^2_{-p/q;0,a}<\frac{1}{4}$ for $a\in(0,\frac{1}{2})\setminus I_1'$.

On I_1', we have $\langle -pa\rangle=-pa+\frac{1}{2}(p+1)\geq\frac{1}{2}\geq qa-\frac{1}{2}(q-1)=\langle qa\rangle$ and

$$B_{p,q;0}(a)\leq-\left(a-\frac{1}{2}+\frac{1}{2q}\right)\left(a-\frac{1}{2}+\frac{1}{2p}\right)+\frac{1}{4p^2q^2}.$$

By applying (32), we have

$$\sigma^2_{-p/q;0,a}\leq a(1-a)-2\left(a-\frac{1}{2}+\frac{1}{2q}\right)\left(a-\frac{1}{2}+\frac{1}{2p}\right)+\frac{1}{2(p^2q^2-1)}=h_{\mathrm{XI}}(a),$$

for $a\in I_1'$. Because of

$$h_{\mathrm{XI}}(a)=-3a^2+\left(3-\frac{p+q}{pq}\right)a-2\left(\frac{1}{2}-\frac{1}{2q}\right)\left(\frac{1}{2}-\frac{1}{2p}\right)+\frac{1}{2(p^2q^2-1)},$$

h_{XI} has maximum at $\check{a}=\frac{1}{2}-\frac{p+q}{6pq}$.

If $p\geq 2q$, then

$$\check{a}-\left(\frac{1}{2}-\frac{1}{2p}\right)=\frac{2q-p}{6pq}\leq 0,$$

and hence h_{XI} is decreasing in I_1'. Thereby we have

$$\sigma^2_{-p/q;0,a}\leq h_{\mathrm{XI}}(a)\leq h_{\mathrm{XI}}\left(\frac{1}{2}-\frac{1}{2p}\right)=h_{\mathrm{I}}\left(\frac{1}{2}-\frac{1}{2p}\right)<\frac{1}{4}$$

for $a\in I_1'$. By continuity, the above estimate can be also proved for $a=\frac{1}{2}$, Hence $\varSigma^*_{-p/q}<\frac{1}{2}$ in this case.

If $p<2q$,

$$\begin{aligned}h_{\mathrm{XI}}(\check{a})-\frac{1}{4}&=\frac{(p^2-4pq+q^2)(p^2q^2-1)+6p^2q^2}{12p^2q^2(p^2q^2-1)}\\&=\frac{p^3q^2(p-2q)+p^2q^3(q-p)-(p-q)^2+(6p^2q^2+2pq-p^3q^3)}{12p^2q^2(p^2q^2-1)}\\&<0\end{aligned}$$

by $6p^2q^2+2pq-p^3q^3\leq 6p^2q^2+2pq-15p^2q^2<0$. Hence $h_{\mathrm{XI}}(a)\leq h_{\mathrm{XI}}(\check{a})<\frac{1}{4}$ for $a\in I_1'$ and $\varSigma^*_{-p/q}<\frac{1}{2}$ in this case.

7 Proof of (6)

Suppose that θ is given by (4). Because we have already proved $\Sigma_\theta \le \Sigma_{|\theta|}$, we here prove $\frac{1}{2} < \Sigma_\theta$. In case when (8) holds, then by (3), we have $\frac{1}{2} < \Sigma_\theta$. Hence we must prove $\frac{1}{2} < \Sigma_{-p/q}$ in the case when one of the p and $q > 1$ is even, and also in the case when $p = 2$ and $q = 1$.

Firstly, assume that p is odd and q is even. We have $\langle (-p)^k \frac{1}{2} \rangle = \frac{1}{2}$ and $\langle q^k \frac{1}{2} \rangle = 0$ for $k \ge 1$. Put

$$\begin{aligned} f_k(a') &= \widetilde{V}(\langle (-p)^k a' \rangle, \langle (-p)^k \tfrac{1}{2} \rangle, \langle q^k a' \rangle, \langle q^k \tfrac{1}{2} \rangle) \\ &= V(\langle (-p)^k a' \rangle, \langle q^k a' \rangle) - V(\tfrac{1}{2}, \langle q^k a' \rangle) \end{aligned}$$

for $k \ge 1$. Since f_k is continuous and piecewisely continuously differentiable, and satisfies $f_k(0) = 0$, we have

$$f_k(a') = \int_0^{a'} D^+ f_k(t)\, dt,$$

where $D^+ f_k$ denotes the left derivative of f_k. We have

$$\begin{aligned} D^+ f_k(a') = (-p)^k &(\mathbf{1}(\langle (-p)^k a' \rangle < \langle q^k a' \rangle) - \langle q^k a' \rangle) \\ &+ q^k (\mathbf{1}(\langle (-p)^k a' \rangle > \langle q^k a' \rangle) - \langle (-p)^k a' \rangle) \\ &- q^k (\mathbf{1}(\langle q^k a' \rangle < \tfrac{1}{2}) - \tfrac{1}{2}), \end{aligned}$$

and hence $|D^+ f_k(a')| \le p^k + \frac{3}{2} q^k$.

We have $D^+ f_k(a') \to \frac{1}{2}(-q)^k$ as $a' \downarrow 0$. Actually, it is verified by $\langle (-p)^k a' \rangle \uparrow 1$ and $\langle q^k a' \rangle \downarrow 0$ if k is odd, and by $\langle (-p)^k a' \rangle \downarrow 0$, $\langle q^k a' \rangle \downarrow 0$, and $\langle (-p)^k a' \rangle > \langle q^k a' \rangle$ for small enough $a' > 0$ if k is even. Hence we have

$$2 \sum_{k=1}^{K} \frac{1}{(-pq)^k} D^+ f_k(a') \to \sum_{k=1}^{K} \frac{1}{p^k} = \frac{1 - 1/p^K}{p - 1} \quad \text{as } a' \downarrow 0.$$

On the other hand, we have

$$\left| 2 \sum_{k=K+1}^{\infty} \frac{1}{(-pq)^k} D^+ f_k(a') \right| \le \sum_{k=K+1}^{\infty} \left(\frac{2}{q^k} + \frac{3}{p^k} \right) = \frac{2}{q^K (q-1)} + \frac{3}{p^K (p-1)}.$$

Take large enough K satisfying

$$\frac{2}{q^K (q-1)} + \frac{3}{p^K (p-1)} \le \frac{1}{4} \frac{1 - 1/p^K}{p - 1},$$

and take small enough $A > 0$ satisfying

$$2t \le \frac{1}{4}\frac{1-1/p^K}{p-1}, \quad 2\sum_{k=1}^{K}\frac{1}{(-pq)^k}D^+ f_k(t) \ge \frac{3}{4}\frac{1-1/p^K}{p-1}, \quad (0 < t < A).$$

By noting $V(\frac{1}{2}-a', \frac{1}{2}-a') = \frac{1}{4}-a'^2 = \frac{1}{4}-\int_0^{a'} 2t\,dt$, we have

$$\begin{aligned}\sigma^2_{-p/q;a',1/2} &= \frac{1}{4} + \int_0^{a'}\left(-2t + \left(2\sum_{k=1}^{K} + 2\sum_{k=K+1}^{\infty}\right)\frac{D^+ f_k(t)}{(-pq)^k}\right)dt \\ &\ge \frac{1}{4} + \frac{1}{4}\frac{1-1/p^K}{p-1}a' > \frac{1}{4}\end{aligned}$$

for $0 < a' < A$. Hence we have $\Sigma_{-p/q} > \frac{1}{2}$.

Secondly assume that p is even and $q \ge 3$ is odd. In this case, we have $\langle(-p)^k\frac{1}{2}\rangle = 0$, $\langle q^k\frac{1}{2}\rangle = \frac{1}{2}$, and

$$f_k(a') = V(\langle(-p)^k a'\rangle, \langle q^k a'\rangle) - V(\langle(-p)^k a'\rangle, \tfrac{1}{2}), \quad (k \ge 1).$$

We have

$$\begin{aligned}D^+ f_k(a') &= (-p)^k(\mathbf{1}(\langle(-p)^k a'\rangle < \langle q^k a'\rangle) - \langle q^k a'\rangle) \\ &\quad + q^k(\mathbf{1}(\langle(-p)^k a'\rangle > \langle q^k a'\rangle) - \langle(-p)^k a'\rangle) \\ &\quad - (-p)^k(\mathbf{1}(\langle(-p)^k a'\rangle < \tfrac{1}{2}) - \tfrac{1}{2})\end{aligned}$$

and hence $|D^+ f_k(a')| \le \frac{3}{2}p^k + q^k$.

If k is odd, $D^+ f_k(a') \to \frac{1}{2}(-p)^k$ as $a' \downarrow 0$, and if k is even, $D^+ f_k(a') \to q^k - \frac{1}{2}(-p)^k$ as $a' \downarrow 0$. Therefore

$$\begin{aligned}&2\sum_{l=1}^{L}\left(\frac{-1}{(pq)^{2l-1}}D^+ f_{2l-1}(a') + \frac{1}{(pq)^{2l}}D^+ f_{2l}(a')\right) \\ &\to \sum_{l=1}^{L}\left(\frac{1}{q^{2l-1}} + \frac{2}{p^{2l}} - \frac{1}{q^{2l}}\right) = \frac{1-1/q^{2L}}{q+1} + 2\frac{1-1/p^{2L}}{p^2-1}.\end{aligned}$$

On the other hand, we have

$$\left|2\sum_{l=L+1}^{\infty}\left(\frac{-D^+ f_{2l-1}(a')}{(pq)^{2l-1}} + \frac{D^+ f_{2l}(a')}{(pq)^{2l}}\right)\right| \le \frac{3}{q^{2L}(q-1)} + \frac{2}{p^{2L}(p-1)}.$$

Take large enough L satisfying

$$\frac{3}{q^{2L}(q-1)}+\frac{2}{p^{2L}(p-1)}\leq\frac{1}{4}\frac{1-1/q^{2L}}{q+1},$$

and take small enough $A>0$ satisfying

$$2t\leq\frac{1}{4}\frac{1-1/q^{2L}}{q+1},\quad 2\sum_{l=1}^{L}\left(\frac{-D^{+}f_{2l-1}(t)}{(pq)^{2l-1}}+\frac{D^{+}f_{2l}(t)}{(pq)^{2l}}\right)\geq\frac{1-1/q^{2L}}{q+1}$$

for $0<t<A$. Then we have

$$\begin{aligned}&\sigma^2_{-p/q;a',1/2}\\&\quad=\frac{1}{4}+\int_0^{a'}\left(-2t+\left(2\sum_{l=1}^{L}+2\sum_{l=L+1}^{\infty}\right)\left(\frac{-D^{+}f_{2l-1}(t)}{(pq)^{2l-1}}+\frac{D^{+}f_{2l}(t)}{(pq)^{2l}}\right)\right)dt\\&\quad\geq\frac{1}{4}+\frac{1}{2}\frac{1-1/q^{2L}}{q+1}a'>\frac{1}{4}\end{aligned}$$

for $0<a'<A$. Hence we have $\Sigma_{-p/q}>\frac{1}{2}$.

Thirdly assume that $p=2$ and $q=1$. Since we have

$$\begin{aligned}\widetilde{V}\left(\left\langle(-2)^{6l}\tfrac{1}{7}\right\rangle,\left\langle(-2)^{6l}\tfrac{6}{7}\right\rangle,\tfrac{1}{7},\tfrac{6}{7}\right)&=\widetilde{V}\left(\tfrac{1}{7},\tfrac{6}{7},\tfrac{1}{7},\tfrac{6}{7}\right)=\tfrac{10}{49},\\
\widetilde{V}\left(\left\langle(-2)^{6l+1}\tfrac{6}{7}\right\rangle,\left\langle(-2)^{6l+1}\tfrac{1}{7}\right\rangle,\tfrac{1}{7},\tfrac{6}{7}\right)&=\widetilde{V}\left(\tfrac{2}{7},\tfrac{5}{7},\tfrac{1}{7},\tfrac{6}{7}\right)=\tfrac{6}{49},\\
\widetilde{V}\left(\left\langle(-2)^{6l+2}\tfrac{1}{7}\right\rangle,\left\langle(-2)^{6l+2}\tfrac{6}{7}\right\rangle,\tfrac{1}{7},\tfrac{6}{7}\right)&=\widetilde{V}\left(\tfrac{4}{7},\tfrac{3}{7},\tfrac{1}{7},\tfrac{6}{7}\right)=\tfrac{-2}{49},\\
\widetilde{V}\left(\left\langle(-2)^{6l+3}\tfrac{6}{7}\right\rangle,\left\langle(-2)^{6l+3}\tfrac{1}{7}\right\rangle,\tfrac{1}{7},\tfrac{6}{7}\right)&=\widetilde{V}\left(\tfrac{1}{7},\tfrac{6}{7},\tfrac{1}{7},\tfrac{6}{7}\right)=\tfrac{10}{49},\\
\widetilde{V}\left(\left\langle(-2)^{6l+4}\tfrac{1}{7}\right\rangle,\left\langle(-2)^{6l+4}\tfrac{6}{7}\right\rangle,\tfrac{1}{7},\tfrac{6}{7}\right)&=\widetilde{V}\left(\tfrac{2}{7},\tfrac{5}{7},\tfrac{1}{7},\tfrac{6}{7}\right)=\tfrac{6}{49},\\
\widetilde{V}\left(\left\langle(-2)^{6l+5}\tfrac{6}{7}\right\rangle,\left\langle(-2)^{6l+5}\tfrac{1}{7}\right\rangle,\tfrac{4}{7},\tfrac{3}{7}\right)&=\widetilde{V}\left(\tfrac{4}{7},\tfrac{3}{7},\tfrac{1}{7},\tfrac{6}{7}\right)=\tfrac{-2}{49},\end{aligned}$$

we have

$$\sigma^2_{-2;1/7,6/7}=\frac{10}{49}+2\left(\frac{1}{2}\cdot\frac{6}{49}+\frac{1}{4}\cdot\frac{-2}{49}+\frac{1}{8}\cdot\frac{10}{49}\right)\left(1+\frac{1}{8}+\frac{1}{8^2}+\cdots\right)=\frac{130}{7^3}>\frac{1}{4}.$$

Hence we have $\Sigma_{-2}\geq\frac{\sqrt{910}}{49}>\frac{1}{2}$.

Although we conjecture that this is the right value of Σ_{-2}, unfortunately we do not have a proof for it. We shall return to this evaluation in future.

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